Deep on Goldbach's conjecture

Frank Vega[1*]

[1*]Software Department, CopSonic, 1471 Route de Saint-Nauphary, Montauban, 82000, Tarn-et-Garonne, France.

Corresponding author(s). E-mail(s): vega.frank@gmail.com;

Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. The Goldbach's conjecture has been verified for every even number $\boldsymbol{N \leq 4 \cdot 10^{18}}$. In this note, we prove that for every even number $\boldsymbol{N \geq 4 \cdot 10^{18}}$, if there is a prime $\boldsymbol{p}$ and a natural number $\boldsymbol{m}$ such that $\boldsymbol{n < p < N - 1}$, $\boldsymbol{p + m = N}$, $\boldsymbol{\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n}$ and $\boldsymbol{p}$ is coprime with $\boldsymbol{m}$, then $\boldsymbol{m}$ is necessarily a prime number when $\boldsymbol{N = 2 \cdot n}$ and $\boldsymbol{\sigma(m)}$ is the sum-of-divisors function of $\boldsymbol{m}$. The previous inequality $\boldsymbol{\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n}$ holds whenever $\boldsymbol{\frac{N}{e^{\gamma} \cdot m \cdot \log\log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n}$ also holds and $\boldsymbol{m \geq 11}$ is an odd number, where $\boldsymbol{\gamma \approx 0.57721}$ is the Euler-Mascheroni constant and $\boldsymbol{\log}$ is the natural logarithm. This implies that the Goldbach's conjecture is true when the Riemann hypothesis is true.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$ is the sum of a prime and a product of at most two primes [2]. A natural number is called k-almost prime if it has k prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let N be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where p denotes a prime and P_2 denotes an almost prime with at most two prime factors [3]. The Goldbach's conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$ [4]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

Theorem 1 *For every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$. The previous inequality $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot m \cdot \log \log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ also holds and $m \geq 11$ is an odd number, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and* $\log$ *is the natural logarithm. This implies that the Goldbach's conjecture is true when the Riemann hypothesis is true.*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 4 \cdot 10^{18}$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n - k, n + k)$

where $n = \frac{N}{2}$, $k < n - 1$ is a natural number, $n + k$ and $n - k$ are coprime integers and $n + k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n - k$ is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n - k$ is not a prime. Certainly, we see that $(n - k) + (n + k) = N$ and thus, the inequality

$$2 \cdot ((n-k) + (n+k)) + \varphi((n-k) \cdot (n+k)) < \sigma((n-k) \cdot (n+k))$$

holds when $n - k$ is not a prime. That is equivalent to

$$2 \cdot ((n-k) + (n+k)) + \varphi(n-k) \cdot \varphi(n+k) < \sigma(n-k) \cdot \sigma(n+k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n - k) \cdot (n + k)$ to obtain that

$$2 \cdot \left(\frac{(n-k) + (n+k)}{(n-k) \cdot (n+k)} \right) + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k} < s(n-k) \cdot s(n+k).$$

We know that

$$s(n-k) \cdot s(n+k) > 1$$

since $s(m) > 1$ for every natural number $m > 1$ [5]. Moreover, we could see that

$$2 \cdot \left(\frac{(n-k) + (n+k)}{(n-k) \cdot (n+k)} \right) = \frac{2}{n+k} + \frac{2}{n-k}$$

and therefore,

$$1 > \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1 > \frac{2}{2 \cdot 10^{18}} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}$$

when $n + k$ is prime and $n - k$ is composite for $N \geq 4 \cdot 10^{18}$. Indeed, when $n + k$ is prime and $n - k$ is composite, then $n + k > 2 \cdot 10^{18}$ and $n - k \geq 9$ for $N \geq 4 \cdot 10^{18}$. Under our assumption, all these pairs of positive integers $(n - k, n + k)$ imply that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

holds whenever $n = \frac{N}{2}$, $k < n - 1$ is a natural number, $n + k$ and $n - k$ are coprime integers and $n + k$ is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k)).$$

Since $n + k$ is prime, then

$$\begin{aligned}
\frac{\varphi(n+k)}{1 + n^{0.889}} &= \frac{n+k-1}{1 + n^{0.889}} \\
&\geq \frac{n}{1 + n^{0.889}} \\
&\geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-1) + \frac{2.5}{\log\log(n-1)} \right)^2 \\
&\geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-k) + \frac{2.5}{\log\log(n-k)} \right)^2
\end{aligned}$$

$$
\begin{aligned}
&> 2 \cdot \left(\frac{n-k}{\varphi(n-k)}\right)^2 \\
&= \frac{n-k}{\varphi(n-k)} \cdot 2 \cdot \prod_{q \mid (n-k)} \left(\frac{q}{q-1}\right) \\
&> s(n-k) \cdot 2 \cdot \prod_{q \mid (n-k)} \left(\frac{q}{q-1}\right) \\
&= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q \mid (n-k)} \left(1 - \frac{1}{q}\right)} \\
&= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}
\end{aligned}
$$

when we know that $\frac{b}{\varphi(b)} < e^{\gamma} \cdot \log\log(b) + \frac{2.5}{\log\log(b)}$ holds for every odd number $b \geq 3$ [6]. Moreover, we have

$$\frac{n}{1 + n^{0.889}} \geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-1) + \frac{2.5}{\log\log(n-1)}\right)^2$$

for every natural number $n \geq 2 \cdot 10^{18}$ under the supposition that $N \geq 4 \cdot 10^{18}$. Certainly, the function

$$f(x) = \frac{x}{1 + x^{0.889}} - 2 \cdot \left(e^{\gamma} \cdot \log\log(x-1) + \frac{2.5}{\log\log(x-1)}\right)^2$$

is strictly increasing and positive for every real number $x \geq 2 \cdot 10^{18}$ because of its derivative is greater than 0 for all $x \geq 2 \cdot 10^{18}$ and it is positive in the value of $2 \cdot 10^{18}$. Furthermore, it is known that $\prod_{q \mid b} \left(\frac{q}{q-1}\right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$ for every natural number $b \geq 2$ [5]. Finally, we would have that

$$-\frac{1}{2} \cdot \varphi(n-k) \cdot \varphi(n+k) < -\sigma(n-k) \cdot (1 + n^{0.889})$$

and so,

$$N < \frac{1}{2} \cdot \sigma(n-k) \cdot \sigma(n+k) - \sigma(n-k) \cdot (1 + n^{0.889}).$$

We would have

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 < \frac{\sigma(n+k)}{2}$$

which is

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} < n.$$

In this way, we obtain a contradiction when we assume that $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. By reductio ad absurdum, the natural number $n-k$ is necessarily prime when $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. Moreover, we know that $\sigma(b) < e^{\gamma} \cdot b \cdot \log\log b$ holds for every odd number $b \geq 11$ [5]. Consequently, the inequality $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot (n-k) \cdot \log\log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ also holds and $(n-k) \geq 11$ is an odd number. In 2014, Dudek proved that the Riemann hypothesis implies that for all $x \geq 2$ there is a prime p satisfying [7]

$$x - \frac{4}{\pi}\sqrt{x}\log x < p \leq x.$$

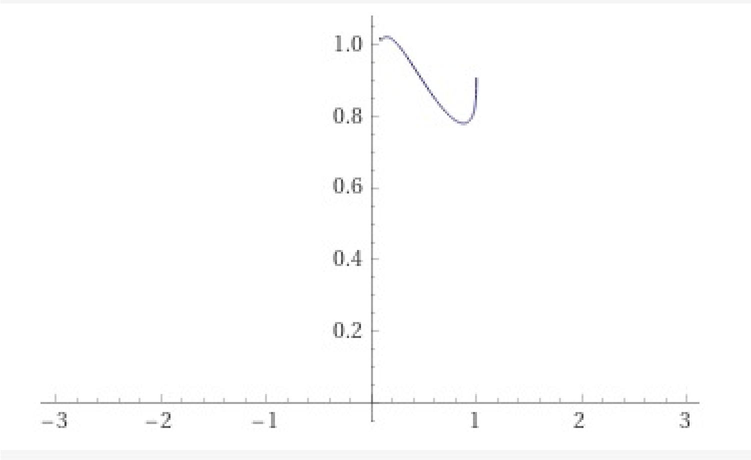

Fig. 1 Plot of function $H_4(x)$ [8]

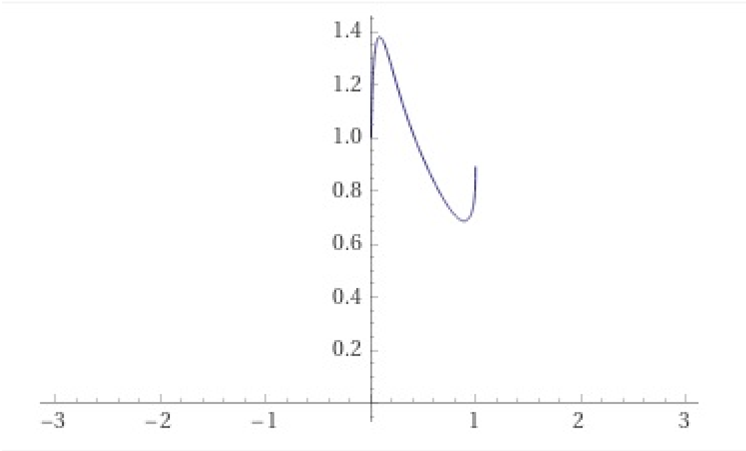

Fig. 2 Plot of function $H_8(x)$ [9]

In this way, there is always a prime $n+k$ for $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$. However, we know the inequality $\frac{2 \cdot n}{e^{\gamma} \cdot (n-k) \cdot \log\log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ holds for all positive integers $n \geq 2 \cdot 10^{18}$ and $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$ since the function $H_a(x) = \frac{x}{(x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x) \cdot \log\log(x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x)} + x^{0.889} + 1 + \frac{x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x - 1}{2} - x$ is positive for all $x \geq 2 \cdot 10^{18}$ and $a \in \{4, 8\}$ (See Figures 1 and 2). Certainly, we know that $H_a(n) \leq \frac{2 \cdot n}{e^{\gamma} \cdot (n-k) \cdot \log\log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} - n$ for all positive integers $n \geq 2 \cdot 10^{18}$ and $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$, where we select the appropriated value of $4 \leq a \leq 8$ according to the value of k. $\square$

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