ORDER TYPES OF SHIFTS OF MORPHIC WORDS

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Abstract

The shifts of an infinite word $W = a_0a_1\cdots$ are the words $W_i = a_ia_{i+1}\cdots$. As a measure of the complexity of a word W, we consider the order type of the set of shifts, ordered lexicographically. We consider morphic words (fixed points of a morphism under a coding) that are not ultimately periodic. Our main result in this setting is that if the first letter of W appears at least twice in W, then the shifts of the aperiodic image of W under a coding are dense in the sense that there is a shift strictly between any two shifts. In particular, any purely morphic binary word is either ultimately periodic or its shifts are dense. As a concrete example, we give an explicit order-preserving bijection between the shifts of the Thue–Morse word and $(0, 1] \cap \mathbb{Q}$. We then give special consideration to morphisms on 3 letters whose shifts do not contain an infinite decreasing sequence.

1. Introduction

A recurring theme in the combinatorics on words is classifying words according to their complexity. For example, the subword complexity of a word $W = a_0a_1a_2\cdots$ is the function

$$C(n) := \#\{a_ia_{i+1}\cdots a_{i+n-1} : i \geq 0\}.$$

The subword complexity is a bounded function if and only if W is ultimately periodic, and $C(n) = N^n$ almost surely if W is a random word, where N is the size of the alphabet. Other measures of complexity in the literature include the abelian complexity and Kolmogorov complexity. See [2].

In this work, we consider the order-type complexity $ot(W)$. We say that two totally ordered sets $\mathcal{S}$ and $\mathcal{T}$ have the same *order type* if there exists a bijection $f : \mathcal{S} \to \mathcal{T}$ such that f is order-preserving, i.e., $s_i < s_j$ implies $f(s_i) < f(s_j)$ for all $s_i, s_j \in \mathcal{S}$. If for all $s_i, s_j \in \mathcal{S}$ such that $s_i < s_j$, there exists s_k for which $s_i < s_k < s_j$, then $\mathcal{S}$ is *dense*. If $\mathcal{S}$ is infinite and countable, then $\mathcal{S}$ is dense if

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and only if $\mathcal{S}$ has the same order type as the order type of $(0,1)\cap\mathbb{Q}$, $(0,1]\cap\mathbb{Q}$, $[0,1)\cap\mathbb{Q}$, or $[0,1]\cap\mathbb{Q}$, depending on whether or not $\mathcal{S}$ contains a maximum or minimum element. Given a right-infinite word W, we define $W_i = a_i a_{i+1}\cdots$ to be the *i-shift of W* for integers $i \geq 0$ and $\{W_i\}$ to be the set of all shifts of W. We can see that $ot(W)$, defined to be the order type of $\{W_i\}$, is finite if W is ultimately periodic, while $ot(W)$ is dense almost surely if W is a random word. Closely related ideas such as infinite permutations and permutation complexity of infinite words have also been studied. See [6, 8–10, 13], for example.

Our main result (Theorem 2) gives the order type of aperiodic morphic words W over a finite alphabet $\Sigma_m = \{0,\ldots,m-1\}$. In particular, $ot(W)$ is dense if the first letter of the purely-morphic pre-image W' of W occurs at least twice in W', but $\{W_i\}$ cannot contain both a maximum element and a minimum element. This implies that any aperiodic binary purely morphic word must have a dense order type (Corollary 7). As an example, we give an explicit ordering for the shifts of the Thue–Morse word in Section 3, showing that the first 2^k shifts are interleaved by the next 2^k shifts (Theorem 1). There is a maximum shift and no minimum shift, so $ot(\mathbf{t})$ is the same as the order type of $(0,1]\cap\mathbb{Q}$.

The special case of words over a finite alphabet was considered in [5], where it was shown that if the order type is an ordinal, then it is either finite or at least ω^2, and can be larger than any particular countable ordinal. In Section 5, we show that $ot(W)$ varies when we consider a 3-letter alphabet, and we give the explicit forms of aperiodic uniform morphic words whose shifts do not contain either an infinite decreasing sequence or an infinite increasing sequence in terms of their associated uniform morphisms in Theorems 3-6, and confirming that the lower bound on the order type in [5] can be attained for uniform morphic words.

2. Definitions and Terminology

We begin with some preliminary definitions. Let $W = a_0a_1a_2\cdots$ be a right-infinite word. We say that W is a *word over the alphabet* $\mathcal{A}$ if $a_i \in \mathcal{A}$ for all $i \geq 0$. An element in $\mathcal{A}$ is called a *letter*. In this paper, we will assume that $\mathcal{A}$ is finite, and $\mathcal{A}$ is *optimal*, i.e., if $\alpha \in \mathcal{A}$, then there exists i such that $a_i = \alpha$. The i-shift of W is the word $W_i = a_ia_{i+1}a_{i+2}\cdots$, and we use $\{W_i\}$ to denote the set of all i-shifts (or simply shifts) of W. A *proper shift* of W is an i-shift of W with $i \geq 1$. We also assume that $\mathcal{A}$ is *totally ordered*, i.e., if α and β are distinct letters in $\mathcal{A}$, then either $\alpha < \beta$ or $\beta < \alpha$. We order $\{W_i\}$ lexicographically and we let $\mathcal{A} = \Sigma_m = \{0,1,\ldots,m-1\}$ if $|\mathcal{A}| = m$ for the implicit order among its letters.

Using terminology from [1], we define $W[i..j]$ to be the factor $a_i\cdots a_j$. If $i > j$, then $W[i..j] = \varepsilon$, the empty word. The *i-th letter of W* is $W[i] = W[i..i] = a_i$. We also define the *i-shift of a finite word* $u = b_0\cdots b_l$ to be $u_i = b_i\cdots b_l$ if $i \leq l$

and ε if $i > l$. We denote the set of all finite words over $\mathcal{A}$ by $\mathcal{A}^*$, which forms a free monoid with concatenation as the operation. The set of all non-empty finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^+$, while $\mathcal{A}^\omega$ is the set of all right-infinite words over $\mathcal{A}$. Let $\mathcal{W} \subseteq \mathcal{A}^*$. If l is a positive integer, then $\mathcal{W}^l$ denotes the set of all words that are concatenations of l elements in $\mathcal{W}$, or an l-concatenation of elements in $\mathcal{W}$, while $\mathcal{W}^\omega$ denotes the set of all infinite words that are concatenations of elements in W. Let $u, v \in \mathcal{A}^*$. We say that a finite word is of the form $(u, v)^l$ if it is an l-concatenation of u and v, and an infinite word is of the form $(u, v)^\omega$ if it is an infinite concatenation of u and v. Let w and p be words. We say that p is a *prefix* of w if there exists a word s such that $w = ps$.

An *ultimately periodic word* is an infinite word such that there exist $i \geq 0$ and $l \geq 1$ for which $W[i..i+l-1] = W[i+bl..i+l-1+bl]$ for all $b \geq 0$. Equivalently, an ultimately periodic word is of the form uv^ω for some $u \in \mathcal{A}^*$ and $v \in \mathcal{A}^+$, where v^ω denote the right-infinite word $vvv\cdots$. An *aperiodic word* is an infinite word that is not ultimately periodic. In this paper, we are only interested in aperiodic infinite words since $\{W_i\}$ is finite if W is ultimately periodic.

A *morphism* ϕ on $\mathcal{A}$ is a map from $\mathcal{A}^*$ to $\mathcal{A}^*$ such that $\phi(uv) = \phi(u)\phi(v)$ for all $u, v \in \mathcal{A}^*$, and we can extend this definition to infinite words by defining $\phi(W) = \phi(a_0)\phi(a_1)\phi(a_2)\cdots$ for all $W \in \mathcal{A}^\omega$. A morphism ϕ is *prolongable at* α if $\phi(\alpha) = \alpha u$ for some $u \in \mathcal{A}^+$. If ϕ is prolongable at α and

$$W = \phi^\omega(\alpha) = \alpha u \phi(u)\phi^2(u)\cdots\phi^t(u)\cdots$$

is an infinite word, i.e., W is the limiting word obtained by applying ϕ repeatedly to α, then W is a *purely morphic word*, and W is the *fixed point of* ϕ *starting with* α. Hence, $W = \phi(W)$, and $\phi(W_i)$ is also a shift of W for any i. If $W' = \sigma(W)$ is the image of a purely morphic word under some coding (i.e., a letter-to-letter morphism) $\sigma : \Sigma_m \to \Sigma_{m'}$, then W' is a *morphic word*.

The *length* of a finite word $u = b_0 \cdots b_l$, denoted by $|u|$, is $l + 1$, whereas $|\varepsilon| = 0$. We say that a morphism ϕ is *n-uniform* if $|\phi(\alpha)| = n$ for all $\alpha \in \mathcal{A}$. We will assume that $n \geq 2$. We say that W is *n-uniform morphic* if W is a fixed point under an n-uniform morphism ϕ. One may check that $\phi(W_i) = W_{ni}$ in this case. The Thue–Morse word

$$\mathbf{t} := 01101001100101\cdots$$

is the fixed point of the 2-uniform morphism τ on Σ_2 which maps 0 to 01 and 1 to 10, starting with 0.

If u and v are finite words of equal length or are infinite words, with $u \neq v$, then *the letters of first distinction between* u *and* v are defined as $u[d]$ and $v[d]$ such that $d \geq 0$ is the minimum value for which $u[d] \neq v[d]$. We say that d is the *position of first distinction between* u *and* v. We say that $u < v$ if $u[0..d-1] = v[0..d-1]$ and $u[d] < v[d]$. Equivalently, $u < v$ if $u = xy_1z_1$ and $v = xy_2z_2$ for some $x, y_1, y_2 \in \mathcal{A}^*$ and $z_1, z_2 \in \mathcal{A}^* \cup \mathcal{A}^\omega$ such that $y_1 < y_2$ and $|y_1| = |y_2| > 0$. A uniform morphism ϕ

on $\mathcal{A}$ *preserves the order in* $\mathcal{A}' \subseteq \mathcal{A}$ if $\alpha < \beta$ implies $\phi(\alpha) < \phi(\beta)$ for all $\alpha, \beta \in \mathcal{A}'$, and ϕ preserves the order in $\mathcal{A}^\omega$ if $u < v$ implies $\phi(u) < \phi(v)$ for all $u, v \in \mathcal{A}^\omega$.

A totally ordered set S is said to *have the order type* ω if there exists an order-preserving bijection from S to $\mathbb{N}$. The backwards order type of ω is ω^*. The order type of l elements is l. If σ_1 and σ_2 are order types, then the order type $\sigma_1 + \sigma_2$ is the order type σ_1 followed by σ_2 in increasing order. If the order type is σ_2 copies of σ_1, we write the order type as $\sigma_1 \cdot \sigma_2$. We denote the order type $\sigma \cdot \sigma$ by σ^2. We say that $\sigma \preceq ot(W)$ if there is an order-preserving embedding from $\{W_i\}$ into any linear ordering of order type σ. Specifically, $\omega \preceq ot(W)$ if $\{W_i\}$ contains an infinite increasing sequence, and $\omega^* \preceq ot(W)$ if $\{W_i\}$ contains an infinite decreasing sequence.

3. The Thue–Morse Word

We explore the order type of the Thue–Morse word $\mathbf{t}$, one of the most studied words, and prove Theorem 1 in this section. In particular, we will show that $ot(\mathbf{t})$ is dense.

We begin with some general properties that are also relevant in Section 4 and Section 5. These lemmas are somewhat obvious but useful in aiding us with the proof.

The following lemma states that, if W is aperiodic, then not only is $\{W_i\}$ infinite, but each of its elements is unique.

Lemma 1. *The set of all shifts of an aperiodic word W is infinite. In particular, $W_i = W_j$ if and only if $i = j$.*

This leads to a very useful consequence.

Corollary 1. *If $W = \phi^\omega(\alpha)$ is the aperiodic fixed point of an n-uniform morphism ϕ on Σ_m starting with α, then no proper shift of W is a fixed point of ϕ.*

Proof. Otherwise, say $W_i = \phi^\omega(\beta)$ for some $i \geq 1$ and $\beta \in \Sigma_m$, then $W_{ni} = \phi(W_i) = \phi(\phi^\omega(\beta)) = \phi^\omega(\beta) = W_i$, which is impossible by Lemma 1. □

We give a result from Richomme [11] and restate it using our notations:

Lemma 2 ([11])**.** *Let $n \geq 1$. An n-uniform morphism on Σ_m is an order-preserving morphism if and only if it preserves the order in Σ_m.*

This gives us a simple way to determine whether or not a given uniform morphism is order-preserving, and leads to the following convenient property related to powers of such morphisms. This can be proved by induction, and we omit the proof.

Corollary 2. *Let ϕ be an n-uniform morphism on Σ_m such that ϕ preserves the order in Σ_m. Suppose that $u, v \in \Sigma_m^\omega$ or $u, v \in \Sigma_m^l$ for some $l \geq 1$. If $u < v$, then $\phi^t(u) < \phi^t(v)$ for any integer $t \geq 1$.*

Next, we give the connection between order-preserving uniform morphisms and the length of the common prefixes between words under them.

Lemma 3. *Let $u \in \Sigma_m^\omega$ and ϕ be an n-uniform morphism on Σ_m such that $\phi(\beta)[0] = \beta$ for some $\beta \in \Sigma_m$. If $u[0] = \beta$ and $u \neq \phi^\omega(\beta)$, then $\phi(u)$ and $\phi^\omega(\beta)$ share a common prefix of length at least n times the length of the common prefix of u and $\phi^\omega(\beta)$.*

Proof. Suppose that the position of first distinction between u and $\phi^\omega(\beta)$ is d for some $d \geq 1$. Then

$$\begin{aligned}\phi(u)[0..nd-1] =& \phi(u[0..d-1]) \\ =& \phi(\phi^\omega(\beta)[0..d-1]) \\ =& \phi^\omega(\beta)[0..nd-1].\end{aligned}$$

Hence the length of the common prefix between $\phi(u)$ and $\phi^\omega(\beta)$ is at least nd. □

We may relax the condition on ϕ such that ϕ preserves the order of some particular subset of the alphabet. The following corollary shows the existence of an infinite decreasing/increasing sequence through construction.

Corollary 3. *Let $W = \phi^\omega(\alpha)$ be an aperiodic fixed point of an n-uniform morphism ϕ on Σ_m starting with α, and suppose that there exists a positive integer p such that ϕ^p preserves the order in $T = \{\rho \in \Sigma_m : \rho$ occurs infinitely often in $W\}$. Assume that $\phi(\beta)[0] = \beta$ for some $\beta \in \Sigma_m$. If there exists $i \geq 1$ such that $W_i[0] = \beta$, $W_i \in T^\omega$, and $W_i > \phi^\omega(\beta)$, then $(\phi^{tp}(W_i))_{t\geq 0}$ is an infinite decreasing sequence contained in $\{W_i\}$, where the length of the common prefix between $\phi^{tp}(W_i)$ and $\phi^\omega(\beta)$ increases as t increases.*

Similarly, if there exists $i \geq 1$ such that $W_i[0] = \beta$, $W_i \in T^\omega$, and $W_i < \phi^\omega(\beta)$, then $(\phi^{tp}(W_i))_{t\geq 0}$ is an infinite increasing sequence contained in $\{W_i\}$, where the length of the common prefix between $\phi^{tp}(W_i)$ and $\phi^\omega(\beta)$ increases as t increases.

Proof. Suppose that $W[i] = \beta$ and $W_i \in T^\omega$ for some $i \geq 1$. This implies that $\beta \in T$. By Corollary 1, $W_i \neq \phi^\omega(\beta)$. Note that ϕ^p is an n^p-uniform morphism and W remains a fixed point of ϕ^p for all integers $p \geq 1$. Also, $\phi^\omega(\beta)$ must not contain $\gamma \in \Sigma_m \setminus T$, otherwise $\phi^{sp}(\beta) = \phi^\omega(\beta)[0..n^{sp}-1]$ contains γ for some $s \geq 1$, so $W[n^{sp}i..n^{sp}(i+1)-1]$ would contain γ, and hence γ would occur infinitely often, a contradiction.

If $W_i > \phi^\omega(\beta)$, then since ϕ^p preserves the order in T, we can treat T as $\Sigma_{|T|}$ and apply Corollary 2 to obtain $(\phi^p)^t(W_i) = \phi^{tp}(W_i) > \phi^\omega(\beta)$ for all $t \geq 0$. By Lemma 3, the length of the common prefix between $\phi^{tp}(w_i)$ and $\phi^\omega(\beta)$ increases as t increases. In particular, if d is the position of first distinction between $\phi^\omega(\beta)$ and $\phi^{tp}(W_i)$, then $\phi^{tp}(W_i)[d] > \phi^\omega(\beta)[d] = \phi^{(t+1)p}(W_i)[d]$. Hence, we have

$$W_i > \phi^p(W_i) > \phi^{2p}(W_i) > \cdots > \phi^{tp}(W_i) > \cdots,$$

so $(\phi^{tp}(W_i))_{t\geq 0}$ is an infinite decreasing sequence with the desired property.

The proof is similar for the case when $W_i < \phi^\omega(\beta)$. □

Remark 1. We will write the sequence $(\phi^t(W_i))_{t\geq 0}$ as $(\phi^t(W_i))$ when the context is clear.

The next corollary gives the length of the common prefix between two binary words under powers of τ, the Thue–Morse morphism.

Corollary 4. *Suppose that $u, v \in \Sigma_2^\omega$, $u \neq v$, and they agree on a prefix of exactly length l for some $l \geq 1$, then $\tau^t(u)$ and $\tau^t(v)$ agree on a prefix of length exactly $2^t l$ for all $t \geq 0$. Consequently, if $u < v$, then the i-shift of $\tau^t(u)$ is smaller than the i-shift of $\tau^t(v)$ for $0 \leq i \leq 2^t l - 1$.*

Proof. By repeated use of Lemma 3, $\tau^t(u)[0..2^t l-1] = \tau^t(v)[0..2^t l-1]$. Since $\tau(0)[0] = 0 \neq 1 = \tau(1)[0]$, we have $\tau^t(u)[2^t l] = \tau^t(u[l])[0] \neq \tau^t(v[l])[0] = \tau^t(v)[2^t l]$, and hence the length of the common prefix of $\tau^t(u)$ and $\tau^t(v)$ is $2^t l$.

If $u < v$, then $\tau^t(u) < \tau^t(v)$ by Corollary 2 and hence the claim follows. □

It is a well-known fact that $\mathbf{t}$ (and consequently its shifts) is overlap-free [12], i.e., there do not exist $c \in \Sigma_2^+$ and $x \in \Sigma_2^*$ such that $\mathbf{t}$ contains the string $cxcxc$. In [4], Berstel proved that the lexicographically greatest infinite overlap-free word over Σ_2 that starts with 0 is $\mathbf{t}$. We define the complement Thue–Morse word $\bar{\mathbf{t}} = 1001011001\cdots$ to be the fixed point of τ starting with 1. Allouche, Currie, and Shallit proved the dual result of Berstel's in [3], i.e., the lexicographically least infinite overlap-free word over Σ_2 that starts with 1 is $\bar{\mathbf{t}}$. We summarize that in the following proposition:

Proposition 1 ([3, 4])**.** *$\mathbf{t} = \mathbf{t}_0$ is the maximum element in $\{\mathbf{t}_i\}$ that starts with 0, and $\bar{\mathbf{t}}$ is the least element in $\{\mathbf{t}_i\} \cup \{\bar{\mathbf{t}}_i\}$ that starts with 1.*

This gives us the following properties:

Corollary 5. *$\mathbf{t}_1$ is the largest shift of $\mathbf{t}$ starting with 1, and hence the maximum element in $\{\mathbf{t}_i\}$.*

Similarly, $\bar{\mathbf{t}}_1$ is the least element in $\{\mathbf{t}_i\} \cup \{\bar{\mathbf{t}}_i\}$.

Proof. We prove the first part by contradiction. Suppose that there exists $j > 1$ such that $\mathbf{t}_1 < \mathbf{t}_j$. We have $\mathbf{t}[j..j+1] = 11$. Since $\mathbf{t}$ is overlap-free, $\mathbf{t}[j-1] = 0$. Then $\mathbf{t} < \mathbf{t}_{j-1}$, contradicting Proposition 1.

The second part can be proven analogously. □

Corollary 6. *If $\mathbf{t}[i] = 0$, then $(\tau^t(\mathbf{t}_i))_{t\geq 0}$ is an increasing sequence such that the length of the common prefix between $\tau^t(\mathbf{t}_i)$ and $\mathbf{t}$ increases as t increases. If $\mathbf{t}[j] = 1$, then $(\tau^t(\mathbf{t}_j))$ is a decreasing sequence such that the length of the common prefix between $\tau^t(\mathbf{t}_i)$ and $\mathbf{t}$ increases as t increases.*

Proof. By Proposition 1, $\mathbf{t}_i < \mathbf{t}$ for all $i \geq 1$ such that $t_i[0] = 0$. Similarly, $\mathbf{t}_j > \bar{\mathbf{t}}$ for all $j \geq 1$ such that $t_j[0] = 1$ by Corollary 5. Since $\tau(0)[0] = 0$ and $\tau(1)[0] = 1$, the result follows from Corollary 3. □

We will now explain the approach to the proof of Theorem 1. We begin by ordering $\mathbf{t}$ and $\mathbf{t}_1$, which have prefixes 01101001 and 11010011, respectively, from which we have

$$\mathbf{t} < \mathbf{t}_1.$$

This is stage 0 of the process. Next, we include $\mathbf{t}_2$ and $\mathbf{t}_3$, which have prefixes 10100110 and 01001100, respectively, at stage 1, and we obtain

$$\mathbf{t}_3 < \mathbf{t} < \mathbf{t}_2 < \mathbf{t}_1.$$

At stage 2, the shifts $\mathbf{t}_4$, $\mathbf{t}_5$, $\mathbf{t}_6$, and $\mathbf{t}_7$ either fall between the shifts from stage 1 or become the new minimum shift. Each of these positions are occupied by exactly one of $\mathbf{t}_4$, $\mathbf{t}_5$, $\mathbf{t}_6$, and $\mathbf{t}_7$. One may check that

$$\mathbf{t}_5 < \mathbf{t}_3 < \mathbf{t}_6 < \mathbf{t} < \mathbf{t}_4 < \mathbf{t}_2 < \mathbf{t}_7 < \mathbf{t}_1.$$

In general, we will show that each of the shifts $\mathbf{t}_{2^{k+1}}, \mathbf{t}_{2^{k+1}+1}, \cdots, \mathbf{t}_{2^{k+2}-1}$ either falls between the shifts from stage k or becomes the new minimum at stage $k+1$ for $k \geq 0$ in the following lemma.

Lemma 4. *Let* $\mathbf{t}$ *be the Thue–Morse word. Then the following are true:*

1. *When* $k \geq 0$, $\mathbf{t} < \mathbf{t}_{2^{k+1}} < \mathbf{t}_{2^k}$.
2. *When* $k \geq 0$, $\mathbf{t}_{2^{k+1}+1} < \mathbf{t}_{2^k+1}$.
3. *When* $k \geq 1$, $\mathbf{t}_{2^k+2^{k-1}+i} < \mathbf{t}_{2^{k+1}+2^k+i} < \mathbf{t}_i$ *for* $0 \leq i \leq 2^{k-1} - 1$.
4. *When* $k \geq 1$, $\mathbf{t}_{2^k+i} < \mathbf{t}_{2^{k+1}+2^k+2^{k-1}+i} < \mathbf{t}_{2^{k-1}+i}$ *for* $0 \leq i \leq 2^{k-1} - 1$.
5. *When* $k \geq 2$, $\mathbf{t}_* < \mathbf{t}_{2^{k+1}+i} < \mathbf{t}_{2^k+i}$ *for* $2 \leq i \leq 2^k - 1$, *where*

$$* = \begin{cases} i + 2^{\lfloor \log_2(i) \rfloor - 1} & \text{if } i < 2^{\lfloor \log_2(i) \rfloor} + 2^{\lfloor \log_2(i) \rfloor - 1} \\ i - 2^{\lfloor \log_2(i) \rfloor - 1} & \text{if } i \geq 2^{\lfloor \log_2(i) \rfloor} + 2^{\lfloor \log_2(i) \rfloor - 1} \end{cases}.$$

Proof. We prove each statement individually as follows.

1. Suppose that $k \geq 0$. Since $\mathbf{t}_{2^{k+1}} = \tau^{k+1}(\mathbf{t}_1)$ and $\mathbf{t}_1[0] = 1$, $\mathbf{t}_0[0] = 0 < 1 = \mathbf{t}_{2^{k+1}}[0]$. In addition, $\mathbf{t}_{2^{k+1}} = \tau(\mathbf{t}_{2^k})$, and hence $\mathbf{t}_{2^{k+1}} < \mathbf{t}_{2^k}$ by Corollary 6.
2. Since $\mathbf{t}_{2^{k+1}}[0] = \mathbf{t}_{2^k}[0] = 1$ for $k \geq 0$, this follows from the previous case and Corollary 4.

3. Suppose that $k \geq 1$. Observe that $\mathbf{t}_3 < \mathbf{t}$ and $\mathbf{t}_3$ agrees with $\mathbf{t}$ on a prefix of length 2. Since $\mathbf{t}_{2^{k+1}+2^k} = \tau^k(\mathbf{t}_3)$ and $\mathbf{t} = \tau^k(\mathbf{t})$ for $k \geq 1$, we have $\mathbf{t}_{2^{k+1}+2^k+i} < \mathbf{t}_i$ for $0 \leq i \leq 2^{k+1}-1$ by Corollary 4, and hence this inequality remains true for $0 \leq i \leq 2^{k-1}-1$.

   Note that $\tau(\mathbf{t}_{2^k+2^{k-1}}) = \mathbf{t}_{2^{k+1}+2^k}$. When $k=1$, $\mathbf{t}_{2^k+2^{k-1}}[0] = \mathbf{t}_3[0] = 0$, so $\mathbf{t}_{2^k+2^{k-1}} < \mathbf{t}_{2^{k+1}+2^k}$ by Corollary 6. Since $\mathbf{t}_6$ has a prefix 0110, $\mathbf{t}_3$ and $\mathbf{t}_6$ share a common prefix of length 2, so $\mathbf{t}_{2^k+2^{k-1}+i} < \mathbf{t}_{2^{k+1}+2^k+i}$ for $0 \leq i \leq 2^k-1$ by Corollary 4 and remains as such for $0 \leq i \leq 2^{k-1}-1$.

4. From the proof of the third statement, we have $\mathbf{t}_{2^{k+1}+2^k+i} < \mathbf{t}_i$ for $2^{k-1} \leq i \leq 2^k-1$, so $\mathbf{t}_{2^{k+1}+2^k+2^{k-1}+i} < \mathbf{t}_{2^{k-1}+i}$ for $0 \leq i \leq 2^{k-1}-1$.

   When $k=1$, we have $\mathbf{t}_{2^k} = \mathbf{t}_2$, and $\mathbf{t}_{2^{k+1}+2^k+2^{k-1}} = \mathbf{t}_7$ with prefix 1100, so $\mathbf{t}_2 < \mathbf{t}_7$ and they agree on a prefix of length 1. For $k \geq 1$, since $\mathbf{t}_{2^k} = \tau^{k-1}(\mathbf{t}_2)$ and $\mathbf{t}_{2^{k+1}+2^k+2^{k-1}} = \tau^{k-1}(\mathbf{t}_7)$, we have $\mathbf{t}_{2^k+i} < \mathbf{t}_{2^{k+1}+2^k+2^{k-1}+i}$ for $0 \leq i \leq 2^{k-1}-1$ by Corollary 4.

5. Finally, suppose that $k \geq 2$. We start with the inequality on the left side. We first show that $\mathbf{t}_j > \mathbf{t}_3$ for all j such that the binary expansion of j is of the form 10^t10 (or $j = 2^p+2$ for some $p \geq 3$), and $\mathbf{t}_l > \mathbf{t}_2$ for all l such that the binary expansion of l is of the form 10^t11 (or $l = 2^p+3$ for some $p \geq 3$), for some positive integer t.

   Observe that $\mathbf{t}_1$ and $\bar{\mathbf{t}}$ share a common prefix of length 1, and hence $\mathbf{t}_{2^p}$ and $\bar{\mathbf{t}}$ share a common prefix of length 2^p by Corollary 4. In particular, $\mathbf{t}_8$ and $\bar{\mathbf{t}}$ share a common prefix of length 8. Hence, $\mathbf{t}_{2^p+2}$ has a prefix 010110 for all $p \geq 3$, so $\mathbf{t}_{2^p+2} > \mathbf{t}_3$ and they share a common prefix of length 3.

   Similarly, $\mathbf{t}_{2^p+3}$ has a prefix 10110 for all $p \geq 3$, so $\mathbf{t}_{2^p+3} > \mathbf{t}_2$ and they share a common prefix of length 3.

   Let i be an integer such that $2 \leq i \leq 2^k-1$. If $i < 2^{\lfloor \log_2(i) \rfloor} + 2^{\lfloor \log_2(i) \rfloor - 1}$, then the binary expansion of i is $10u$ for some $u \in \Sigma_2^*$, and hence the binary expansion of $2^{k+1}+i$ is 10^t10u, where $t \geq 1$. If $u = \varepsilon$, then $i = 2$ and $i + 2^{\lfloor \log_2(i) \rfloor - 1} = 3$, so $\mathbf{t}_{2^{k+1}+i} > \mathbf{t}_3$ as shown previously. If $|u| = g$ for some $g \geq 1$, then observe that $g = k-t-1$, so $k-t-g = 1$. Also, $g = \lfloor \log_2(i) \rfloor - 1$. Since $k-g+1 = t+2 \geq 3$, we have

$$\mathbf{t}_{2^{k-g+1}+2^{k-t-g}} > \mathbf{t}_3$$

   by the conclusion in the second paragraph. Moreover, notice that $2^{g+1} \leq i \leq 2^{g+1}+2^g-1$, and hence $0 \leq i-2^{g+1} \leq 2^g-1 < 2^g \cdot 3 - 1$, so by Corollary 4, we have $\tau^g(\mathbf{t}_{2^{k-g+1}+2^{k-t-g}}) > \tau^g(\mathbf{t}_3)$, and consequently

$$\mathbf{t}_{2^{k+1}+i} = \mathbf{t}_{2^{k+1}+2^{k-t}+(i-2^{g+1})} > \mathbf{t}_{2^g3+(i-2^{g+1})} = \mathbf{t}_{i+2^{\lfloor \log_2(i) \rfloor - 1}}.$$

If $i \geq 2^{\lfloor \log_2(i) \rfloor} + 2^{\lfloor \log_2(i) \rfloor - 1}$, then the binary expansion of i is $11v$, where $v \in \Sigma_2^*$, and hence the binary expansion of $2^{k+1} + i$ is $10^t 11v$. Going through a similar argument as in the previous case, we have $i = 3$ and $\mathbf{t}_{2^{k+1}+i} > \mathbf{t}_2$ when $v = \varepsilon$. When $|v| = g$ for some $g \geq 1$, observe that we still have $g = k - t - 1$ with $t + 2 \geq 3$. Therefore,

$$\mathbf{t}_{2^{k-g+1}+2^{k-t-g}+2^{k-t-g-1}} > \mathbf{t}_2$$

by the conclusion in the third paragraph. Since $2^{g+1} + 2^g \leq i \leq 2^{g+2} - 1$, we have $0 \leq i - (2^{g+1} + 2^g) \leq 2^g - 1 < 2^{g+1} - 1$. We may conclude that $\tau^g(\mathbf{t}_{2^{k-g+1}+2^{k-t-g}+2^{k-t-g-1}}) > \tau^g(\mathbf{t}_2)$, and consequently

$$\mathbf{t}_{2^{k+1}+i} = \mathbf{t}_{2^{k+1}+2^{k-t}+2^{k-t-1}+(i-2^{g+1}-2^g)} > \mathbf{t}_{2^{g+1}+(i-2^{g+1}-2^g)} = \mathbf{t}_{i-2^{\lfloor \log_2(i) \rfloor - 1}}$$

by Corollary 4 once again.

For the inequality on the right, note that $\mathbf{t}_{2^{k+1}} = \tau(\mathbf{t}_{2^k})$ and $\mathbf{t}_{2^k}[0] = 1$ for all $k \geq 1$. The result follows from Corollary 4 and Corollary 6.

□

Since each of the upper bounds in the five cases above is unique and each of the lower bounds (with $\mathbf{t}_{2^{k+1}+1}$ having no lower bound since it is the smallest on each stage) is also unique, we obtain the following result.

Theorem 1. *For $k \geq 1$, the shifts $\mathbf{t}_i$ ($2^k \leq i \leq 2^{k+1} - 1$) of the Thue–Morse word interleave with $\mathbf{t}_j$ ($0 \leq j \leq 2^k - 1$).*

4. Dense Shifts of Morphic Words

We have looked at a specific example of an aperiodic binary word in Section 3 whose order type is dense. In this section, we give a generalization to aperiodic morphic words W. In particular, we consider the case when W is the image of a purely morphic word W' under some coding such that the first letter of W' occurs more than once in W'. We will show that $ot(W)$ must be dense.

We begin with a simple observation regarding the occurrence of α in W.

Proposition 2. *Let W be a word over Σ_m, and suppose that $W = \phi^\omega(\alpha)$ is an aperiodic fixed point of a morphism ϕ that is prolongable on $\alpha \in \Sigma_m$. Then α occurs at least twice in W if and only if α occurs in $\phi(\gamma)$ for some $\gamma \in \Sigma_m$ other than at $\phi(\alpha)[0]$. In particular, α occurs either infinitely often in W or only at $W[0]$.*

Proof. Let $\phi(\alpha) = \alpha u$ for some $u \in \Sigma_m^+$. For the forward direction, if α occurs only at $\phi(\alpha)[0]$, then α does not occur in u and consequently $\phi^t(u)$ for all positive integer t. Therefore, α occurs only at $W[0]$.

For the backward direction, if $\phi(\alpha)[i] = \alpha$ with $i \neq 0$, then $W[0] = W[i] = \alpha$. If $\phi(\beta)[j] = \alpha$ for some $\beta \neq \alpha$ and $j \geq 0$, then, since there is some $k \geq 1$ such that $W[k] = \beta$, we have $W[k'] = \alpha$ for some $k' > k$ corresponding to $\phi(W[k])[j]$. In either case, α occurs at least twice in W.

Finally, whenever $W[l] = \alpha$ for some $l \geq 1$, we have $W[l'] = \alpha$ for some $l' > l$ corresponding to the position of $\phi(W_l)[0]$ in W. Therefore, α must occur infinitely often if α occurs more than once in W. □

We shall now prove the main results.

Theorem 2. *Let W be an aperiodic morphic word over Σ_m. Suppose that W is the image of a purely morphic word W' over $\Sigma_{m'}$ under some coding σ, and suppose that $W'[0]$ occurs at least twice in W'. Then $ot(W)$ is dense. In particular, $ot(W)$ is the same as the order type of $\mathbb{Q} \cap (0, 1)$, $\mathbb{Q} \cap (0, 1]$, or $\mathbb{Q} \cap [0, 1)$.*

Proof. Let i and j be two distinct non-negative integers such that $W_i < W_j$. We want to show that there exists an integer l such that $W_i < W_l < W_j$.

Since W is aperiodic, W' must also be aperiodic. Let $\alpha = W'[0]$ for some $\alpha \in \Sigma_{m'}$. Since α occurs at least twice in W', we can pick $k \geq 1$ such that $W'[k] = \alpha$. We know that $W'_k \neq W'$ by Lemma 1. Let d be the position of first distinction between W_i and W_j and let $M = \max\{i, j\}$. Since ϕ is prolongable on α, we may pick q large enough such that $L = |\phi^q(\alpha)| > M + d$. By Lemma 1, we also have $\phi^q(W'_k) \neq W'$. Let ℓ be the integer such that $W'_\ell = \phi^q(W'_k)$. We have

$$W' = \phi^q(\alpha)u \quad \text{and} \quad W'_\ell = \phi^q(\alpha)v$$

for some distinct $u, v \in \Sigma_{m'}$. If $W < W_\ell$, then $\sigma(u) < \sigma(v)$. Let $l = \ell + i$. Since $L > M + d$, we have $L > i$, so

$$W_i = \sigma(\phi^q(\alpha))[i..L-1]\sigma(u) \quad \text{and} \quad W_l = \sigma(\phi^q(\alpha))[i..L-1]\sigma(v),$$

and hence $W_i < W_l$. Moreover, $L > M + d$ also implies that $L - i - 1 \geq d$, so $W_l[0..d] = W_i[0..d] < W_j[0..d]$. Therefore, $W_i < W_l < W_j$. If $W_\ell < W$, then let $l = \ell + j$, and we have $W_i < W_l < W_j$ by a similar argument.

Finally, we will show that $ot(W)$ cannot be the same as the order type of $\mathbb{Q} \cap [0, 1]$, i.e., $\{W_i\}$ cannot have both a maximum element and a minimum element. Suppose to the contrary that there exist non-negative integers a, b such that $W_a \leq W_i \leq W_b$ for any i. Using a similar strategy as above, we pick $k \geq 1$ such that $W'[k] = \alpha$. Let Q be large enough such that $|\phi^Q(\alpha)| > \max\{a, b\}$: we have $\phi^Q(\alpha)_a$ and $\phi^Q(\alpha)_b$ being non-empty. Once again, $\phi^Q(W'_k) \neq W'$ by Lemma 1. Let t be the integer such that $W'_t = \phi^Q(W'_k)$. If $W < W_t$, then $W_b < W_{t+b}$, so W_b cannot be the largest element in $\{W_i\}$. If $W_t < W$, then $W_{t+a} < W_a$, so W_a cannot be the least element in $\{W_i\}$, and hence the result follows. □

Corollary 7. *For any aperiodic binary purely morphic word W, $ot(W)$ is dense.*

Proof. Let W be the aperiodic fixed point of a morphism ϕ starting with $\alpha \in \Sigma_2 = \{\alpha, \beta\} = \{0, 1\}$. It suffices to show that α must occur at least twice, and the result follows from Theorem 2 by letting σ be the identity coding. For a contradiction, suppose not. Then we have $\phi(\alpha) = \alpha\beta^k$ for some $k \geq 1$ and $\phi(\beta)$ does not contain α by Proposition 2. If $\phi(\beta) = \varepsilon$, then W would be finite, which is a contradiction. If $\phi(\beta) = \beta^l$ for some $l \geq 1$, then W would be the ultimately periodic word $\alpha\beta^\omega$, resulting in another contradiction. □

Let us consider some follow-up questions to these results. If the pre-image of W under σ has its starting letter occurring only once in it, is $ot(W)$ still dense? If W is an aperiodic binary morphic word, is $ot(W)$ still dense? The answer to both is no. Consider the fixed point W of the uniform morphism defined by $0 \mapsto 012$, $1 \mapsto 111$, and $2 \mapsto 222$, starting with 0, and the coding σ defined by $0, 1 \mapsto 0$ and $2 \mapsto 1$. The order type of the resulting aperiodic binary morphic word $\sigma(W)$ is not dense. However, there are morphic words W whose order type is dense, but the starting letter of its pre-image W occurs only once in W. For example, consider the fixed point W of the uniform morphism defined by $0 \mapsto 02$, $1 \mapsto 12$, and $2 \mapsto 20$, starting with 1, and the coding defined by $0, 1 \mapsto 0$ and $2 \mapsto 1$. The resulting morphic word is the Thue–Morse word, whose order type is dense by Theorem 1. A more detailed explanation of the first example is presented in Section 5.3.

Remark 2. We may apply Theorem 2 to the Fibonacci word

$$\mathbf{f} := 01001010010010\cdots$$

defined as the fixed point of the morphism μ on Σ_2 which maps 0 to 01 and 1 to 0, starting with 0. Since $\mathbf{f}$ is purely morphic and 0 occurs at least twice in $\mathbf{f}$, we may conclude that $ot(\mathbf{f})$ is dense. In particular, $ot(\mathbf{f})$ is the same as the order type of $\mathbb{Q} \cap (0, 1)$, as $\{\mathbf{f}_i\}$ does not contain a maximum or minimum element: given any shift $\mathbf{f}_i$, there exists large enough n such that the Zeckendorf representations of j and k are $(10)^n$ and $(10)^n 1$, respectively, and $\mathbf{f}_j < \mathbf{f}_i < \mathbf{f}_k$.

5. Non-dense Shifts of Some Uniform Morphic Ternary Words

In the previous section, we saw that $ot(W)$ is dense if W is an aperiodic binary purely morphic word. This is not necessarily the case when we increase the size of the alphabet, as there is no guarantee that the starting letter would occur more than once in W. For example, if ϕ is a 2-uniform morphism on Σ_3 defined by $0 \mapsto 01$, $1 \mapsto 12$, $2 \mapsto 22$, then one may check that

$$W = \phi^\omega(0) = 0112122212222222\cdots$$

is an aperiodic word, but $\{W_i\}$ does not contain an infinite decreasing sequence since the length of the strings of 2 steadily increases as we go further to the right in W (see Theorem 3), and hence $ot(W)$ cannot be dense. In this section, we will look at different examples of aperiodic uniform morphic ternary words whose order types are not dense.

Let $\Sigma_3 = \{\alpha, \beta, \gamma\}$. In this section, we assume that W is an aperiodic n-uniform morphic word on Σ_3 starting with α. We will also assume that $\beta < \gamma$ in Section 5.1 and Section 5.2. Once again, Σ_3 is the optimal alphabet, as mentioned in Section 2.

By Theorem 2, we know that $\{W_i\}$ is dense if α occurs at least twice in W, and hence we will focus on cases where α occurs only once in W, in which case, β and γ must occur infinitely often since W is aperiodic. In Section 5.1 and Section 5.2, we will look at words whose shifts do not contain an infinite decreasing sequence or an infinite increasing sequence. In Section 5.3, we will discuss the order types of those particular types of words, as well as look at examples of other morphic words.

5.1. $\phi(\beta)$ and $\phi(\gamma)$ Have Different Starting Letters

Suppose that $\{\phi(\beta)[0], \phi(\gamma)[0]\} = \{\beta, \gamma\}$. Since α occurs only once in W, we have $\phi(\beta), \phi(\gamma) \in \{\beta, \gamma\}^+$ by Proposition 2. Observe that $\phi^2(\alpha)[0] = \phi(\alpha)[0] = \alpha$. We can deduce that $\phi^2(\rho)[0] = \rho$ for all $\rho \in \Sigma_3$ from the following lemma by restricting ϕ to $\{\beta, \gamma\}$.

Lemma 5. *Let ϕ be an n-uniform morphism on Σ_2 with $\phi(0) \neq \phi(1)$. Then ϕ^2 preserves the order in Σ_2.*

Proof. If $\phi(0) < \phi(1)$, then ϕ^2 preserves the order in Σ_2 by Corollary 2. If $\phi(1) < \phi(0)$, then let $0 \leq d \leq n-1$ be the position of first distinction between $\phi(0)$ and $\phi(1)$, so we have

$$\phi(0)[0..d-1] = \phi(1)[0..d-1] \quad \text{and} \quad \phi(1)[d] = 0 < 1 = \phi(0)[d].$$

Therefore,

$$\phi^2(0)[0..nd-1] = \phi(\phi(0)[0..d-1])\phi(1) \text{ and } \phi^2(1)[0..nd-1] = \phi(\phi(1)[0..d-1])\phi(0).$$

This implies that $\phi^2(0) < \phi^2(1)$, and so ϕ^2 preserves the order in Σ_2. □

Since $\phi^\omega(\alpha) = (\phi^2)^\omega(\alpha)$, the set of shifts of $\phi^\omega(\alpha)$ is equal to the set of shifts of $(\phi^2)^\omega(\alpha)$, so consider without loss of generality a uniform morphism ψ with $\psi(\alpha)[0] = \alpha$, $\psi(\beta)[0] = \beta$, and $\psi(\gamma)[0] = \gamma$.

We will determine properties of ψ such that $\{W_i\}$ does not contain an infinite decreasing sequence. Analogous results apply to the case where $\{W_i\}$ does not contain an infinite increasing sequence. By Corollary 3, we need to make sure that $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are greater than all shifts of their corresponding starting letters.

We also consider the following lemma on properties of ϕ.

Lemma 6. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3, and a fixed point of some n-uniform morphism ϕ on Σ_3. Assume that $\beta < \gamma$, and that α occurs only at $W[0]$.*

Suppose that $\phi(\gamma)[0] = \gamma$, and that the fixed point $\phi^\omega(\gamma)$ is larger than all shifts of W starting with γ, and $\phi(\gamma) \neq \gamma^n$. Then

(i) $\phi(\beta)[n-1] = \beta$.

(ii) If $W[0..1] = \alpha\gamma$, then $\phi(\alpha)[n-1] = \beta$.

(iii) If $\gamma\gamma$ is a factor of W, then $\phi(\gamma)[n-1] = \beta$.

Similarly, suppose that $\phi(\beta)[0] = \beta$, and that the fixed point $\phi^\omega(\beta)$ is smaller than all shifts of W starting with β, and $\phi(\beta) \neq \beta^n$. Then

(i) $\phi(\gamma)[n-1] = \gamma$;

(ii) if $W[0..1] = \alpha\beta$, then $\phi(\alpha)[n-1] = \gamma$;

(iii) if $\beta\beta$ is a factor of W, then $\phi(\beta)[n-1] = \gamma$.

Proof. We will prove the first case. The proof of the second case is analogous.

Since α occurs only at $W[0]$, we have $\phi(\alpha)_1, \phi(\beta), \phi(\gamma) \in \{\beta, \gamma\}^+$ by Proposition 2. If $\phi(\gamma) \neq \gamma^n$, then β occurs in $\phi(\gamma)$. In particular, there exists $1 \leq k \leq n-1$ such that $\phi(\gamma)[j] = \gamma$ for all $j < k$ and $\phi(\gamma)[k] = \beta$. Since W is aperiodic, $\beta\gamma$ must be a factor of W.

To show (i), suppose to the contrary that $\phi(\beta)[n-1] = \gamma$. We can then pick i such that $W[i..i+1] = \beta\gamma$ and obtain

$$\begin{aligned} W[n(i+1)-1..n(i+1)+k-1] =& \phi(W[i..i+1])[n-1..n+k-1] \\ =& \phi(W[i])[n-1]\phi(W[i+1])[0..k-1] \\ =& \phi(\beta)[n-1]\phi(\gamma)[0..k-1] \\ =& \gamma^{k+1}, \end{aligned}$$

while $\phi^\omega(\gamma)[0..k] = \gamma^k\beta$, and we would have $\phi^\omega(\gamma) < W_{n(i+1)-1}$ with $W_{n(i+1)-1}[0] = \gamma$, a contradiction.

For (ii) and (iii), if $\alpha\gamma$ or $\gamma\gamma$ is a factor of W, then the result follows from the same argument above and replacing β with α or γ, respectively. □

We now turn our focus to ψ. We can establish a connection between the fixed points $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$.

Lemma 7. *Let $W = \psi^\omega(\alpha)$ be an aperiodic word over Σ_3, and a fixed point of some n-uniform morphism ψ on Σ_3. Suppose that α occurs only at $W[0]$, and that $\psi(\rho)[0] = \rho$ for all $\rho \in \Sigma_3$. Assume that $\beta < \gamma$. If $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are larger than all shifts starting with β and γ, respectively, then $\psi^\omega(\beta)_1 = \psi^\omega(\gamma)$.*

Proof. If $\psi(\gamma) = \gamma^n$, then there must be an α or β followed by a string of γ of length at least n in W. If it is α, then $\psi(\alpha) = \alpha\gamma^{n-1}$ since W is the only shift starting with α, but that is impossible because W would be the ultimately periodic word $\alpha\gamma^\omega$. Therefore, it must be a β. Since $\psi^\omega(\beta)$ is larger than all shifts starting with β, we have $\psi(\beta) = \beta\gamma^{n-1}$, and hence the result follows.

If $\psi(\gamma) \neq \gamma^n$, then $\psi(\beta)[n-1] = \beta$ by Lemma 6. By means of contradiction, we first suppose that $\psi^\omega(\gamma) < \psi^\omega(\beta)_1$. This implies that $\psi(\beta)[1] = \gamma$. Let $d \geq 1$ be the position of first distinction between $\psi^\omega(\gamma)$ and $\psi^\omega(\beta)_1$. We have $\psi^\omega(\gamma)[0..d-1] = \psi^\omega(\beta)[1..d]$ and $\psi^\omega(\gamma)[d] < \psi^\omega(\beta)[d+1]$. Pick any W_i such that $W[i] = \beta$ and let $q = \lceil \log_n(d+2) \rceil$. Then $W_{n^q i+1} = \psi^q(W_i)_1$ and the length of the common prefix between $W_{n^q i}$ and $\psi^\omega(\beta)$ is at least $d+2$ by Lemma 3. Therefore, $W_{n^q i+1}[0..d-1] = \psi^\omega(\gamma)[0..d-1]$ and $W_{n^q i+1}[d] = \psi^\omega(\beta)[d+1] > \psi^\omega(\gamma)[d]$, so we have $W_{n^q i+1} > \psi^\omega(\gamma)$, which is a contradiction.

Next, suppose that $\psi^\omega(\beta)_1 < \psi^\omega(\gamma)$. Once again, we let $d \geq 0$ be the position of first distinction between $\psi^\omega(\gamma)$ and $\psi^\omega(\beta)_1$, and we have $\psi^\omega(\beta)[d+1] < \psi^\omega(\gamma)[d]$. Since $\beta\gamma$ is a factor of W, we can pick j such that $W_j[0..1] = \beta\gamma$. Let $r = \lceil \log_n(d+1) \rceil$, then the length of the common prefix between $\psi^r(W_{j+1})$ and $\psi^\omega(\gamma)$ is at least $d+1$ by Lemma 3. Since $\psi(\beta)[n-1] = \beta$, we have $W[n^r j + n^r - 1] = \psi^r(W[j])[n^r - 1] = \beta$. Since $W_{n^r j + n^r}[0..d] = \psi^r(W_{j+1})[0..d] = \psi^\omega(\gamma)[0..d]$, we have $W_{n^r j + n^r - 1} > \psi^\omega(\beta)$, which is another contradiction. □

If we keep the order between β and γ, but adjust the proof of Lemma 7 by switching the positions of β and γ and considering $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ to be smaller than all shifts starting with β and γ, respectively, then we have a similar property below.

Lemma 8. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3 and a fixed point of some n-uniform morphism ϕ on Σ_3. Suppose that α occurs only at $W[0]$, and that $\psi(\rho)[0] = \rho$ for all $\rho \in \Sigma_3$. Assume that $\beta < \gamma$. If $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are smaller than all shifts starting with β and γ, respectively, then $\psi^\omega(\beta) = \psi^\omega(\gamma)_1$.*

The next four lemmas give some forms of uniform morphism ϕ and properties of W such that the set of shifts of $W = \phi^\omega(\alpha)$ does not contain an infinite decreasing sequence or an infinite increasing sequence.

Lemma 9. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3 and a fixed point of some n-uniform morphism ϕ on Σ_3. Assume that $\beta < \gamma$. Suppose that α occurs only at $W[0]$. If $\phi(\beta)$ contains exactly one β and $\phi(\gamma) = \gamma^n$, then $\phi(\alpha)$ must contain β, and $\{W_i\}$ has no infinite decreasing sequence.*

Proof. Observe that β must occur in $\phi(\alpha)$ since W is aperiodic. We will pick an arbitrary shift W_i with $i \geq 1$ and show that it is impossible to build an infinite decreasing sequence from the shifts of W with W_i as the first term. Note that the inclusion of W would not change the result. Since W is aperiodic and α occurs only at $W[0]$, we have $W_i \in \{\beta, \gamma\}^\omega$. Consider $\gamma^f \beta \gamma^l \beta$, a prefix of W_i, where $f, l \geq 0$. We will prove our assertion by induction on f.

When $f = 0$, we have $W[i] = \beta$. If $W_j < W_i$ for some integer j, then W_j has a prefix of the form $\beta \gamma^{l_j} \beta$ with $l_j \leq l$. Let $q = \max\{1, \lceil \log_n(l+1) \rceil\}$. We have $n^q - 1 \leq l$. We see that $\phi^q(\gamma) = \gamma^{n^q}$. Since $\phi(\beta)$ contains exactly one β, we have $\phi(\beta) = \gamma^{\ell_1} \beta \gamma^{\ell_2}$ with $0 \leq \ell_1, \ell_2 \leq n-1$ and $\ell_1 + \ell_2 = n-1$. Therefore,

$$\phi^q(\beta) = \gamma^{\ell_1 \sum_{k=0}^{q-1} n^k} \beta \gamma^{\ell_2 \sum_{k=0}^{q-1} n^k},$$

where $\ell_1 \sum_{k=0}^{q-1} n^k + \ell_2 \sum_{k=0}^{q-1} n^k = n^q - 1$. Observe that for all $j \geq n^q$, W_j is a shift of some concatenation of $\phi^q(\beta)$ and $\phi^q(\gamma)$, so if $W[j] = \beta$, then $l_j \geq n^q - 1$. This implies that $W_j < W_i$ only if $j < n^q$. Therefore, there can only be finitely many shifts of W smaller than W_i, so no infinite decreasing sequence can begin with W_i.

Now, let $F \geq 0$. Suppose that for all $f \leq F$, any decreasing sequence with W_i as the first term is finite. Consider $f = F + 1$. Let j be an integer such that $W_j < W_i$. Then either W_j has a prefix of the form $\gamma^f \beta \gamma^{l_j} \beta$ with $l_j \leq l$, or W_j has a prefix of the form $\gamma^{f_j} \beta$ with $f_j < f$. In the first case, we know from the case when $f = 0$ that there are only finitely many shifts with prefix β followed by a string of γ of length at most l. Therefore, there can only be finitely many shifts with prefix $\gamma^f \beta \gamma^{l_j} \beta$. To build an infinite decreasing sequence with W_i as the first term, we must eventually pick a shift from the second case. Let $W_{j'}$ be such a shift for some integer j'. Suppose that $W_{j'}$ has a prefix of the form $\gamma^{f'} \beta$ for some integer f' with $0 \leq f' < f$. By the inductive hypothesis, any decreasing sequence with $W_{j'}$ as the first term must be finite. Since a finite sequence remains finite when added a finite number of terms, any decreasing sequence with W_i as the first term must also be finite. Therefore, the set of shifts of W does not contain an infinite decreasing sequence. □

Making similar adjustment as we did in Lemma 8, we have the following similar property.

Lemma 10. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3 and a fixed point of some n-uniform morphism ϕ on Σ_3. Assume that $\beta < \gamma$. Suppose that α occurs only at $W[0]$. If $\phi(\gamma)$ contains exactly one γ and $\phi(\beta) = \beta^n$, then $\phi(\alpha)$ must contain γ, and $\{W_i\}$ has no infinite increasing sequence.*

Lemma 11. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3 and a fixed point of some n-uniform morphism ϕ on Σ_3 with $n \geq 3$ being odd. Assume that $\beta < \gamma$. Suppose that α occurs only at $W[0]$, and that $\phi(\beta) = \beta(\gamma\beta)^{\frac{n-1}{2}}$ and $\phi(\gamma) = \gamma(\beta\gamma)^{\frac{n-1}{2}}$. If $\gamma\gamma$ is not a factor of W, then $\beta\beta$ must be a factor of W and $\{W_i\}$ has no infinite decreasing sequence.*

Proof. Since $\gamma\gamma$ is not a factor of W and W is aperiodic, $\beta\beta$ must be a factor of W and occurs infinitely often in W. Specifically, if $W[j..j+1] = \beta\beta$ for some integer j, then $W[n(j+1)-1..n(j+1)] = \beta\beta$ since $\phi(\beta\beta)[n-1..n] = \beta\beta$. Moreover, $W_n \in \{\phi(\beta), \phi(\gamma)\}^\omega$ and $\phi(\gamma)\phi(\gamma)$ is not a factor of W_n since $\gamma\gamma$ is not a factor of W. Therefore, either W_n is a concatenation of $\beta\beta$ and strings of the form $\gamma(\beta\gamma)^k$ for some positive integer k, or $W[n] = \beta$ and W_{n+1} is a concatenation of $\beta\beta$ and strings of the form $\gamma(\beta\gamma)^k$ for some positive integer k. Note that $\beta\beta\beta$ is not a factor of W_n based on the form of $\phi(\beta)$ and $\phi(\gamma)$.

We restrict our focus to $\{W_i\}_{i\geq n}$, since $\{W_i\}_{i\geq 0}$ contains an infinite decreasing sequence if and only if $\{W_i\}_{i\geq n}$ contains one. Similar to the strategy in the proof of Lemma 9, we will pick an arbitrary shift W_i with $i \geq n$ and show that it is impossible to build an infinite decreasing sequence from the shifts of W with W_i as the first term.

We first consider a shift W_i with prefix β. Then W_i has a prefix of the form $(\beta\gamma)^f\beta\beta\gamma(\beta\gamma)^l\beta\beta$, where $f \geq 0$ and $l \geq \frac{n-3}{2}$. We will induct on f. When $f = 0$, W_i has a prefix $\beta\beta$. Suppose that $j \geq n$. If $W_j < W_i$, then W_j must have prefix $\beta\beta$ and the length of the string between the first and second occurrences of $\beta\beta$ in W_j must be at most $2l+1$. Let $q = \lceil \log_n(2l+4) \rceil$. Observe that $\phi^q(\beta)$ and $\phi^q(\gamma)$ both have a factor of the form $\gamma(\beta\gamma)^k$ of length at least $n^q - 2$, which is greater than $2l+1$. Hence, for all $j \geq n^q$, if W_j has a prefix $\beta\beta$, then $W_i < W_j$. Therefore, there are only finitely many shifts W_j smaller than W_i, so no infinite decreasing sequence can begin with W_i.

Next, let $F \geq 0$. Suppose that for all $f \leq F$, any decreasing sequence with W_i as the first term must be finite. Consider $f = F + 1$. Let j be an integer such that $W_j < W_i$. Then either W_j has a prefix of the form $(\beta\gamma)^f\beta\beta\gamma(\beta\gamma)^{l_j}\beta\beta$ with $l_j \leq l$, or it has a prefix of the form $(\beta\gamma)^{f_j}\beta\beta$ with $f_j < f$. In the first case, since there are finitely many shifts with prefix $\beta\beta\gamma(\beta\gamma)^{l_j}\beta\beta$, where $l_j \leq l$, there can only be finitely many shifts with prefix $(\beta\gamma)^f\beta\beta\gamma(\beta\gamma)^{l_j}\beta\beta$. This implies that we must eventually pick a shift from the second case to build an infinite decreasing sequence with W_i as the first term. Let $W_{j'}$ be such a shift for some integer j'. By the inductive hypothesis, any decreasing sequence with $W_{j'}$ as the first term must be finite. Hence, any decreasing sequence with W_i as the first term must also be finite.

Next, we consider a shift W_i with prefix γ. Then W_i has a prefix of the form $\gamma(\beta\gamma)^f\beta\beta\gamma(\beta\gamma)^l\beta\beta$, where $f \geq 0$ and $l \geq \frac{n-3}{2}$. The proof of this case is very similar to the proof of the previous case, and will be omitted. Note that when $f = 0$, $W_j < W_i$ implies that either W_j has a prefix of the form $\gamma\beta\beta\gamma(\beta\gamma)^{l_j}\beta\beta$ with $l_j \leq l$, or it has a prefix β, which we have already shown cannot be the first term of an infinite decreasing sequence in the previous case. □

Once again, by adjusting the previous proof and using Lemma 10, we have a similar property below.

Lemma 12. *Let $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3, and a fixed point of some n-uniform morphism ϕ on Σ_3 with $n \geq 3$ being odd. Assume that $\beta < \gamma$. Suppose that α occurs only at $W[0]$, and that $\phi(\beta) = \beta(\gamma\beta)^{\frac{n-1}{2}}$ and $\phi(\gamma) = \gamma(\beta\gamma)^{\frac{n-1}{2}}$. If $\beta\beta$ is not a factor of W, then $\gamma\gamma$ must be a factor of W and $\{W_i\}$ has no infinite increasing sequence.*

Next, we consider a classical result in combinatorics on words, which we state below. See Proposition 1.3.4 in [7].

Proposition 3 ([7])**.** *Two words $x, y \in \mathcal{A}^+$ are conjugate if and only if there exists a $z \in \mathcal{A}^*$ such that*

$$xz = zy.$$

More precisely, the equation holds if and only if there exist $u, v \in \mathcal{A}^*$ such that

$$x = uv, \quad y = vu, \quad z \in u(vu)^*.$$

We will now identify the forms of ψ which result in $\{W_i\}$ not containing an infinite decreasing sequence.

Theorem 3. *Suppose that an n-uniform morphism ψ satisfies $\psi(\rho)[0] = \rho$ for all $\rho \in \Sigma_3$, and that α occurs only once in the aperiodic word $W = \psi^\omega(\alpha)$ over Σ_3. Assume that $\beta < \gamma$. Then, $\{W_i\}$ has no infinite decreasing sequence if and only if $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are larger than all shifts starting with β and γ, respectively. In particular, ψ satisfies one of the following:*

1. *β occurs in $\psi(\alpha)$, $\psi(\beta) = \beta\gamma^{n-1}$, and $\psi(\gamma) = \gamma^n$,*
2. *$\beta\beta$ is a factor of $\psi(\alpha)$, but $\gamma\gamma$ is not; $\psi(\beta) = \beta(\gamma\beta)^{\frac{n-1}{2}}$, $\psi(\gamma) = \gamma(\beta\gamma)^{\frac{n-1}{2}}$, with $n \geq 3$ being odd, and $\psi(\alpha)[1] = \beta$ or $\psi(\alpha)[n-1] = \beta$.*

Proof. Since α occurs only once in W, we have $\psi(\alpha)[1..n-1], \psi(\beta), \psi(\gamma) \in \{\beta, \gamma\}^+$ by Proposition 2.

The forward direction follows immediately from Corollary 3. For the other direction, we consider the following cases:

1. If $\psi(\gamma) = \gamma^n$, then $\psi(\beta) = \beta\gamma^{n-1}$ by Lemma 7. As noted in the proof of Lemma 9, β must occur in $\psi(\alpha)$. By Lemma 9, we obtain the first form of ψ, with $\{W_i\}$ not containing an infinite decreasing sequence.

2. If $\psi(\gamma) \neq \gamma^n$, then β occurs in $\psi(\gamma)$. By Lemma 6, $\psi(\beta)[n-1] = \beta$. Hence, it follows from Lemma 7 that $\psi(\beta)[1] = \psi(\gamma)[0] = \gamma$ and $\psi(\beta)[n-1] = \psi(\gamma)[n-2] = \beta$. This implies that $n \geq 3$.

   By Lemma 7, $\psi^\omega(\beta)_1 = \psi^\omega(\gamma)$ which implies that $\psi(\beta)_1\psi(\gamma)[0] = \psi(\gamma)$, so $\psi(\gamma)[n-1] = \psi(\gamma)[0] = \gamma$. We note that $\psi^2(\beta)[0..2n-1] = \psi(\beta)\psi(\gamma)$. Since

$$\psi^2(\gamma)[2n-2] = \psi^2(\beta)[2n-1] = \psi(\gamma)[n-1] = \gamma \neq \psi(\gamma)[n-2],$$

we can conclude that $\psi^2(\gamma)[n..2n-1] \neq \psi(\gamma)$, and hence $\psi(\gamma)[1]$ must be β, which implies that $\psi^2(\gamma)[2n-1] = \beta$. Hence, $\psi^\omega(\beta)[2n] = \psi^\omega(\gamma)[2n-1] = \beta$. Let $\psi(\beta) = \beta\gamma u\beta$ for some $u \in \{\beta, \gamma\}^*$. We have $\psi(\gamma) = \gamma u\beta\gamma$. Observe that

$$\psi(\beta)\gamma\beta = \beta\psi(\gamma)\beta = \psi^\omega(\beta)[n-1..2n] = \psi^2(\gamma)[n-2..2n-1] = \beta\gamma\psi(\beta).$$

By Proposition 3, we have $\psi(\beta) \in \beta(\gamma\beta)^*$. Since $n \geq 3$, $\psi(\beta) \in \beta(\gamma\beta)^+$, and so $\psi(\gamma) \in \gamma(\beta\gamma)^+$. Therefore, n must be odd, and $\psi(\beta) = \beta(\gamma\beta)^{\frac{n-1}{2}}$, $\psi(\gamma) = \gamma(\beta\gamma)^{\frac{n-1}{2}}$.

Note that $\psi(\alpha)$ cannot be $\alpha(\gamma\beta)^{\frac{n-1}{2}}$ or $\alpha(\beta\gamma)^{\frac{n-1}{2}}$; otherwise, the resulting word W would be $\alpha(\gamma\beta)^\omega$ or $\alpha(\beta\gamma)^\omega$, respectively, which are ultimately periodic. Since $\psi(\gamma)[n-1] = \gamma$, by Lemma 6, $\gamma\gamma$ cannot be a factor of W. Since $\psi(\alpha) \notin \alpha(\beta\gamma)^* \cup \alpha(\gamma\beta)^*$, this implies that $\beta\beta$ is not a factor of W. Note that $\gamma\gamma$ would be a factor of W if $\psi(\alpha)[1] = \gamma$ and $\psi(\alpha)[n-1] = \gamma$. Hence $\phi(\alpha)[1] = \beta$ or $\psi(\alpha)[n-1] = \beta$. Finally, by Lemma 11, the fact that $\gamma\gamma$ is not a factor of W implies that $\{W_i\}$ has no infinite decreasing sequence.

This concludes the proof. □

Using Lemma 8, 10, and 12, and an analogous argument as given in the proof of Theorem 3, we have the following result.

Theorem 4. *Suppose that an n-uniform morphism ψ satisfies $\psi(\rho)[0] = \rho$ for all $\rho \in \Sigma_3$, and that α occurs only once in the aperiodic word $W = \psi^\omega(\alpha)$ over Σ_3. Assume that $\beta < \gamma$. Then, $\{W_i\}$ has no infinite increasing sequence if and only if $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are smaller than all shifts starting with β and γ, respectively. In particular, ψ satisfies one of the following forms:*

1. *γ occurs in $\psi(\alpha)$, $\psi(\beta) = \beta^n$ and $\psi(\gamma) = \gamma\beta^{n-1}$,*
2. *$\gamma\gamma$ is a factor of $\psi(\alpha)$, but $\beta\beta$ is not; $\psi(\beta) = \beta(\gamma\beta)^{\frac{n-1}{2}}$, $\psi(\gamma) = \gamma(\beta\gamma)^{\frac{n-1}{2}}$, with $n \geq 3$ being odd, and either $\psi(\alpha)[1] = \gamma$ or $\psi(\alpha)[n-1] = \gamma$.*

Remark 3. Although Theorem 3 gives the forms of ψ, or ϕ^2, those are the only forms of ϕ as well. In other words, $\{W_i\}$ must contain an infinite decreasing sequence if $\phi(\beta)[0] = \gamma$ and $\phi(\gamma)[0] = \beta$. To see that, we first consider the first form of ψ. Assume that $\sqrt{n} \in \mathbb{Z}$. If $\phi^2(\beta) = \beta\gamma^{n-1}$, then we would have

$$\phi(\gamma) = \phi(\phi(\beta)[0]) = \psi(\beta)[0..\sqrt{n}-1] = \beta\gamma^{\sqrt{n}-1}$$

with $\sqrt{n} \geq 2$. Also,

$$\phi(\beta) = \phi(\phi(\gamma)[0]) = \psi(\gamma)[0..\sqrt{n}-1] = \gamma^{\sqrt{n}}.$$

However, that would imply that $\phi^2(\beta) = \phi(\gamma^{\sqrt{n}}) = (\beta\gamma^{\sqrt{n}-1})^{\sqrt{n}}$, which is a contradiction.

Consider the second form of ψ. One can easily check that $\phi(\beta) = \gamma(\beta\gamma)^{\frac{\sqrt{n}-1}{2}}$ and $\phi(\gamma) = \beta(\gamma\beta)^{\frac{\sqrt{n}-1}{2}}$ to obtain the expressions for $\psi(\beta)$ and $\psi(\gamma)$. $\phi(\alpha)$ cannot contain $\beta\beta$, otherwise $\psi(\alpha)$ would contain $\gamma\gamma$. $\phi(\alpha)$ cannot contain $\gamma\gamma$ either, otherwise $\psi(\alpha)[0..\sqrt{n}-1]$ would also contain $\gamma\gamma$. Therefore, we have $\phi(\alpha) = \alpha(\beta\gamma)^{\frac{\sqrt{n}-1}{2}}$ or $\alpha(\gamma\beta)^{\frac{\sqrt{n}-1}{2}}$. The first case would give $\psi(\alpha)[\sqrt{n}-1..\sqrt{n}] = \gamma\gamma$, while the second case would give both $\psi(\alpha)[1] = \gamma$ and $\psi(\alpha)[n-1] = \gamma$. In both cases, the conditions on $\psi(\alpha)$ given in Theorem 3 are violated.

5.2. $\phi(\beta)$ and $\phi(\gamma)$ Have Same Starting Letter

We now consider the case when $\phi(\beta)[0] = \phi(\gamma)[0] \in \{\beta, \gamma\}$. We first point out that $\phi(\beta) \neq \phi(\gamma)$, otherwise, W would be of the form $\phi(\alpha)(\phi(\beta), \phi(\gamma))^\omega = \phi(\alpha)\phi(\beta)^\omega$, which is ultimately periodic.

Observe that one of $\phi^\omega(\beta)$ and $\phi^\omega(\gamma)$ is not a fixed point of ϕ. We will identify the forms of ϕ that do not result in an infinite decreasing sequence in the set of shifts of $W = \phi^\omega(\alpha)$. We will first look at a property in the case when $\phi(\beta) < \phi(\gamma)$ and $\phi^\omega(\gamma)[0] = \gamma$, which requires a bit more work.

If $\phi^\omega(\gamma)$ is not larger than all shifts starting with γ, then $\{W_i\}$ contains an infinite decreasing sequence by Corollary 3. Otherwise, we consider the following property.

Lemma 13. *Let ϕ be an n-uniform morphism on Σ_3 and $W = \phi^\omega(\alpha)$ be an aperiodic word over Σ_3. Suppose that α occurs only once in W, $\beta < \gamma$, and that $\phi(\beta) < \phi(\gamma)$ with $\phi(\beta)[0] = \phi(\gamma)[0] = \gamma$. Also assume that the fixed point $\phi^\omega(\gamma)$ is larger than all shifts of W starting with γ. If β occurs more than once in $\phi(\beta)$ with its first occurrence at position $f \geq 1$, and $\beta\beta$ is not a factor of $\phi(\beta)$ or $\phi(\gamma)$, then $(\phi(\beta)\phi(\gamma))_f[0..n] < \beta\phi(\gamma)$.*

Proof. We first note that W_n is a concatenation of $\phi(\beta)$ and $\phi(\gamma)$ since α only occurs at $W[0]$, and that $\phi(\beta), \phi(\gamma) \in \{\beta, \gamma\}^n$ by Proposition 2. Suppose that $\phi(\gamma) = \gamma^n$. Since β occurs in $\phi(\beta)_{f+1}$, it follows that $(\phi(\beta)\phi(\gamma))_f[0..n] < \beta\gamma^n = \beta\phi(\gamma)$.

We prove by contradiction for the case when $\phi(\gamma) \neq \gamma^n$. First, suppose that $(\phi(\beta)\phi(\gamma))_f[0..n] > \beta\phi(\gamma)$. We can pick i such that $W[i..i+1] = \beta\gamma$, and we would have $W_{ni+f+1} = \phi(W_i)_{f+1} > \phi^\omega(\gamma)$. Since $\beta < \gamma$, we must have $W_{ni+f+1}[0] = \gamma$, which is a contradiction.

Next, suppose that $(\phi(\beta)\phi(\gamma))_f[0..n] = \beta\phi(\gamma)$. Since $\phi(\beta)[0] = \phi(\gamma)[0] = \gamma$ and $\beta\beta$ is not a factor of $\phi(\beta)$ or $\phi(\gamma)$, $\beta\beta$ cannot be a factor of W_n. Hence, $\beta\gamma$ and $\gamma\gamma$ must be factors of W, lest W be ultimately periodic. Let $u, v \in \{\beta, \gamma\}^+$ such that $\phi(\beta)[f+1..n-1] = \phi(\gamma)[0..n-f-2] = v$ and $\phi(\gamma)[0..f] = \phi(\gamma)[n-f-1..n-1] = u$. We have

$$\phi(\beta) = \gamma^f \beta v \quad \text{and} \quad \phi(\gamma) = vu.$$

By Lemma 6, $\phi(\beta)[n-1] = \phi(\gamma)[n-1] = \beta$, so $v[n-f-2] = u[f] = \beta$. Since $\phi(\beta) < \phi(\gamma)$, we have $|v| \geq |\gamma^f\beta| = f+1$ and $v[0..f] \geq \gamma^f\beta$. Also, $u = \phi(\gamma)[0..f] =$

$v[0..f]$, so $v[0..f] = \gamma^f \beta$. We may write $v = \gamma^f \beta v'$ for some $v' \in \{\beta, \gamma\}^*$. Then,

$$\phi(\beta) = \gamma^f \beta \gamma^f \beta v' \quad \text{and} \quad \phi(\gamma) = \gamma^f \beta v' \gamma^f \beta.$$

We will show that both $\phi(\beta)$ and $\phi(\gamma)$ have infinite lengths, which would lead to a contradiction. To see that, suppose to the contrary that there exists a largest integer $L \geq 1$ such that $\phi(\beta) = (\gamma^f \beta)^{L+1} w$ and $\phi(\gamma) = (\gamma^f \beta)^L w \gamma^f \beta$ for some $w \in \{\beta, \gamma\}^*$. We must have $w \neq \varepsilon$, otherwise, we would have $\phi(\beta) = \phi(\gamma)$. Observe that w ends with a β since $\phi(\beta)[n-1] = \beta$. Consider the prefix $\gamma^p \beta$ of w. If $p < f$, then $\phi(\beta) > \phi(\gamma)$, a contradiction. If $p > f$, then γ^{f+1} is a factor of W. In particular, we can find an integer k such that $W[k] = \gamma$, and we would have

$$\begin{aligned} W[nk + f + 1..nk + (L+1)(f+1) - 1] &= \phi(\gamma)[nk + f + 1..nk + (L+1)(f+1) - 1] \\ &= (\gamma^f \beta)^{L-1} \gamma^{f+1}, \end{aligned}$$

so $W_{nk+f+1} > \phi^\omega(\gamma)$. Therefore, we must have $p = f$. This implies that $w = \gamma^f \beta w'$ for some $w' \in \{\beta, \gamma\}^*$, and we would have $\phi(\beta) = (\gamma^f \beta)^{L+2} w'$ and $\phi(\gamma) = (\gamma^f \beta)^{L+1} w' \gamma^f \beta$, and hence L is not the largest such integer, and the claim follows. □

We also give a condition that would guarantee the existence of an infinite decreasing sequence. Let u be a word over Σ_m and ϕ be an n-uniform morphism on Σ_m. Define the notation $\phi_s(u)$ to be the s-shift of $\phi(u)$, or $\phi_s(u) = (\phi(u))_s$. We have the following lemma.

Lemma 14. *Suppose that ϕ is an n-uniform morphism on Σ_m that preserves the order in Σ_m, and that there exist some $\rho \in \Sigma_m$ and $0 \leq s \leq n-1$ such that $\phi(\rho)[s] = \rho$. If W is an infinite word over Σ_m such that $W[0] = \rho$ and $\phi_s(W) < W$, then $((\phi_s)^t(W))_{t \geq 0}$ is an infinite decreasing sequence.*

Proof. We will prove that $(\phi_s)^{t+1}(W) < (\phi_s)^t(W)$ and $(\phi_s)^{t+1}(W)[0] = (\phi_s)^t(W)[0] = \rho$ for all $t \geq 0$ by induction, and the result follows. When $t = 0$, since $\phi(\rho)[s] = \rho$ and $s \leq n-1$, we have $\phi_s(W)[0] = \phi(\rho)[s] = \rho$. The inequality is true by assumption.

Now suppose that $(\phi_s)^{k+1}(W) < (\phi_s)^k(W)$ and $(\phi_s)^{k+1}(W)[0] = (\phi_s)^k(W)[0] = \rho$ for some $k \geq 0$. Let d be the position of first distinction between W and $\phi_s(W)$. Then there exist $u \in \Sigma_m^*$ and $\rho_1, \rho_2 \in \Sigma_m$ with $\rho_1 < \rho_2$, such that $(\phi_s)^k(W)[0..d] = \rho u \rho_2$ and $(\phi_s)^{k+1}(W)[0..d] = \rho u \rho_1$. Hence, we have

$$\phi((\phi_s)^k(W)[0..d]) = \phi(\rho)\phi(u)\phi(\rho_2) \quad \text{and} \quad \phi((\phi_s)^{k+1}(W)[0..d]) = \phi(\rho)\phi(u)\phi(\rho_1).$$

Therefore, $(\phi_s)^{k+1}(W)$ has a prefix $\phi(\rho)_s \phi(u) \phi(\rho_2)$ and $(\phi_s)^{k+2}(W)$ has a prefix $\phi(\rho)_s \phi(u) \phi(\rho_1)$. This implies that $(\phi_s)^{k+2}(W) < (\phi_s)^{k+1}(W)$ since ϕ preserves the order between ρ_1 and ρ_2. Finally, we have $(\phi_s)^{k+2}(W)[0] = (\phi_s)^{k+1}(W)[0] = \phi(\rho)[s] = \rho$. □

We will now show that the form of ϕ that does not result in an infinite decreasing sequence in $\{W_i\}$ is similar to the first form given in Theorem 3.

Theorem 5. *Let $W = \phi(\alpha)$ be an aperiodic word over Σ_3, and a fixed point of an n-uniform morphism ϕ on Σ_3 such that $\phi(\beta)[0] = \phi(\gamma)[0]$. Suppose that α occurs only once in W, and $\beta < \gamma$. Then $\{W_i\}$ does not contain an infinite decreasing sequence if and only if $\phi(\beta)$ contains exactly one β and $\phi(\gamma) = \gamma^n$.*

Proof. As we have mentioned in the beginning of this section, W is ultimately periodic if $\phi(\beta) = \phi(\gamma)$, and so we need only to consider cases where $\phi(\beta) \neq \phi(\gamma)$. Also note that $\phi(\beta), \phi(\gamma) \in \{\beta, \gamma\}^n$ by Proposition 2, and that $W_i \in \{\beta, \gamma\}^\omega$ for all $i \geq 1$.

We start with the case where $\phi(\beta) > \phi(\gamma)$. We restrict ϕ to the 2-letter alphabet $\{\beta, \gamma\}$. By Lemma 5, we have $\phi^2(\beta) < \phi^2(\gamma)$. Since $\phi(\beta)[0] = \phi(\gamma)[0]$, we must also have $\phi^2(\beta)[0] = \phi^2(\gamma)[0]$. If $\phi(\beta)[0] = \beta$, then $\phi^2(\beta)[0] = \phi^2(\gamma)[0] = \beta$. We may pick i such that $W[i] = \gamma$, and we have $W_{n^2 i} = \phi^2(W_i) > \phi^\omega(\beta)$. Hence, $\{W_i\}$ contains an infinite decreasing sequence by Corollary 3. If $\phi(\beta)[0] = \gamma$, then we may pick i such that $W[i] = \beta$, and we have $W_{ni} = \phi(W_i) > \phi^\omega(\gamma)$ since $\phi(\beta) > \phi(\gamma)$. Hence, $\{W_i\}$ contains an infinite decreasing sequence by Corollary 3.

Next, consider the case where $\phi(\beta) < \phi(\gamma)$. Suppose that $\phi(\beta)[0] = \beta$. Similar to the previous case, we can pick i such that $W[i] = \gamma$, and we have $W_{ni} = \phi(W_i) > \phi^\omega(\beta)$, so $\{W_i\}$ contains an infinite decreasing sequence by Corollary 3. Now, suppose that $\phi(\beta)[0] = \gamma$. Observe that β must occur in $\phi(\beta)$, otherwise $\phi(\beta) = \gamma^n$ and $\phi(\beta) \not< \phi(\gamma)$. If $\beta\beta$ is a factor of $\phi(\beta)$, say $\phi(\beta)[l..l+1] = \beta\beta$ for some integer l where $1 \leq l \leq n-2$, then we pick an integer i such that $W_i[0..1] = \beta\gamma$, which must exist since W is aperiodic. We have $\phi_l(W_i) < W_i$. Since ϕ restricted to $\{\beta, \gamma\}$ is order-preserving, $((\phi_l)^t(W_i))$ is an infinite decreasing sequence by Lemma 14.

If $\beta\beta$ is not a factor of $\phi(\beta)$, then we may also assume that $\beta\beta$ does not occur in $\phi(\gamma)$, otherwise $\beta\beta$ would occur in $\phi^2(\beta)$. Consider the subcase where β occurs at least twice in $\phi(\beta)$, and that $\phi(\gamma) = \gamma^n$. We can pick i such that $W[i..i+1] = \beta\gamma$. Let the positions of the first and last β in $\phi(\beta)$ be f and l, respectively, we hence obtain $W_{ni+l}[0..n] = \beta\gamma^n$. By Lemma 13, $\phi_f(W_{ni+l}) < W_{ni+l}$. By Lemma 14, $((\phi_f)^t(W_{ni+l}))$ is an infinite decreasing sequence. Now, consider the next subcase where $\phi(\gamma) \neq \gamma^n$. We may assume that $\phi^\omega(\gamma)$ is larger than all shifts that starts with γ in light of Corollary 3 once again. Observe that $\beta\beta$ does not occur in W_n since $\beta\beta$ occurs in neither $\phi(\beta)$ nor $\phi(\gamma)$, and $\phi(\beta)[0] = \phi(\gamma)[0] = \gamma$. As a result, $\gamma\gamma$ must occur in W since W is aperiodic. We have $\phi(\beta)[n-1] = \phi(\gamma)[n-1] = \beta$, or else $\{W_i\}$ contains an infinite decreasing sequence by Lemma 6. Pick $j > 1$ such that $W[j] = \gamma$, then $W[j-1] \neq \alpha$ and $W_{nj-1}[0..n] = \beta\phi(\gamma)$. Since

$$\phi(W_{nj-1})[0..2n-1] = (\phi(\beta)\phi^2(\gamma))[0..2n-1] = \phi(\beta)\phi(\gamma),$$

we have $\phi_f(W_{nj-1}) < W_{nj-1}$ by Lemma 13. By Lemma 14 once again, we obtain an infinite decreasing sequence $((\phi_f)^t(W_{nj-1}))$.

Suppose that $\phi(\beta)$ contains exactly one β. If $\phi(\gamma)$ contains a β, then $\phi^2(\beta)$ contains at least two β, so $\{W_i\}$ contains an infinite decreasing sequence as we have seen above. Suppose that $\phi(\gamma) = \gamma^n$. By Lemma 9, $\{W_i\}$ does not contain an infinite decreasing sequence. □

Once again, we may apply Lemma 10 and properties analogous to Lemma 13 and Lemma 14 to obtain the following result.

Theorem 6. *Let $W = \phi(\alpha)$ be an aperiodic word over Σ_3, and a fixed point of an n-uniform morphism ϕ on Σ_3 such that $\phi(\beta)[0] = \phi(\gamma)[0]$. Suppose that α occurs only once in W. Then $\{W_i\}$ does not contain an infinite increasing sequence if and only if $\phi(\gamma)$ contains exactly one γ and $\phi(\beta) = \beta^n$.*

5.3. Some Examples of Order Type of Shifts

We end this section with a discussion of some order types of $\{W_i\}$ if $W[0]$ occurs only once in a uniform morphic word W, as well as the effect of some coding have on a uniform morphic word. We begin with the cases discussed in Theorem 3 and Theorem 5.

1. Suppose that $W = \psi^\omega(\alpha)$ is a fixed point of the first form of morphisms given in Theorem 3 or the similar form in Theorem 5. Let $i \geq 1$. An i-shift of W has a prefix of the form $\gamma^{f_i}\beta\gamma^{l_i}\beta$, where $f_i, l_i \geq 0$. The ordering of these shifts depends on both f_i and l_i, both of which can be arbitrarily large. Observe that $W_k < W_j$ if $f_k < f_j$, or $f_k = f_j$ and $l_k < l_j$. We first fix a value for f_i and consider the ordering of such shifts. The order type of those shifts would be ω since there are finitely many shifts with $l_i = a$ for each infinitely many possible values for a, as demonstrated in the proof of Lemma 9. As f_i can take on all non-negative integer values, the order type of $\{W_i\}_{i\geq 1}$ is ω^2. Therefore, the order type of $\{W_i\}$ is ω^2 if $\alpha = 0$ or 1, and $\omega^2 + 1$ if $\alpha = 2$.

2. The order type of $W = \psi^\omega(\alpha)$ that is a fixed point of the second form of morphisms in Theorem 3 is slightly different. In particular, for $i \geq n$, an i-shift of W has a prefix of the form $(\beta\gamma)^{f_i}\beta\beta\gamma(\beta\gamma)^{l_i}\beta\beta$ or $\gamma(\beta\gamma)^{f_i}\beta\beta\gamma(\beta\gamma)^{l_i}\beta\beta$, where $f_i \geq 0$ and $l_i \geq 0$. We first consider shifts starting with β. When we fix f_i, there are only finitely many shifts with $l_i = a$ for each possible value of a, as seen in the proof of Lemma 11. Since f_i takes on all non-negative integer values, the order type of shifts starting with β is ω^2. Similarly, for shifts starting with γ, we also have order type ω^2. For $1 \leq i \leq n-1$, W_i starts with either β or γ. Since $\psi^\omega(\beta)$ and $\psi^\omega(\gamma)$ are larger than all shifts of their respective starting letters and ψ preserves the order in $\{\beta, \gamma\}$, we have $W_i < \psi(W_i)$ by Corollary 3, and hence there is no maximum element

in $\{W_i\}_{i\geq 1}$. Therefore, the order type of $\{W_i\}$ is $\omega^2 \cdot 2$ if $\alpha = 0$ or 1, and $\omega^2 \cdot 2 + 1$ if $\alpha = 2$.

3. Using analogous arguments from the first two examples, we can show that if $W = \psi(\alpha)$ is a fixed point of the first form of morphisms in Theorem 4 or the type in Theorem 6, the order type of $\{W_i\}$ is $(\omega^*)^2$ if $\alpha = 1$ or 2, and $1+(\omega^*)^2$ if $\alpha = 0$. If $W = \psi(\alpha)$ is a fixed point of the second form of morphisms in Theorem 4, the order type of $\{W_i\}$ is $(\omega^*)^2 \cdot 2$ if $\alpha = 1$ or 2, and $1 + (\omega^*)^2 \cdot 2$ if $\alpha = 0$.

In the first three examples, $\{W_i\}$ does not have an infinite decreasing sequence or an infinite increasing sequence. We now look at a few examples of words whose shifts contain both an infinite decreasing sequence and an infinite increasing sequence. The order types of their shifts vary, as one might expect.

1. Consider the fixed point r of the uniform morphism defined by $0 \mapsto 012$, $1 \mapsto 111$, $2 \mapsto 222$, starting with 0:

$$r = 012111222111111111222222222 1 \cdots .$$

Obviously, r is the minimum element in $\{r_i\}$. If $r[i] = 1$, then r_i has a prefix of the form 1^a2^b1. In particular, for each $a \geq 1$ and $b \geq 3^{\lceil \log_3(a) \rceil}$ such that b is a power of 3, there is a unique shift with that corresponding prefix. Therefore, the order type of the set of shifts starting with 1 is $\omega \cdot \omega^*$. Similarly, if $r[i] = 2$, then r_i has a prefix of the form 2^c1^d2. In particular, for each $c \geq 1$ and $d \geq 3^{\lceil \log_3(a) \rceil + 1}$ such that d is a power of 3, there is a unique shift with that corresponding prefix. Hence, the order type of $\{r_i\}_{i\geq 1}$ is $\omega^* \cdot \omega$, and therefore, $ot(r) = 1 + \omega \cdot \omega^* + \omega^* \cdot \omega$, which is not dense.

2. Consider the fixed point $\mathbf{u}$ of the uniform morphism defined by $0 \mapsto 02$, $1 \mapsto 12$, $2 \mapsto 20$, starting with 1:

$$\mathbf{u} = 122020022002 \cdots .$$

If we define a coding $\sigma_0 : \Sigma_3 \mapsto \Sigma_2$ by $0, 1 \mapsto 0$ and $2 \mapsto 1$, then one would quickly realize that $\sigma_0(\mathbf{u})$ is the Thue–Morse word. We may view W as the Thue–Morse Word over $\{0, 2\}$, but with $\mathbf{u}[0] = 1$ instead of 0. By Proposition 1, we know that $\mathbf{t}$ is the maximum element of $\{\mathbf{t}_i\}$ starting with 0, and hence $\mathbf{u}$ remains to be larger than all other shifts starting with 0, but smaller than all shifts starting with 2. As a result, $ot(\mathbf{u})$ is the same as the order type of $(0, 1] \cap \mathbb{Q}$, which is dense.

3. Consider the fixed point of the uniform morphism defined by $0 \mapsto 01$, $1 \mapsto 10$, $2 \mapsto 21$, starting with 2:

$$\mathbf{v} = 211010011001 \cdots .$$

This is the Thue–Morse word $\mathbf{t}$ with the first 0 replaced by 2. Observe that $\mathbf{v}$ is larger than all of its shifts. By Corollary 5, we know that $\mathbf{v}_1$ is the largest shift in $\{\mathbf{v}_i\}_{i\geq 1}$. Hence, the order type of $\{\mathbf{v}_i\}_{i\geq 1}$ is the same as the order type of $(0,1]\cap\mathbb{Q}$, and so $ot(\mathbf{v})$ is the same as the order type of $((0,1]\cap\mathbb{Q})\cup\{2\}$ and is (barely) not dense.

Finally, we give some examples of the effect of coding on the order type of an aperiodic word.

1. For any aperiodic word W over some alphabet $\mathcal{A}$, if a coding $\Sigma : \mathcal{A} \to \Sigma_m$ sends all letters in $\mathcal{A}$ that appear infinitely often in W to the same letter in Σ_m, then $\sigma(W)$ is ultimately periodic, and hence $ot(\sigma(W))$ is finite.

2. Consider an aperiodic morphic word W that is the image of a purely morphic word W' under a coding σ, with the first letter of W' appearing twice in W'. By Theorem 2, $ot(W)$ is dense. If σ_1 is another coding such that $\sigma_1(W)$ is aperiodic, then $ot(\sigma_1(W))$ must also be dense since $\sigma_1(W)$ is the image of W' under the coding $\sigma_1 \circ \sigma$. However, the order type may change. For example, let $\sigma : \Sigma_2 \to \Sigma_2$ be defined by $0 \mapsto 1$ and $1 \mapsto 0$. Then, $\bar{\mathbf{t}} = \sigma(\mathbf{t})$. However, $ot(\mathbf{t})$ is the same as the order type of $\mathbf{Q}\cap(0,1]$, while $ot(\bar{\mathbf{t}})$ is the same as the order type of $\mathbf{Q}\cap[0,1)$.

3. Suppose that W is an aperiodic morphic word over Σ_m whose first letter only occurs once in W. Suppose that $\sigma : \Sigma_m \to \Sigma_n$ is a coding such that $\sigma(W)$ is aperiodic. Denseness of $ot(W)$ does not imply denseness of $ot(\sigma(W)$, or vice versa. Consider $\mathbf{u}$ and $\mathbf{v}$ from previous examples. Let $\sigma_1 : \Sigma_3 \to \Sigma_3$ be defined by $0 \mapsto 0$, $1 \mapsto 2$, and $2 \mapsto 1$. We can see that $\mathbf{u} = \sigma_1(\mathbf{v})$ and $\mathbf{v} = \sigma_1(\mathbf{u})$, and only $ot(\mathbf{u})$ is dense.

4. We mentioned at the end of Section 4 that aperiodic binary morphic word need not have a dense order type. Consider r in a previous example and the coding $\sigma : \Sigma_3 \to \Sigma_2$ defined by $0, 1 \mapsto 0$ and $2 \mapsto 1$. We have

$$\sigma(W) = 0010001110000000001111111110\cdots.$$

The shifts that start with 1 or 2 are now shifts starting with 0 and 1, respectively, and so the order type given by the proper shifts of $\sigma(W)$ is $\omega\cdot\omega^*+\omega^*\cdot\omega$. Since $\sigma(W)$ now starts with 0, it becomes the smallest shift of $\omega_2(W)$ starting with 00. Moreover, since $1+\omega=\omega$, we still have $ot(\sigma(W)) = \omega\cdot\omega^*+\omega^*\cdot\omega$, which is not dense.

6. Open Problems

We have barely scratched the surface on the topic of order type of words. Here are a few problems that may interest the reader:

- We have looked at some examples related to codings in Section 5.3. What else can we say about the effects of different coding have on the order type of words? For example, are there certain properties of order type that are preserved under certain coding?
- A question was posed below Theorem 3 in [5] concerning voids in order types. We pose a similar question on morphic words: what countable order types can/cannot be achieved?
- What is the order type of shifts of other type of words, like the Kolakoski word?

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