A Short Disproof of the Riemann Hypothesis

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ABSTRACT

The **Riemann Hypothesis** is one of the most important unsolved problems in Mathematics and its validity will have a great consequence on the precise calculation of the number of primes. Riemann developed an explicit formula relating the number of primes with the hypothesized *non-trivial zeros* of the Riemann zeta function. Riemann hypothesis states that all the non-trivial zeros of the zeta function have real part equal to one-half.

Despite many attempts to solve it for about 150 years, no one have so far succeeded. The Riemann hypothesis is based on the **existence of the zeros o**f the zeta function. If it can be shown, that such zeros do not exist, then the Riemann Hypothesis is false or not valid.

Introduction

The Riemann zeta "function" is shown below

(1) $$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s = \sigma + \omega i$$

where n is the independent variable and s is a complex constant with real part σ and imaginary part ω

$$\Re(s) = \sigma \text{ and } \Im(s) = \omega,$$

respectively, and i is the imaginary unit equal to $\sqrt{-1}$. A positive real number associated with any complex quantity is known as its modulus, usually denoted by $|\zeta(s)|$. The quantity σ has a damping effect on $\zeta(s)$ while ω acts as a filter that can remove some of its components. Thus, the values of σ and ω have a great effect on the convergence of the infinite series in (1).

Since s is constant,

$$\zeta'(s) = \frac{d\zeta}{ds} = \frac{0}{0} \text{ and } \int_s^s \zeta(s)\,ds = 0.$$

The role of s is simply to ensure that the sum in (1) remains finite $|(\zeta(s))| < \infty$.

$\zeta(s)$ Has No Zeros

FIRST DISPROOF

$\zeta(s)$ is related to the distribution of prime numbers for one obtains from (1) the infinite product,

(2) $$\zeta(s) = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})(1-7^{-s}) \cdots} = \prod_p^{\infty} \frac{1}{1-p^{-s}} \qquad \sigma > 1.$$

This had driven Riemann to obtained a formula for relating the supposedly *non-trivial zeros* of $\zeta(½ + \omega i)$ with the number of primes given a certain number. A simple inspection of (2) and one can easily conclude that such zeros are nowhere to be found.

The infinite product in (2) runs through all the prime numbers p and is widely known as the Euler product. The modulus of (2) is given by

(3) $$|\zeta(\sigma + \omega i)| = \prod_p^{\infty} \frac{1}{\sqrt{1 - 2p^{-\sigma}\cos(\omega \log p) + p^{-2\sigma}}}$$

For $\sigma > 0$ and, for all ω and p

(4) $$1 - 2p^{-\sigma}\cos(\omega\log p) + p^{-2\sigma} > 0$$

since the least value of (4) is attained when $\cos(\omega\log p) = 1$ resulting in (4) still greater than zero,

$$(1 - p^{-\sigma})^2 > 0$$

Each individual term in (4) converges absolutely for $\sigma > 0$ while their product converges conditionally if $0 < \sigma \le 1$, and their product converges absolutely for $\sigma > 1$. In fact, the divergent nature of $\zeta(s)$ at $0 < \sigma \le 1$ proves the existence of the infinity of primes but at $\sigma \le 0$ it is completely invalid. Also, as a consequence of (4), the zeta function has no zeros and its modulus is always greater than zero

$$\zeta(s) \ne 0 \qquad |\zeta(s)| > 0, \quad \sigma > 0,$$

therefore, the **Riemann hypothesis is false or not valid**.

SECOND DISPROOF

If $\sigma > 1$, the series (1) converges absolutely

$$|\zeta(\sigma+\omega i)| = \left|\sum_{n=1}^{\infty} n^{-\sigma+\omega i}\right| \le 1+2^{-\sigma}+3^{-\sigma}+4^{-\sigma}+5^{-\sigma}+\cdots+n^{-\sigma} = \sum_{n=1}^{\infty} n^{-\sigma},$$

while if $0 < \sigma \le 1$ and $\omega \ne 0$, the series is said to be conditionally convergent. It may converge if σ is large enough. But, how large?

The series (1) can be express as

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-s\log n} = \sum_{n=1}^{\infty} e^{-\sigma\log n - i\omega\log n} = \sum_{n=1}^{\infty} n^{-\sigma}(\cos\omega\log n - i\sin\omega\log n),$$

where $\log n$ is the natural logarithm of n and its modulus is

$$|\zeta(\sigma + \omega i)| = \sqrt{\left(\sum_{n=1}^{\infty} n^{-\sigma}\cos\omega\log n\right)^2 + \left(\sum_{n=1}^{\infty} n^{-\sigma}\sin\omega\log n\right)^2},$$

or

$$|\zeta(\sigma+\omega i)| = \sqrt{\sum_{n=1}^{\infty} n^{-2\sigma} + 2\sum_{n=2}^{\infty} n^{-\sigma}\cos\omega\log n + 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-\sigma}(n+k)^{-\sigma}\cos\omega\log\left(\frac{n+k}{n}\right)}.$$

Also

$$|\zeta(\sigma+\omega i)| = \sqrt{S_1 + S_2 + S_3} = \sqrt{S},$$

where

$$S_1 = \sum_{n=1}^{\infty} n^{-2\sigma}, \quad S_2 = 2\sum_{n=2}^{\infty} n^{-\sigma}\cos\omega\log n, \quad S_3 = 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-\sigma}(n+k)^{-\sigma}\cos\omega\log\left(\frac{n+k}{n}\right),$$

and $S = S_1 + S_2 + S_3$.

The first series S_1 is independent of ω and converges absolutely if $\sigma > ½$; the second series S_2 converges absolutely if $\sigma > 1$; and the double sum S_3 converges quickly due to its highly damped coefficients. The value of the sum S must be greater than or equal to zero ($S \geq 0$) in order for (1) to be valid; since if $S < 0$, the modulus $|\zeta(\sigma+\omega i)|$ will not be a real number.

It is known that $|\zeta(\sigma+\omega i)|$ is always greater than zero if $\sigma > 1$. The sum $S_1 + S_2$, will converge to a value say, A, and is unlikely to reduce $|\zeta(\sigma+\omega i)|$ to zero due to S_1 providing a fixed positive value for a given value of σ: that is $S_1 > |A|$ so that $|\zeta(\sigma+\omega i)| = \sqrt{S_1 + A} > 0$. Hence, $\zeta(s)$ has no zeros if $\sigma > 1$.

If $½ < \sigma \leq 1$, S_1 converges and the sum $S_1 + S_2$ converges for $\omega \neq 0$. Hence $|\zeta(\sigma+\omega i)|$ is conditionally convergent if $½ < \sigma \leq 1$ ($\omega \neq 0$). One can also conclude that $\zeta(s)$ has no zeros for the same reason given above.

If $0 < \sigma \leq ½$ and for every ω, S_1 diverges and it doesn't matter whether the sum $S_1 + S_2$ converges or not, $\zeta(s)$ is undefined. It is also interesting to note that $\zeta\left(\frac{1}{2} + \omega i\right)$ is **undefined** which proves that the **Riemann Hypothesis is not valid**.

If $\sigma \leq 0$ and for every ω, S_1 diverges very rapidly and $\zeta(s)$ is undefined.

Therefore,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \begin{cases} \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0 \\ \sigma > 1. \end{cases}$$

It has no zeros and its modulus is always greater than zero,

$$|\zeta(s)| > 0 \begin{cases} \text{If } \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0 \\ \text{If } \sigma > 1\,. \end{cases}$$

REFERENCE

[1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Magnitude*.

LINKS

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis