Entropic and Newtonian Forces in a Maxwell-Boltzmann Gas

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In this note we try to understand the link between entropic and Newtonian forces in a Maxwell-Boltzmann (MB) gas. We begin by considering particles in a box with no Newtonian potential. This particles are endowed with a special property e which is conserved in stochastic "interactions" among the particles i.e. e1+e2=e3+e4 and f(e1)f(e2) = f(e3)f(e4). Taking ln of the latter and linking to the former yields the usual MB distribution exp(-e/T), but we do not assume that e is kinetic energy. In the case of a box, we assume physical density is constant as there is no special information indicating that it is not. In order to achieve this result, one may consider a possible solution with the particles to all have constant velocity or 0 velocity which is not the MB case.

Next, we consider the same situation, but impose a function of V(x) such that: e(x1)+V(x1) = e(x2) + V(x2). This function is not assumed to be a potential and e is not assumed to be kinetic energy of a particle. Using the idea of information we find that P(x,e) = C exp(-e/T)exp(-V(x)/T). This, however, suggests some kind of density at x, but the ratios of exp(-e/T) remain unchanged at any x so the distribution in e is not changing in x. It seems that only particle number at x may change and this seems to imply a change in velocity(speed) at x. In other words, V(x) forces velocities of the particles to change with x i.e. there is acceleration. Furthermore, we argue that e must be linked to v. In such a case, the reaction e1+e2=e3+e4 is also associated with changing v and so is also linked to the idea of Newtonian force subject to statistical equilibrium considerations and hence called an entropic force.

We try to further argue that -dV/dx also behaves like a force and must balance a d/dx ln(density), thus Newtonian mechanics seems to apply to chunks of statistical material even though particles are moving in and out of such a chunk. Finally given that e is linked to v and -v we try to use statistical arguments to find the exact relationship between the two. We argue that e must be equivalent to v to a single even power with 2 being the simplest. We further argue that the de/dv is then of the form of a flux so that e=bvv may be associated with a more general expression E=mo + bvv and de/dv with mov the first order flux (i.e. from special relativity).

Box with Gas Particles and No V(x)

We first consider a box in which there are particles which have an unknown property e (which we do not consider to be kinetic energy) and undergo stochastic 2 body changes i.e.

$$e1+e2 = e3+e4 \quad ((1a)) \quad \text{with} \quad n(e1)n(e2) = n(e3)n(e4) \quad ((1b))$$

with n(e) being the probability to find a particle with e at x. Given that the box is the same throughout we assume n(e) is not a function of x.

Taking ln of ((1a)) and linking with ((1b)) yields:

P(e) = C exp(-e/T) where <e average> = Sum over i C ei exp(-ei/T) to defined T ((2))

If spatial density is constant, one may picture particles at rest or moving at constant speed, say in a one dimensional picture. Thus, this is quite different from the full Maxwell-Boltzmann gas situation.

Box with Gas Particles with V(x)

Next, we assume the above scenario, but add a function V(x) (not considered to be a potential) such that:

$$e(x1) + V(x1) = e(x2) + V(x2) \quad ((3))$$

In other words, the property e changes in x according to the function V(x). Then ((2)) implies by information arguments which do not distinguish the total values on the RHS and LHS of ((3)) that:

$$P(x,e) = C2 \exp(-e/T) \exp(-V(x)/T) \quad ((4))$$

One may note that ratios at a fixed x are the same for any x because exp(-V(x)/T) cancels out. Thus the e distribution is the same at all x points. As a result, exp(-V(x)/T) must distinguish between x values and it seems the only way to do this is by changing particle number at x. This implies the particles must have different speeds at various x values so that dt, ,the time spent in a dx, differs. Thus V(x) which changes the unknown property e, must also change speed.

One may next ask: Is it possible for the change in e with x and the change in v with x to be independent? This does not seem possible because exp(-V(x)/T) is obtained using strictly information arguments and the variable e, yet the result must be consistent with the speed changing. Thus the two cannot be unrelated. If we assume that e is positive, then:

A given e is linked with both -v and +v where v is speed. ((5))

We assume a 1-1 link between e and |v|. This immediately has consequences for the box with no V(x) considered in the first section. In the case that e is linked with v, the reaction e1+e2=e3+e4 must necessarily involve changes in v or Newtonian force. The probability exp(-e/T) follows strictly from information considerations (equilibrium) and so the average of this force (called pressure) is called an entropic force. Then there exists an average entropic force which is the same at every x point in the box with no V(x). This idea may be extended by applying the pressure (even though it is not fully defined here) to a small box within the larger box. The pressure is the same on all walls of the box and so it does not move, even though particles stream out and in. As a result, its mass is the same (although its physical constituents) continually change. This is in keeping with a Newtonian balance of forces for a stationary object, but now the force may be entropic and the mass may constitute a system which is linked with flows in and out such that there is no net loss of mass and <momentum> average = 0 i.e. no overall flow or a static picture.

A question which arises is whether a similar scenario holds for the gas with the V(x) function. It was seen that a particle must accelerate according to V(x), so it is linked with a Newtonian acceleration or force. There is also the pressure or entropic force in the picture. If one considers:

$$\frac{d}{dx} C1\exp(-V(x)/T) = - dV/x \ (1/T)\ C1 \exp(-V(x)/T) \rightarrow \frac{d}{dx} \ln(\text{density}) = -(1/T)dV/dx \quad ((6))$$

then one sees the form ln(density) which is identical in form to ln(n(e)) in section 1. There it was equated to e. Here it is linked with density i.e.

$$\ln(C1) - V(x)/T = \ln(\text{density}) \text{ or } \ T \ln(\text{density}(x)) + V(x) = T\ln(C1) \quad ((7))$$

((7)) is a conservation equation which differs from ((3)).

One may consider the similarity with thermodynamics i.e.

E-TS = Free energy = Sum over i ei C1 exp(-ei/T) + T Sum over i C1exp(-ei/T) ln[C1exp(-ei/T)]

= Tln(C1) ((8))

Comparing ((7)) and ((8)) V(x) is like E and S is like: - ln(density(x)).

What about pressure balance? We argued above that if e is linked to v then the reaction e1+e2=e3+e4 is linked with a force which changes v values. Thus we argued there is a force in the problem which we called an entropic force, but we did not define it. We simply stated that it depends on the distribution Cexp(-ei/T) at each x. If density values change, one may argue that this force scales as density as the distribution remains unchanged. Doubling the particles with the same distribution should create double the pressure. Thus

Pressure= C3(T) density(x) where C3(T) is a constant. Pressure is an entropic force depending on the statistical constant T.

((6)), however, gives a balance equation for two pressures i.e.

Pressure(x+dx)-P(x) = dx C3(T) d/dx density = -C3(T) (1/T) density dV/dx ((9))

Thus the Newtonian picture of a little box with density(x) acted on by a force -dV/dx seems to balance with a pressure difference given by:

Pressure= T density(x) ((10)) (one may note that T is really a constant multiplied by T to give it the correct units if T is measured in Kelvin.)

Thus one does not need to know the exact expressions creating pressure in terms of individual momenta or velocity in order to find the form of overall pressure or to find that -dV/dx acts like a force.

Form of e

In the above, we assumed first that e was an unknown property and then deduced that it must be linked to speed. We argued that -v and v both link to a single e, thus e should be an even function in v. This is still a quite general expression.

The form of pressure in ((10)) i.e. Pressure = T density is consistent with the e dependent form of pressure satisfying:

Pressure = Sum over i a ei C1 exp(-ei/T) because Average energy = Sum over i ei C exp(-ei/T).

So aei = force part * v where a is an unknown constant ((11))

Pressure, however, should be proportional to flux (i.e. speed v) times a force or force averaged over time.

((3)) implies that: de/dx = de/dv dv/dt dt/dx = -dV/dx or

de/dv (1/v) v = Integral dt (-dV/dx)= averaged force over time ((12))

de/dv = a e/v ((13)) with a unknown

((13)) implies a single even power of v for e, it seems, the simplest being 2.

If the power 2 is taken, then e goes as vv or de/dv goes as v, which is of the form of a flux (although one needs the factor in front of v). This is suggestive of a quantity e and a related flux if masses are used in the factors. In fact, such arguments follow from special relativity, but the power of 2 anticipates them. It is also associated with the vector v being linked to its modulus squared. Higher powers of 2 do not have this property.

Thus it seems that one may almost determine the relationship between e and v nonrelativistically using statistical equilibrium considerations alone.

Conclusion

In conclusion, we try to consider an entropic force by starting with the notion of a particle with a property e such that interactions occur between the particles with e being conserved i.e. e1+e2=e3+e4 and n(e1)n(e2)=n(e3)n(e4). We argue this leads to n(e) = C exp(-e/T) where T is a parameter fixed by the average value of e. Next we introduce a function V(x) such that e(x) + V(x) = constant ((A)) for a single particle. Using information arguments, this leads to n(e,x) = C2 exp(-e/T) exp(-V(x)/T), but ratios remain the same as the V(x) piece cancels out at any x. Thus the distribution does not change by moving from one x to another, so what does? We argue that only the particle number at x may change. This suggests there must be different speeds in the picture i.e. e must be linked to v and -v. Using ((A)) we argue that -dV/dx acts as a force and there exists a pressure (entropic force) of the form pressure = constant T density(x) such that there is a balance. We then try to find the form of e in terms of v by considering pressure to be linked to a flux times a force or averaged force. This leads to the conclusion that e must be a single even power of v. We suggest the power of 2 because this leads to an e=bvv term and de/dv which is in the form of a flux. This suggests that e=bvv might be part of a more general expression e= mo + bvv and p=mov the first order flux i.e. a power of 2 leads to special relativistic notions.