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Applications of Laplace Transform Method to the Fractional Linear Integro Differential Equations

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Abstract. In this study, the Laplace Transformation Method (LTM) was used to get the exact solution to linear fractional integro-differential equations, which play a significant role in fractional differential equations . The Caputo fractional derivative is considered throughout this study. The examples considered demonstrate the effectiveness and applicability of LTM, which was used to provide the exact solution of the linear fractional integro differential equations.

Key Words and Phrases: Fractional calculus; Linear fractional integro differential equation; Laplace transform method.

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1. Introduction

Applications of fractional differential equations in mathematics and engineering have generated a great deal of attention. Particularly, coupled conduction, convection, and radiation problems are among the several physical phenomena for which fractional integro-differential equations are widely used in mathematical modeling [1]-[3].

In fractional differential equations, the linear fractional integro differential equation plays a significant role. The linear fractional integro-differential equation with the Caputo derivative is presented in the following manner [4];

$$^{c}D^{\alpha}y(t) = f(t) + \int_0^t K(t,s)y(s)ds, \qquad 0 \leq t \leq 1 \tag{1}$$

subject to the initial conditions

$$y^{i}(0) = \gamma_i, \quad i = 0, 1, \cdots, n-1 \quad , n \in \mathbb{N} \tag{2}$$

where y(t) is an unknown function of the independent variable t and $^{c}D^{\alpha}$ is the αth order derivative of y in the sense of Caputo fractional differential operator, $n-1 < \alpha \leq n$.

In applied sciences, mathematical physics, and problem-solving in engineering, integral transformations play a significant role [5]. One of the most helpful and efficient techniques for solving integral equations, ordinary differential equations, and partial differential equations is the use of integral transformation methods. These transformations have become widely employed in the recent past to solve fractional differential equations. Due to the fact that these transforms convert differential equations into algebraic equations.

The Laplace transform is an integral transform; It is an important transformation used in physics, mechanics, engineering, telecommunications, mathematics and other applied sciences. It was described by the famous mathematician Laplace. While this transformation provides great convenience in solving differential equations, it is also a method that can be used in the mathematical solution of physics. It has been seen that the equations obtained in this way also contain the initial conditions [6].

There are some studies concerning approximate solutions of fractional linear integro-differential equations in the literature. The Taylor expansion approximate method for solving linear fractional integro-differential equations was introduced [7]. It was proposed and investigated to solve general linear fractional integro-differential equations numerically using the spectral Jacobi-collocation technique [4]. For the solution of fractional linear integro-differential equations with a linear variable order, the collocation method based on the Haar wavelet is presented [8].

In this study, LTM was applied to find exact solution of linear fractional integro-differential equations. With the use of LT, the fractional differential equation is first transformed into an algebraic equation in this method, and the desired solution is then discovered using the inverse LT.

2. Fractional Calculus

In this section, fundamental fractional calculus concepts which employed throughout the study were given.

Gamma Function: One of the most fundamental functions in fractional calculus is the gamma function. For positive n, this function is defined by the Euler integral. The gamma function is convergent for positive values of n.

The Gamma function is defined by the improper integral [9]

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \qquad \Gamma : (0, \infty) \longrightarrow \mathbb{R} \tag{3}$$

The following equations are provided by the gamma function.

(i) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) $\Gamma(n+1) = n\Gamma(n)$

(iii) $\Gamma(n+1) = n!$

Beta Function: For the Beta function, which is defined as Euler's second integral, with m and n positive values,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \qquad \beta : (0,\infty)x(0,\infty) \longrightarrow \mathbb{R} \tag{4}$$

is defined as. For any non-positive m or n values, this integral is divergent. Besides, the relation showing the relationship between Beta and Gamma function is given as [9]

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{5}$$

The Caputo fractional derivative: There are various definitions of fractional derivatives in the literature. In this study, the Caputo fractional derivative definition was used. Because for initial value problems, Caputo's fractional derivative definition is the one that gives to most appropriate the initial conditions to the physical cases. Let f function can be continuously differentiable m times. The Caputo fractional derivative of function f is defined by the integral [10]

$$^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt \tag{6}$$

where α any positive integer and m is a positive integer such that $m \in \mathbb{N}$, $m-1 < \alpha < m$.

3. Laplace Transform Method

In this section, some properties and definition of LT which forms the basis of LTM were given.

Definition: If $f(t)$ is defined over interval $[0,\infty)$, the Laplace transform of $f(t)$, denoted as $F(s)$, is given as follow in [11, 12]:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}f(t)dt. \tag{7}$$

The Inverse Laplace Transform of $F(s)$ is defined as

$$f(t) = \mathcal{L}^{-1}[F(s)]. \tag{8}$$

The Laplace transform existence theorem states that, if $f(t)$ is piecewise continuous on every finite interval in $[0,\infty)$ satisfying

$$|f(t)| \leq Me^{at}$$

for all t in $[0,\infty)$, then $\mathcal{L}[f(t)]$ exists for all $s > a$.

The Laplace transform is also unique, in the sense that, given two functions $f_1(t)$ and $f_2(t)$ with the same transform so that

$$\mathcal{L}[f_1(t)] = \mathcal{L}[f_2(t)] = F(s).$$

Properties: The Laplace transform has many important properties. Some of these features are given below.

Linearity property:

Let LT of $f(t)$ be $F(u)$ and LT of $g(t)$ be $G(u)$. LT of linear sum of functions $f(t), g(t)$ is given as follow:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(u) + bG(u)$$

where $a, b \in \mathbb{R}$.

Convolution property:

Let LT of $f(t)$ be $F(u)$ and LT of $g(t)$ be $G(u)$. Convolution property of the Laplace transform is given as follow:

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}\left[\int_0^t F(\tau)G(t-\tau)d\tau\right] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s)$$

Laplace transform of integration:

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{F(s)}{s} \tag{9}$$

Laplace transform of derivative:

$$\begin{aligned}
\mathcal{L}[f'(t)] &= sF(s) - f(0) \\
\mathcal{L}[f''(t)] &= s^2F(s) - sf(0) - f'(0) \\
&\vdots \\
\mathcal{L}[f^{(n)}(t)] &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\end{aligned}$$

Laplace transform of Caputo fractional derivative:

$$\mathcal{L}[{}^CD^\alpha f(t)] = \frac{s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}{s^{n-\alpha}} \tag{10}$$

where $n \in \mathbb{N},\ n-1 < \alpha \le n$.

Laplace transforms of some fundamental functions are given in Table 1.

Table 1: Laplace transforms of some functions.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$t^n,\ n=1,2,3,\cdots$	$\frac{n!}{s^{n+1}}$
$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

4. Illustrative Examples

In this section, the application of LTM to fractional linear integro differential equation is presented for three test problems.

Example 1. As the first example, linear fractional integro differential equation was considered as following form:

$$^{c}D^{1/2}y(t) = y(t) + \frac{8}{3\Gamma(1/2)}t^{3/2} - t^2 - \frac{1}{3}t^3 + \int_0^t y(s)ds$$

and initial condition

$$y(0) = 0$$

The algebraic equation that results from applying LT on this equation is as follows:

$$\mathcal{L}\left[{}^{c}D^{1/2}y(t)\right] = \mathcal{L}\left[y(t) + \frac{8}{3\Gamma(1/2)}t^{3/2} - t^2 - \frac{1}{3}t^3 + \int_0^t y(s)ds\right]$$

$$s^{1/2}Y(s) = Y(s) + \frac{2}{s^{5/2}} - \frac{2}{s^3} - \frac{2}{s^4} + \frac{Y(s)}{s}$$

$$Y(s)(s^{3/2}-s-1) = s\left(\frac{2}{s^{5/2}}-\frac{2}{s^3}-\frac{2}{s^4}\right)$$

$$Y(s)(s^{3/2}-s-1) = \frac{2s^{3/2}-2s-2}{s^3}$$

$$Y(s) = \frac{2}{s^3}$$

where $\mathcal{L}[y(t)] = Y(u)$. The exact solution to the initial value problem is founded by calculating the inverse LT of this algebraic equation.

$$\mathcal{L}^{-1}\left[Y(u)\right] = \mathcal{L}^{-1}\left[\frac{2}{s^3}\right]$$

$$y(t) = t^2$$

Example 2. As the second example, linear fractional integro differential equation was considered as following form:

$${}^{c}D^{\sqrt{3}}y(t) = \frac{2}{\Gamma(3-\sqrt{3})}t^{2-\sqrt{3}} + 2\sin t - 2t + \int_0^t \cos(t-s)y(s)ds$$

and initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

The algebraic equation that results from applying LT on this equation is as follows:

$$\mathcal{L}\left[{}^{c}D^{\sqrt{3}}y(t)\right] = \mathcal{L}\left[\frac{2}{\Gamma(3-\sqrt{3})}t^{2-\sqrt{3}} + 2\sin t - 2t + \int_0^t \cos(t-s)y(s)ds\right]$$

$$s^{\sqrt{3}}Y(s) = \frac{2}{\Gamma(3-\sqrt{3})}s^{\sqrt{3}-3}\Gamma(3-\sqrt{3}) + \frac{2}{s^2+1} - \frac{2}{s^2} + \frac{s}{s^2+1}Y(s)$$

$$Y(s)\left(s^{\sqrt{3}}(s^2+1)-s\right) = (s^2+1)\left(2s^{\sqrt{3}-3} + \frac{2}{s^2+1} - \frac{2}{s^2}\right)$$

$$Y(s)(s^{2+\sqrt{3}} + s^{\sqrt{3}} - s) = \frac{2s^{\sqrt{3}+1} + 2s^{\sqrt{3}-1}}{s^2}$$

$$Y(s) = \frac{2}{s^3}$$

where $\mathcal{L}[y(t)] = Y(u)$. The exact solution to the initial value problem is founded by calculating the inverse LT of this algebraic equation.

$$\mathcal{L}^{-1}[Y(u)] = \mathcal{L}^{-1}\left[\frac{2}{s^3}\right]$$

$$y(t) = t^2$$

Example 3. As the third example, linear fractional integro differential equation was considered as following form:

$$^{c}D^{6/5}y(t) = \frac{6}{\Gamma(14/5)}t^{9/5} + t^3 + 3t^2 + 6t + 6e^t + \int_0^t e^{(t-s)}y(s)ds$$

and initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

The algebraic equation that results from applying LT on this equation is as follows:

$$\mathcal{L}\left[{}^{c}D^{6/5}y(t)\right] = \mathcal{L}\left[\frac{6}{\Gamma(14/5)}t^{9/5} + t^3 + 3t^2 + 6t + 6e^t + \int_0^t e^{(t-s)}y(s)ds\right]$$

$$s^{6/5}Y(s) = \frac{6}{s^{14/5}} + \frac{6}{s^4} + \frac{6}{s^3} + \frac{6}{s^2} + \frac{6}{s} - \frac{6}{s-1} + \frac{Y(s)}{s-1}$$

$$Y(s)(s^{11/5} - s^{6/5} - 1) = 6\left(\frac{s-1}{s^{14/5}} + \frac{s-1}{s^4} + \frac{s-1}{s^3} + \frac{s-1}{s^2} + \frac{s-1}{s} - 1\right)$$

$$Y(s)(s^{11/5} - s^{6/5} - 1) = 6\left(\frac{(s-1)s^{6/5} - 1}{s^4}\right)$$

$$Y(s) = \frac{6}{s^4}$$

where $\mathcal{L}[y(t)] = Y(s)$. The exact solution to the initial value problem is founded by calculating the inverse LT of this algebraic equation.

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{6}{s^4}\right]$$

$$y(t) = t^3.$$

5. Conclusion

In this study, LTM was used to determine the exact solution to the linear fractional integro-differential equation. To show the viability and effectiveness of the suggested method, three separate test problems for the linear fractional integro differential equation were examined. LTM has been seen to be a very efficient method for determining this equation's exact solution. Because of this, it is expected that this study will make significant contributions to the literature.

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