The Riemann Transform

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ABSTRACT

In his 1859 paper, Bernhard Riemann used the integral equation $\int_0^\infty f(x)\, x^{-s-1}\, dx$ to develop an explicit formula for estimating the number of prime numbers less than a given quantity. It is the purpose of this present work to explore some of the properties of this equation.

Introduction

Consider the integral equation given below

(1) $$F(s) = \int_0^{\infty} f(x)\, x^{-s-1}\, dx$$

Formula (1) is the integral of $f(x)$ times x^{-s-1} for $x = 0$ to ∞ and the resulting integral is denoted by $F(s)$ (or the **transform** of f). It must be assumed that $f(x)$ is such that $|F(s)| < \infty$ and s is a complex constant. Since s is constant,

$$F(s) = \text{constant}, \quad F'(s) = \frac{dF}{ds} = \frac{0}{0}, \quad \text{and} \quad \int_s^s F(s)ds = 0.$$

Example 1: Apply formula (1) to obtain the transform of $f(x) = e^{-x}$.

Solution. Substitute e^{-x} to (1)

$$F(s) = \int_0^{\infty} e^{-x} x^{-s-1} dx = \Gamma(-s), \qquad \Re(s) < 0, \quad \text{since} \quad \Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx, \quad \Re(s) > 0,$$

where $\Gamma(s)$ is the gamma "function" and $\Re(s)$ is the real part of the complex quantity s.

Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $\mu(x - a)$ is 0 for $x < a$, has a jump size 1 at $x = a$ (where it is usually consider as undefined), and is 1 for $x > a$, in a formula:

$$\mu(x-a) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases} \qquad a \geq 0.$$

The transform of $\mu(x - a)$ is

$$F(s) = \int_0^{\infty} x^{-s-1} \mu(x-a) dx = \int_a^{\infty} x^{-s-1} dx = \left. \frac{-x^{-s}}{s} \right|_a^{\infty} \quad ;$$

here the integration begins at $x = a$ (>0) because $\mu(x - a)$ is 0 for $x < a$. Hence

$$F(s) = \frac{a^{-s}}{s} \qquad (a > 0 \quad \text{and} \quad \Re(s) > 0).$$

Example 2: The Riemann Zeta Function is given by

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1,$$

obtain the transform of $\sum_{n=1}^{\infty} \mu(x-n), \quad n = 1,2,3,4,\ldots$

$$F(s) = \int_0^{\infty} \{\mu(x-1) + \mu(x-2) + \mu(x-3) + \ldots\} x^{-s-1} dx = \left.\frac{-x^{-s}}{s}\right|_1^{\infty} + \left.\frac{-x^{-s}}{s}\right|_2^{\infty} + \left.\frac{-x^{-s}}{s}\right|_3^{\infty} + \ldots$$

$$= \frac{1}{s}(1 + 2^{-s} + 3^{-s} + 4^{-s} + \ldots) = \frac{1}{s}\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{s}, \qquad \Re(s) > 1.$$

Example 3: Obtain the transform of $\pi(x) = \sum_p^{\infty} \mu(x-p)$, where p is a prime number, p = 2, 3, 5, 7, 11, ….

$$F(s) = \int_0^{\infty} \left\{\sum_p^{\infty} \mu(x-p) x^{-s-1} dx\right\} = \int_0^{\infty} \{\mu(x-2) + \mu(x-3) + \mu(x-5) + \mu(x-7) + \ldots\} x^{-s-1} dx$$

$$\pi(s) = \frac{1}{s}(2^{-s} + 3^{-s} + 5^{-s} + 7^{-s} + \ldots) = \frac{1}{s}\sum_p^{\infty} p^{-s} \qquad \Re(s) > 1.$$

Dirac's Delta Function

Consider the function

$$f_\tau(x-a) = \begin{cases} 1/\tau & \text{if } a \le x \le a+\tau \\ 0 & \text{otherwise.} \end{cases}$$

Its integral is

$$I = \int_0^{\infty} f_\tau(x-a) dx = \int_a^{a+\tau} \frac{1}{\tau} dx = 1.$$

We let now let τ becomes smaller and smaller and take the limit as $\tau \to 0$ ($\tau > 0$). This limit is denoted by $\delta(x-a)$, that is,

$$\delta(x-a) = \lim_{\tau \to 0} f_\tau(x-a)$$

and obtain

$$\delta(x-a) = \begin{cases} \infty & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(x-a)dx = 1.$$

$\delta(x - a)$ is called the **Dirac delta function** or the **unit impulse function**. For a *continuous* function $f(x)$ one uses the **sifting** property of $\delta(x - a)$,

$$\int_0^\infty f(x)\delta(x-a)dx = f(a).$$

To obtain the transform of $\delta(x - a)$, we write

$$f_\tau(x-a) = \frac{1}{\tau}\{\mu(x-a) - \mu(x-(a+\tau))\}$$

and take the transform

$$F(s) = \int_0^\infty f_\tau(x-a)x^{-s-1}dx = \frac{1}{\tau s}\left[a^{-s} - (a+\tau)^{-s}\right] = a^{-s}\frac{1 - \left(1+\frac{\tau}{a}\right)^{-s}}{\tau s}, \qquad a > 0 \text{ and } \Re(s) > 0.$$

Take the limit as $\tau \to 0$. By l'Hopital's rule, the quotient on the right has the limit $1/a$. Hence, the right side has the limit $a^{-(s+1)}$. The transform of $\delta(x - a)$ define by this limit is

$$F(s) = \int_0^\infty \delta(x-a)x^{-s-1}dx = a^{-(s+1)} \qquad a > 0.$$

Example 4: Obtain the transform of $\sum_{n=1}^{\infty} x\,\delta(x-n)$ and $\sum_{n=1}^{\infty} \delta(x-n)$.

$$\int_0^\infty \left\{\sum_{n=1}^\infty x\,\delta(x-n)\right\}x^{-s-1}dx = \sum_{n=1}^\infty n^{-s} = \zeta(s), \quad \Re(s) > 1,$$

$$\int_0^\infty \left\{\sum_{n=1}^\infty \delta(x-n)\right\}x^{-s-1}dx = \sum_{n=1}^\infty n^{-(s+1)} = \zeta(s+1), \quad \Re(s) > 0.$$

The Riemann Transform

If $f(x)$ is a function defined for all $x \geq 1$, its **Riemann transform** is the integral of $f(x)$ times x^{-s-1} for $x = 1$ to ∞. Let's denote it by $F(s)$ or $R\{f\}$,

(2) $$F(s) = R\{f\} = \int_1^{\infty} f(x)\, x^{-s-1}\, dx\ .$$

The given function $f(x)$ in (2) is called the **inverse transform** of $F(s)$ and is denoted by $R^{-1}\{F\}$; that is,

$$f(x) = R^{-1}\{F\}.$$

Example 5: Let $f(x) = 1$, find $F(s)$.

Solution. From (2) we obtain by integration

$$R\{f\} = R\{1\} = \int_1^{\infty} x^{-s-1} dx = -\frac{1}{s} x^{-s} \Big|_1^{\infty} = \frac{1}{s} \qquad (\Re(s) > 0).$$

Example 6: Let $f(x) = x^a$, where a is constant. Find $F(s)$.

Solution. From (2),

$$R\{x^a\} = \int_1^{\infty} x^a x^{-s-1} dx = -\frac{1}{s-a} x^{-(s-a)} \Big|_1^{\infty} = \frac{1}{s-a} \qquad (\Re(s-a) > 0).$$

THEOREM 1: Linearity of the Riemann Transform

The Riemann transform is a linear operation; that is, for any functions f(x) and g(x) whose transforms exist and any constants a and b the transform of af(x) + bg(x) exists, and

$$R\{af(x) + bg(x)\} = aF(s) + bG(s).$$

Example 7: Find the transforms of $\cosh(a \ln x)$ and $\sinh(a \ln x)$.

Solution. Since $\cosh(a\ln x) = \frac{1}{2}(x^a + x^{-a})$ and $\sinh(a\ln x) = \frac{1}{2}(x^a - x^{-a})$, we obtain from Example 6 and Theorem 1,

$$R\{\cosh(a\ln x)\} = \frac{1}{2}(R(x^a) + R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$R\{\sinh(a\ln x)\} = \frac{1}{2}(R(x^a) - R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \ .$$

Example 8: Let $f(x) = x^{\alpha i}$, where *i* is the imaginary operator $(i = \sqrt{-1})$. Find *F*(*s*).

Solution. From Example 6

$$R\{x^{\alpha i}\} = \frac{1}{s - \alpha i} = \frac{1}{s - \alpha i}\,\frac{s + \alpha i}{s + \alpha i} = \frac{s}{s^2 + \alpha^2} + i\frac{\alpha}{s^2 + \alpha^2}.$$

Example 9: Cosine and Sine

Derive the formulas

$$R\{\cos(\alpha \ln x)\} = \frac{s}{s^2 + \alpha^2} \quad \text{and} \quad R\{\sin(\alpha \ln x)\} = \frac{\alpha}{s^2 + \alpha^2}.$$

Solution. From Example 8 and Theorem 1

$$x^{\alpha i} = \cos(\alpha \ln x) + i\sin(\alpha \ln x)$$

$$R\{x^{\alpha i}\} = R\{\cos(\alpha \ln x)\} + iR\{\sin(\alpha \ln x)\} \quad \text{, thus}$$

$$R\{\cos(\alpha \ln x)\} = \frac{s}{s^2 + \alpha^2} \quad \text{and} \quad R\{\sin(\alpha \ln x)\} = \frac{\alpha}{s^2 + \alpha^2}.$$

THEOREM 2: *s*-Shifting Theorem

If f(x) has the transform F(s) (where s > k for some k), then $x^a f(x)$ has the transform F(s – a) (where s – a > k). In formulas,

$$R\{x^a f(x)\} = F(s-a)$$

or, if we take the inverse on both sides $x^a f(x) = R^{-1}\{F(s-a)\}.$

PROOF: We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_1^\infty x^{-(s-a)-1} f(x)dx = \int_1^\infty x^{-s-1}[x^a f(x)]dx = R\{x^a f(x)\}.$$

Example 10: From Example 9 and the *s*-Shifting theorem one can obtain the Riemann transform for

$$R\{x^a \cos(\alpha \ln x)\} = \frac{s-a}{(s-a)^2+\alpha^2} \quad \text{and} \quad R\{x^a \sin(\alpha \ln x)\} = \frac{\alpha}{(s-a)^2+\alpha^2}\ .$$

Existence and Uniqueness of Riemann Transforms

A function $f(x)$ has a Riemann transform if it does not grow too fast, say, if for all $x \geq 1$ and some constants M and k it satisfies

(3) $$|f(x)| \leq Mx^k.$$

THEOREM 3: Existence Theorem for Riemann Transforms

If f(x) is defined and piecewise continuous on every finite interval on x ≥ 1 and satisfies (3) *for all x ≥ 1 and some constants M and k, then the Riemann transform R{f} exists for all s > k.*

PROOF Since $f(x)$ is piecewise continuous, $x^{-s-1}f(x)$ is integrable over any finite interval on the x-axis,

$$|R\{f\}| = \left|\int_1^\infty f(x)x^{-s-1}\right| \leq \int_1^\infty |f(x)|x^{-s-1}dx \leq \int_1^\infty M x^k x^{-s-1}dx = \frac{M}{s-k}.$$

Uniqueness. If the Riemann transform of a given function exists, it is uniquely determined and if two *continuous* functions have the same transform, they are completely identical

Transforms of Derivatives

THEOREM 4: Riemann Transform of Derivatives
The transforms of the first and second derivatives of f(x) satisfy

(4) $$R(f') = (s+1)F(s+1) - f(1)$$

(5) $$R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1)$$

Formula (4) *holds if f(x) is continuous for all x ≥ 1 and satisfies* (3) *and f'(x) is piecewise continuous on every finite interval for x* ≥ 1. *Formula* (5) *holds if f and f' are continuous for all x* ≥ 1 *and satisfy* (3) *and f'' is piecewise continuous on every finite interval for x* ≥ 1.

PROOF: Using integration by parts on formula (4)

$$R\{f\} = \int_1^\infty f'(x)x^{-s-1}dx = [f(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f(x)x^{-s-2}dx = -f(1) + (s+1)F(s+1).$$

The proof of (5) now follows by applying integration by parts twice on it, that is

$$R\{f''\} = \int_1^\infty f''(x)x^{-s-1}dx = [f'(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f'(x)x^{-s-2}dx$$

$$= -f'(1) + (s+1)\left[f(x)x^{-s-2}|_1^\infty + (s+2)\int_1^\infty f(x)x^{-s-3}dx\right]$$

$$= -f'(1) - (s+1)f(1) + (s+2)(s+1)F(s+2).$$

Repeatedly using integration by parts as in the proof of (5) and using induction, we obtain the following Theorem.

THEOREM 5: Riemann Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $x \geq 1$ and satisfy (2). *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval for $x \geq 1$. Then the transform of $f^{(n)}$ satisfies*

$$R\{f^{(n)}\} = (s+n)(s+n-1)\cdots(s+1)F(s+n) - (s+n-1)(s+n-2)\cdots f(1) -$$

$$(s+n-2)(s+n-3)\cdots f'(1) - \cdots - f^{(n-1)}(1).$$

Example 11: Let $f(x) = x^2$. Then $f(1) = 1$, $f'(x) = 2x$, $f'(1) = 2$, $f''(x) = 2$. Obtain $R\{f\}$, $R\{f'\}$, and $R\{f''\}$.

Solution. $R\{f\} = F(s) = \dfrac{1}{s-2}$, $F(s+1) = \dfrac{1}{s-1}$, $F(s+2) = \dfrac{1}{s}$. Hence, by formulas (4) and (5),

$$R(f') = (s+1)\frac{1}{s-1} - 1 = \frac{2}{s-1} \quad \text{and} \quad R(f'') = (s+2)(s+1)\frac{1}{s} - (s+1) - 2 = \frac{2}{s}.$$

The Riemann Transform and the Laplace Transform

The Laplace transform is the integral of $f(y)$ times e^{-sy} from $y = 0$ to ∞ where $f(y)$ is defined for all $y \geq 0$. It is denoted by $L\{f\}$,

(6) $$L\{f\} = \int_0^\infty f(y) e^{-sy}\, dy.$$

The Riemann transform is given below

(7) $$R\{f\} = \int_1^\infty f(x)\, x^{-s-1}\, dx\,.$$

Replace $x = e^y$ (or $y = \ln x$) in formula (8) and since $x = 1$ to ∞, $y = 0$ (ln1) to ∞ (ln∞).

$$\int_1^\infty f(x)\, x^{-s-1}\, dx = \int_0^\infty f(e^y)\, e^{-sy-y}\, d(e^y) = \int_0^\infty f(y)\, e^{-sy}\, dy\,,$$

which is formula (6).

The Bilateral Laplace Transform

Formula (6) is usually called the **Unilateral** Laplace transform since the integral is evaluated from 0 to ∞. The integral below is known as the Bilateral Laplace transform because the integral is taken from -∞ to ∞,

(8) $$B\{f\} = \int_{-\infty}^\infty f(y) e^{-sy} dy.$$

Now, consider the integral equation

(9) $$\int_0^\infty f(x)\, x^{-s-1}\, dx\,,$$

Replace $x = e^y$ (or $y = \ln x$) in formula (4) and since $x = 0$ to ∞, $y = -\infty$ to ∞, thus

$$\int_0^\infty f(x)\, e^{-sx}\, dx = \int_{-\infty}^\infty f(e^y)\, e^{-ys-y}\, d(e^y) = \int_{-\infty}^\infty f(y)\, e^{-sy}\, dy\,,$$

which is (8).

Riemann Transform: General Formulas

Formula	Name
$F(s) = R\{f(x)\} = \int_1^\infty f(x)\,x^{-s-1}\,dx$ $f(x) = R^{-1}(F(s))$	Definition of Transform Inverse Transform
$R\{af(x) + bg(x)\} = aR\{f(x)\} + bR\{g(x)\}$	Linearity
$R\{x^a f(x)\} = F(s-a)$ $R^{-1}\{F(s-a)\} = x^a f(x)$	*s*-Shifting Theorem

Table: Some Riemann Transforms

	$f(x) = R^{-1}\{F(s)\}$	$F(s) = \int_1^\infty f(x)x^{-s-1}dx$
1	1	$\frac{1}{s}$
2	x	$\frac{1}{s-1}$
3	x^a	$\frac{1}{s-a}$
4	$x^{\alpha i}$	$\frac{1}{s-\alpha i}$
5	$\cos(\alpha \ln x)$	$\frac{s}{s^2+\alpha^2}$
6	$\sin(\alpha \ln x)$	$\frac{\alpha}{s^2+\alpha^2}$
7	$\cosh(a \ln x)$	$\frac{s}{s^2-a^2}$

8	$\sinh(a\ln x)$	$\frac{a}{s^2 - a^2}$
9	$x^b\cos(\alpha\ln x)$	$\frac{s-b}{(s-b)^2 + \alpha^2}$
10	$x^b\sin(\alpha\ln x)$	$\frac{\alpha}{(s-b)^2 + \alpha^2}$
11	$\frac{2}{\ln x}\{1-\cos(\alpha\ln x)\}$	$\ln\left(\frac{s^2+\alpha^2}{s^2}\right)$
12	$\frac{1}{\ln x}\sin(\alpha\ln x)$	$\arctan\frac{\alpha}{s}$
13	$\frac{2}{\ln x}\{1-\cosh(a\ln x)\}$	$\ln\left(\frac{s^2-a^2}{s^2}\right)$
14	$\frac{1}{\ln x}(x^b - x^a)$	$\ln\left(\frac{s-a}{s-b}\right)$

REFERENCE

Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity.* pp. 5-7.