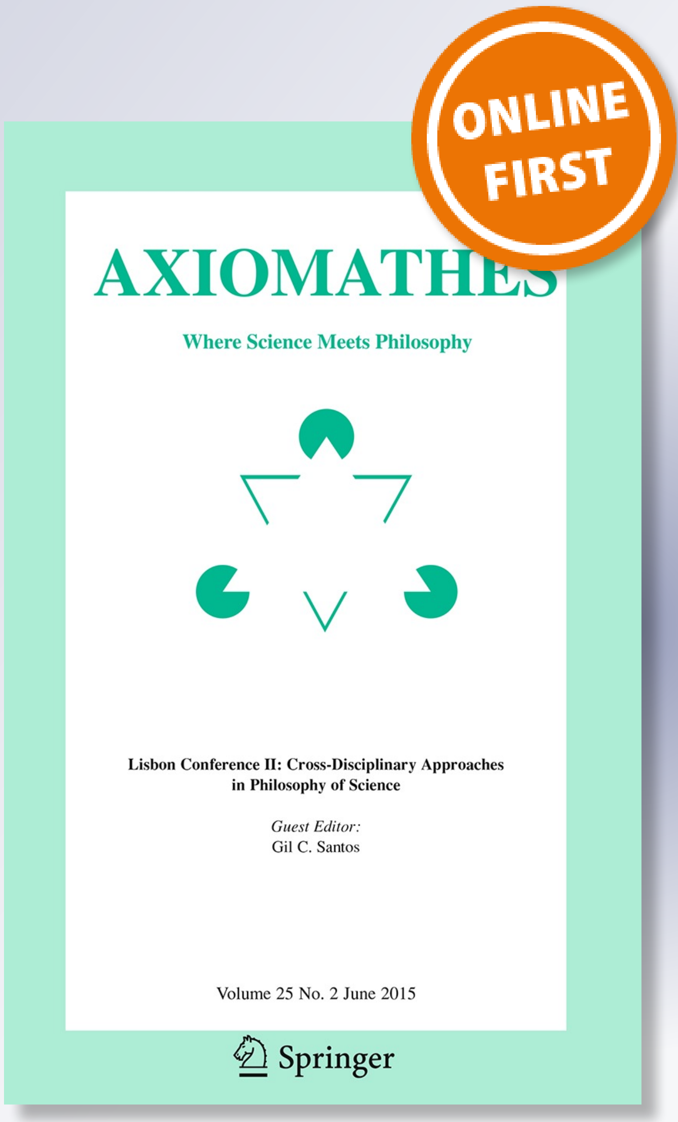
On Adjoint and Brain Functors

David Ellerman

Axiomathes
Where Science Meets Philosophy

ISSN 1122-1151

Axiomathes
DOI 10.1007/s10516-015-9278-7



Springer

Axiomathes
DOI 10.1007/s10516-015-9278-7

ORIGINAL PAPER

On Adjoint and Brain Functors

David Ellerman[1]

Received: 16 June 2015 / Accepted: 3 August 2015

Abstract There is some consensus among orthodox category theorists that the concept of adjoint functors is the most important concept contributed to mathematics by category theory. We give a heterodox treatment of adjoints using heteromorphisms (object-to-object morphisms between objects of different categories) that parses an adjunction into two separate parts (left and right representations of heteromorphisms). Then these separate parts can be recombined in a new way to define a cognate concept, the brain functor, to abstractly model the functions of perception and action of a brain. The treatment uses relatively simple category theory and is focused on the interpretation and application of the mathematical concepts.

Keywords Category theory · Adjoint functors · Heteromorphism · Brain functors

Mathematics Subject Classification 18 · 92

1 Category Theory in the Life and Cognitive Sciences

There is already a considerable but widely varying literature on the application of category theory to the life and cognitive sciences–such as the work of Rosen

✉ David Ellerman
David@ellerman.org

[1] Philosophy Department, University of California at Riverside, Riverside, CA, USA

Published online: 13 August 2015

(1958, 2012) and his followers[1] as well as Ehresmann and Vanbremeersch (2007) and their commentators.[2]

The approach taken here is based on a specific use of the characteristic concepts of category theory, namely universal mapping properties. One such approach in the literature is that of François Magnan and Gonzalo Reyes which emphasizes that "Category theory provides means to circumscribe and study what is universal in mathematics and other scientific disciplines." (Magnan and Reyes 1994, p. 57). Their intended field of application is cognitive science.

> We may even suggest that universals of the mind may be expressed by means of universal properties in the theory of categories and much of the work done up to now in this area seems to bear out this suggestion....
> By discussing the process of counting in some detail, we give evidence that this universal ability of the human mind may be conveniently conceptualized in terms of this theory of universals which is category theory. (Magnan and Reyes 1994, p. 59)

Another current approach that emphasizes universal mapping properties ("universal constructions") is that of Halford and Wilson (1980), Philips and Wilson (2014), Philips (2014).

In addition to the focus on universals, the approach here is distinctive in the use of heteromorphisms–which are object-to-object morphisms between objects if different categories–in contrast to the usual homomorphisms or homs between objects in the same category. By explicitly adding heteromorphisms to the usual homs-only presentation of category theory, this approach can directly represent interactions between the objects of different categories (intuitively, between an organism and the environment). But it is still early days, and many approaches need to be tried to find out "where theory lives."

2 The Ubiquity and Importance of Adjoints

Before developing the concept of a brain functor, we need to consider the related concept of a pair of adjoint functors, an adjunction. The developers of category theory, Saunders MacLane and Samuel Eilenberg, famously said that categories were defined in order to define functors, and functors were defined in order to define natural transformations (Eilenberg and MacLane 1945). A few years later, the concept of universal constructions or universal mapping properties was isolated (MacLane 1948; Samuel 1948). Adjoints were defined a decade later by Kan (1958) and the realization of their ubiquity ["Adjoint functors arise everywhere" (MacLane 1971, p. v)] and their foundational importance has steadily increased over time (Lawvere 1969; Lambek 1981). Now it would perhaps not be too much of an

[1] See Zafiris (2012), Louie (1985) and Louie and Poli (2011) and their references.

[2] See Kainen (2009) for Kainen's comments on the Ehresmann-Vanbremeersch approach, Kainen's own approach, and a broad bibliography of relevant papers.

exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. As Steven Awodey put it:

> The notion of adjoint functor applies everything that we have learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. (Awodey 2006, p. 179)

Other category theorists have given similar testimonials.

> To some, including this writer, adjunction is the most important concept in category theory. (Wood 2004, p. 6)
> The isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas." (Goldblatt 2006, p. 438)
> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. (Taylor 1999, p. 367)

3 Adjoints and Universals

How do the ubiquitous and important adjoint functors relate to the universal constructions? MacLane and Birkhoff succinctly state the idea of the universals of category theory and note that adjunctions can be analyzed in terms of those universals.

> The construction of a new algebraic object will often solve a specific problem in a universal way, in the sense that every other solution of the given problem is obtained from this one by a unique homomorphism. The basic idea of an adjoint functor arises from the analysis of such universals. (MacLane and Birkhoff 1988, p. v)

We can use some old language from Plato's theory of universals to describe those universals of category theory (Ellerman 1988) that solve a problem in a universal or paradigmatic way so that "every other solution of the given problem is obtained from this one" in a unique way.

In Plato's Theory of Ideas or Forms ($\varepsilon\iota\delta\eta$), a property F has an entity associated with it, the universal u_F, which uniquely represents the property. An object x has the property F, i.e., $F(x)$, if and only if (iff) the object x *participates* in the universal u_F. Let μ (from $\mu\varepsilon\theta\varepsilon\xi\iota\varsigma$ or *methexis*) represent the participation relation so

$$\text{“}x\ \mu\ u_F\text{” reads as “}x \text{ participates in } u_F\text{”}.$$

Given a relation μ, an entity u_F is said to be a *universal* for the property F (with respect to μ) if it satisfies the following universality condition:

$$\text{for any } x,\ x\,\mu\,u_F \quad \text{if and only if } F(x).$$

A universal representing a property should be in some sense unique. Hence there should be an equivalence relation ($\approx$) so that universals satisfy a uniqueness condition:

$$\text{if } u_F \text{ and } u'_F \text{ are universals for the same } F, \quad \text{then } u_F \approx u'_F.$$

The two criteria for a *theory of universals* is that it contains a binary relation μ and an equivalence relation $\approx$ so that with certain properties F there are associated entities u_F satisfying the following conditions:

1. *Universality condition*: for any x, $x\ \mu\ u_F$ iff $F(x)$, and
2. *Uniqueness condition*: if u_F and u'_F are universals for the same F [i.e., satisfy (1)], then $u_F \approx u'_F$.

A universal u_F is said to be *non-self-predicative* if it does not participate in itself, i.e., $\neg(u_F\,\mu\,u_F)$. A universal u_F is *self-predicative* if it participates in itself, i.e., $u_F\,\mu\,u_F$.[3] For the sets in an iterative set theory (Boolos 1971), set membership is the participation relation, set equality is the equivalence relation, and those sets are never-self-predicative (since the set of instances of a property is always of higher type or rank than the instances). The universals of category theory form the "other bookend" as always-self-predicative universals. The set-theoretical paradoxes arose from trying to have *one* theory of universals ("Frege's Paradise") where the universals could be *either* self-predicative or non-self-predicative,[4] instead of having two opposite "bookend" theories, one for never-self-predicative universals (set theory) and one for always always-self-predicative universals (category theory).

For the self-predicative universals of category theory (see MacLane and Birkhoff 1988 or MacLane 1971 for introductions), the participation relation is the *uniquely-factors-through* relation. It can always be formulated in a suitable category as:

$$\text{“}x\ \mu\ u_F\text{” means “there exists a unique arrow } x \Rightarrow u_F\text{”}.$$

Then x is said to *uniquely factor through* u_F, and the arrow $x \Rightarrow u_F$ is the unique factor or participation morphism. In the universality condition,

$$\text{for any } x,\ x\ \mu\ u_F \quad \text{if and only if } F(x),$$

[3] A self-predicative universal for some property gives an impredicative definition of having that property. See Louie and Poli (2011, p. 245) where a supremum or least upper bound is referred to as giving an impredicative definition of being an upper bound of a subset of a partial order. Also Makkai (1999) makes a similiar remark about the universal mapping property of the natural number system.

[4] Then the universal for all the non-self-predicative universals would give rise to Russell's Paradox since it could not be self-predicative or non-self-predicative [Russell (2010, p. 80)].

the existence of the identity arrow $1_{u_F} : u_F \Rightarrow u_F$ is the self-participation of the self-predicative universal that corresponds with $F(u_F)$, the self-predication of the property to u_F. In category theory, the equivalence relation used in the uniqueness condition is the isomorphism ($\cong$).

4 The Hom-Set Definition of an Adjunction

We will later use a specific heterodox treatment of adjunctions, first developed by Pareigis (1970) and later rediscovered and developed by Ellerman (2007), which shows that adjoints arise by gluing together in a certain way two universals (left and right representations). But for illustration, we start with the standard Hom-set definition of an adjunction.

The category *Sets* has all sets as objects and all functions between sets as the homomorphisms so for sets a and a', $\mathrm{Hom}(a, a')$ is the set of functions $a \to a'$. In the product category $Sets \times Sets$, the objects are ordered pairs of sets (a, b) and homomorphism $(a, b) \to (a', b')$ is just a pair of functions (f, g) where $f : a \to a'$ and $g : b \to b'$.

For an example of an adjunction, consider the *product functor* $\times : Sets \times Sets \to Sets$ which takes a pair of sets (a, b) to their Cartesian product $a \times b$ (set of ordered pairs of elements from a and b) and takes a homomorphism $(f, g) : (a, b) \to (a', b')$ to $f \times g : a \times b \to a' \times b'$ where for $x \in a$ and $y \in b, f \times g : (x, y) \longmapsto (f(x), g(y))$.

The maps $f : a \to a'$ in *Sets* go from one set to one set and the maps $(f, g) : (a, b) \to (a', b')$ in $Sets \times Sets$ go from a pair of sets to a pair of sets. There is also the idea of a *cone* $[f, g] : c \to (a, b)$ of maps that is a pair of maps $f : c \to a$ and $g : c \to b$ going from one set c (the point of the cone) in *Sets* to a pair of sets (a, b) (the base of the cone) in $Sets \times Sets$. Before the notion of a adjunction was defined by Kan (1958), the product of sets $a \times b$ was defined by its universal mapping property. The projection maps $\pi_a : a \times b \to a$ and $\pi_b : a \times b \to b$ define a canonical cone $[\pi_a, \pi_b] : a \times b \to (a, b)$ that is universal in the following sense. Given any other cone $[f, g] : c \to (a, b)$ from any set c to (a, b), there is a unique homomorphism $\langle f, g \rangle : c \to a \times b$ in *Sets* such that the two triangles in the following diagram commute.

In terms of the self-predicative universals considered in the last section, the property in question is the property of being a cone $[f, g] : c \to (a, b)$ to (a, b) from any set c. The canonical cone of projections $[\pi_a, \pi_b] : a \times b \to (a, b)$ is the self-predicative universal for that property. The participation relation $[f, g]\ \mu\ [\pi_a, \pi_b]$ is defined as "uniquely factoring through" (as in Fig. 1). The universal mapping property of the product can then be restated as the universality condition: For any cone $[f, g]$ from any set to a pair of sets,

$$[f, g]\ \mu\ [\pi_a, \pi_b] \quad \text{if and only if } [f, g] \text{ is a cone to } (a, b).$$

UMP of $a \times b$ stated as a universality condition.

The Hom-set definition of the adjunction for the product functor uses the auxiliary device of a diagonal functor to avoid mentioning the cones and to restrict attention

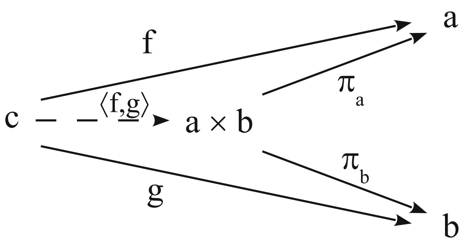


Fig. 1 Universal mapping property for the direct product of sets

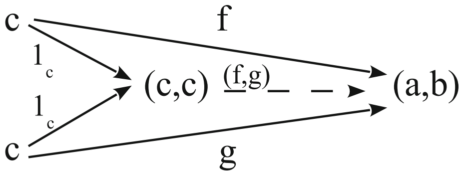


Fig. 2 Universal mapping property for diagonal functor

only to the Hom-sets of the two categories. The *diagonal functor* $\Delta : Sets \rightarrow Sets \times Sets$ in the opposite direction of the product functor just doubles everything so $\Delta(c) = (c, c)$ and $\Delta(f) = (f, f)$. Then the product functor is said to be the *right adjoint* of the diagonal functor, the diagonal functor is said to be the *left adjoint* of the product functor, and the two functors together form an *adjunction* if there is a natural isomorphism between the Hom-sets as follows:

$$\mathrm{Hom}_{Sets\times Sets}(\Delta(c), (a, b)) \cong \mathrm{Hom}_{Sets}(c, a \times b).$$

Hom-set definition of the adjunction between the product and diagonal functors.

The diagonal functor $\Delta : Sets \rightarrow Sets \times Sets$ also has a (rather trivial) UMP that can be stated in terms of cones $c \rightarrow (a, b)$ except now we fix c and let (a, b) vary. There is the canoncial cone $[1_c, 1_c] : c \rightarrow (c, c)$ and it is universal in the following sense. For any cone $[f, g] : c \rightarrow (a, b)$ from the given c to any pair of sets (a, b), there is a unique homomorphism in $Sets \times Sets$, namely $(f, g) : (c, c) \rightarrow (a, b)$ that factors through the canonical cone $c \rightarrow (c, c)$ (Fig. 2).

This product-diagonal adjunction illustrates the general Hom-set definition. Given functors $F : \mathbb{X} \rightarrow \mathbb{A}$ and $G : \mathbb{A} \rightarrow \mathbb{X}$ going each way between categories $\mathbb{X}$ and $\mathbb{A}$, they form an adjunction if there is a natural isomorphism (for objects $X \in \mathbb{X}$ and $A \in \mathbb{A}$):

$$\mathrm{Hom}_{\mathbb{A}}(F(X), A) \cong \mathrm{Hom}_{\mathbb{X}}(X, G(A))$$

Hom – set definition of an adjunction.

To further analyze adjoints, we need the notion of a "heteromorphism."

5 Heteromorphisms and Adjunctions

We have seen that there are two UMPs (often one is trivial like $\Delta(c)$ in the above example) involved in an adjunction and that the object-to-object maps were always

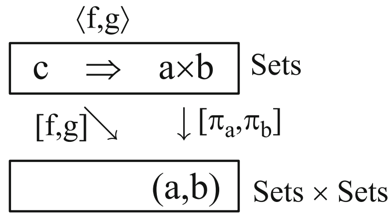


Fig. 3 UMP for the product functor

within one category, e.g., in the "Hom-sets" of one category or the other. Using object-to-object maps between objects of *different* categories (properly called "heteromorphisms" or "chimera morphisms"), the notion of an adjunction can be factored into two representations [or "half-adjunctions" in Ellerman (2006, p. 158)], each of which expresses a universal mapping property.

We have already seen one standard example of a *heteromorphism* or *het*, namely a cone $[f,g]: c \to (a,b)$ that goes from an object in *Sets* to an object in *Sets* × *Sets*. The hets are contrasted with the homs or homomorphisms between objects in the same category. To keep them separate in our notation, we will henceforth use single arrows $\longrightarrow$ for hets and double arrows $\Rightarrow$ for homs.[5] Then the UMP for the product functor can be represented as follows (Fig. 3).

It should be particularly noted that this het-formulation of the UMP for the product does not involve the diagonal functor. If we associate with each $c \in Sets$ and each $(a,b) \in Sets \times Sets$, the set $\mathrm{Het}(c,(a,b))$ of cones or hets $[f,g]: c \to (a,b)$ then this defines a Het-*bifunctor* in the same manner as the usual Hom-bifunctor $\mathrm{Hom}_{Sets}(a,a')$ or $\mathrm{Hom}_{Sets\times Sets}((a,b),(a',b'))$ [see the appendix for more details]. Then the UMP for the product functor gives a natural isomorphism based on the pairing: $[f,g] \mapsto \langle f,g\rangle$, so that the *Sets*-valued functor $\mathrm{Het}(c,(a,b))$ is said to be *represented on the right* by the *Sets*-valued $\mathrm{Hom}_{Sets}(c, a\times b)$:

$$\mathrm{Het}(c,(a,b)) \cong \mathrm{Hom}_{Sets}(c, a\times b)$$

Right representation of the hets $c \to (a,b)$ by the homs $c \Rightarrow a \times b$.

The trivial UMP for the diagonal functor can also be stated in terms of the cone-hets without reference to the product functor (Fig. 4).

This UMP for the diagonal functor gives a natural isomorphism based on the pairing $(f,g) \longmapsto [f,g]$, so the *Sets*-valued functor $\mathrm{Het}(c,(a,b))$ is said to be *represented on the left* by the *Sets*-valued $\mathrm{Hom}_{Sets\times Sets}((c,c),(a,b))$:

$$\mathrm{Hom}_{Sets\times Sets}((c,c),(a,b)) \cong \mathrm{Het}(c,(a,b))$$

Left representation of the hets $c \to (a,b)$ by the homs $(c,c) \Rightarrow (a,b)$.

[5] The hets between objects of different categories are represented as single arrows ($\to$) while the homomorphisms or homs between objects in the same category are represented by double arrows ($\Rightarrow$). The functors between whole categories are also represented by single arrows ($\to$). One must be careful not to confuse a functor $F: \mathbb{X} \to \mathbb{A}$ from a category $\mathbb{X}$ to a category $\mathbb{A}$ with its action on an object $X \in \mathbb{X}$ which would be symbolized $X \longmapsto F(X)$. Moreover since a functor often has a canonical definition, there may well be a canonical het $X \to F(X)$ or $X \leftarrow F(X)$ but such hets are no part of the definition of the functor itself.

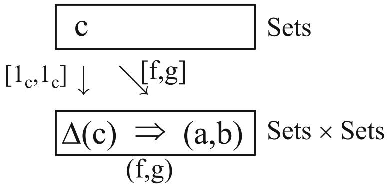


Fig. 4 UMP for the diagonal functor

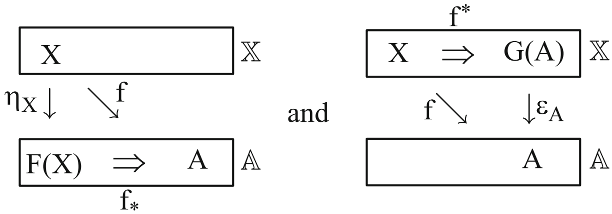


Fig. 5 Left and right representations each involving on one of the adjoints F and G

Then the right and left representations of the hets $\mathrm{Het}(c,(a,b))$ can be combined to obtain as a consequence the Hom-set definition of the adjunction between the product and diagonal functors:

$$\mathrm{Hom}_{Sets\times Sets}((c,c),(a,b)) \cong \mathrm{Het}(c,(a,b)) \cong \mathrm{Hom}_{Sets}(c,a\times b)$$

Heteromorphic presentation of the product-diagonal adjunction.

In the general case of adjoint functors $F:\mathbb{X}\rightleftarrows\mathbb{A}:G$, the hets $\mathrm{Het}(X,A)$ from objects $X\in\mathbb{X}$ to objects $A\in\mathbb{A}$ have left and right representations:

$$\mathrm{Hom}_{\mathbb{A}}(F(X),A) \cong \mathrm{Het}(X,A) \cong \mathrm{Hom}_{\mathbb{X}}(X,G(A))$$

Heteromorphic presentation of a general adjunction.

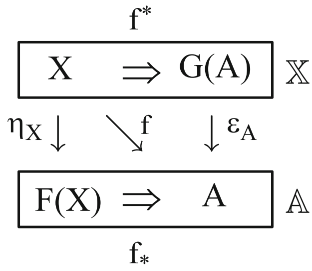
This is the heterodox treatment of an adjunction first published by Pareigis (1970, pp. 60–61) and later rediscovered and developed by Ellerman (2006, p. 130, 2007). It is "heterodox" since the morphisms between the objects of *different* categories are not "officially" recognized in the standard presentations of category theory (e.g., MacLane 1971 or Awodey 2006) even though such hets are a common part of mathematical practice (see the appendix for further discussion). Hence the standard Hom-set definition of an adjunction just deletes the Het-middle-term $\mathrm{Het}(X,A)$ to obtain just the het-free or homs-only presentation of an adjunction.

The important advance of the heteromorphic treatment of an adjunction is that the adjunction can be parsed or factored into two parts, the left and right representations, each of which only involves one of the Hom-functors (Fig. 5).

Moreover, the diagrams for the two representations can be glued together at the diagonal het $\searrow^{f}$ into one diagram to give the simple *adjunctive square diagram* for an adjunction (Fig. 6).

Every adjunction can be represented (up to isomorphism) in this manner (Ellerman 2006, p. 147) so the molecule of an adjunction can be split into two

Fig. 6 Adjunctive square diagram for the het-treatment of an adjunction



atoms, each of which is a (left or right) representation of a Het-functor. This means that the importance and ubiquity of adjunctions (emphasized above) also passes to the atoms, left or right representations, that make up those molecules. Moreover, it should be noted that each left or right representation defines a self-predicative universal as indicated in the previous example of the het-cones $c \to (a, b)$.

The main point of this paper is that those atoms, the left and right representations can be recombined in a new way to define a "recombinant construction" cognate to an adjunction, and that is the concept of a "brain functor."

6 Brain Functors

In many adjunctions, the important fact is expressed by either the left or right representation (e.g., the UMP for the product functor or for the free-group functor considered in the "Appendix"), with no need for the "auxiliary device" (such as a diagonal or forgetful functor) of the other representation used to express the adjunction in a het-free manner.

Another payoff from analyzing the important but molecular concept of an adjunction into two atomic representations is that we can then reassemble those atomic parts in a new way to define the cognate concept speculatively named a "brain functor."

The basic intuition is to think of one category $\mathbb{X}$ in a representation as the "environment" and the other category $\mathbb{A}$ as an "organism." Instead of representations within *each* category of the hets going *one way* between the categories (as in an adjunction), suppose the hets going *both ways* were represented within *one* of the categories (the "organism").

Intuitively, a het from the environment to the organism is say, a visual or auditory stimulus. Then a left representation would play the role of the brain in providing the re-cognition or perception (expressed by the intentionality-of-perception slogan: "seeing is seeing-as") of the stimulus as a perception of, say, a tree where the internal re-cognition is represented by the homomorphism $\Rightarrow$ inside the "organism" category (Fig. 7).

Perhaps not surprisingly, this mathematically models the old philosophical theme in the Platonic tradition that external stimuli do not give knowledge; the stimuli only trigger the internal perception, recognition, or recollection (as in Plato's *Meno*) that

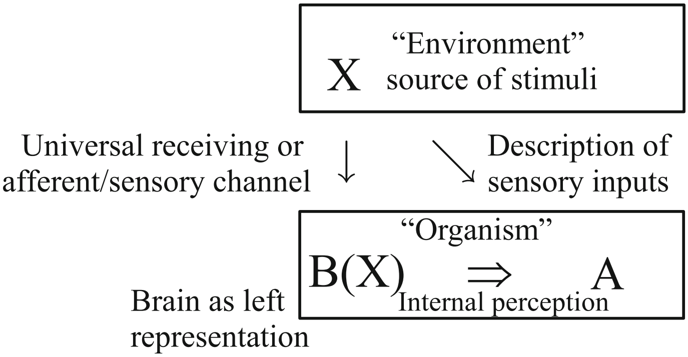


Fig. 7 Perceiving brain presented as a left representation

is knowledge. In *De Magistro* (The Teacher), the neo-Platonic Christian philosopher Augustine of Hippo developed an argument (in the form of a dialogue with his son Adeodatus) that as teachers teach, it is only the student's internal appropriation of what is taught that gives understanding.

> Then those who are called pupils consider within themselves whether what has been explained has been said truly; looking of course to that interior truth, according to the measure of which each is able. Thus they learn,.... But men are mistaken, so that they call those teachers who are not, merely because for the most part there is no delay between the time of speaking and the time of cognition. And since after the speaker has reminded them, the pupils quickly learn within, they think that they have been taught outwardly by him who prompts them. (Augustine, *De Magistro*, Chapter XIV)

The basic point is the active role of the mind in generating understanding (represented by the internal hom). This is clear even at the simple level of understanding spoken words. We hear the auditory sense data of words in a completely strange language as well as the words in our native language. But the strange words bounce off our minds, like @#$%ˆ with no resultant understanding while the words in a familiar language prompt an internal process of generating a meaning so that we understand the words. Thus it could be said that "understanding a language" means there is a left representation for the heard statements in that language, but there is no such internal re-cognition mechanism for the heard auditory inputs in a strange language.

Dually, there are also hets going the other way from the "organism" to the "environment" and there is a similar distinction between mere behavior (e.g., a reflex) and an action that expresses an intention. Mathematically that is described by dualizing or turning the arrows around which gives an acting brain presented as a right representation (Fig. 8).

In the heteromorphic treatment of adjunctions, an adjunction arises when the hets from one category $\mathbb{X}$ to another category $\mathbb{A}$, $\mathrm{Het}(X, A)$ for $X \in \mathbb{X}$ and $A \in \mathbb{A}$, have a right representation, $\mathrm{Het}(X, A) \cong \mathrm{Hom}_{\mathbb{X}}(X, G(A))$, and a left representation,

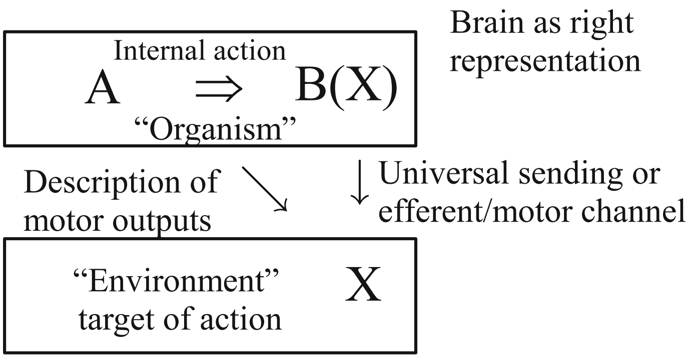


Fig. 8 Acting brain as a right representation

$\mathrm{Hom}_{\mathbb{A}}(F(X), A) \cong \mathrm{Het}(X, A)$. But instead of taking the same set of hets as being represented by two different functors on the right and left, suppose we consider a single functor $B(X)$ that represents the hets $\mathrm{Het}(X, A)$ on the left:

$$\mathrm{Het}(X, A) \cong \mathrm{Hom}_{\mathbb{A}}(B(X), A),$$

and represents the hets $\mathrm{Het}(A, X)$ [going in the opposite direction] on the right:

$$\mathrm{Hom}_{\mathbb{A}}(A, B(X)) \cong \mathrm{Het}(A, X).$$

If the hets each way between two categories are represented by the same functor $B(X)$ as left and right representations, then that functor is said to be a *brain functor*. Thus instead of a pair of functors being adjoint, we have a single functor $B(X)$ with values within one of the categories (the "organism") as representing the two-way interactions, "perception" and "action," between that category and another one (the "environment"). The use of the adjective "brain" is quite deliberate (as opposed to say "mind") since the universal hets going each way between the "organism" and "environment" are part of the definition of left and right representations. In particular, it should be noted how the "turn-around-the-arrows" category-theoretic duality provides a mathematical model for the type of "duality" between:

- sensory or afferent systems (brain furnishing the left representation of the environment to organism heteromorphisms), and
- motor or efferent systems (brain furnishing the right representation of the organism to environment heteromorphisms).

In view of this application, those two universal hets, representing the afferent and efferent nervous systems, might be denoted α_X and ε_X as in the following diagrams for the two representations (Fig. 9).

We have seen how the adjunctive square diagram for an adjunction can be obtained by gluing together the left and right representation diagrams at the common diagonal $\searrow^{f}$. The diagram for a brain functor is obtained by gluing together the diagrams for the left and right representations at the common values of

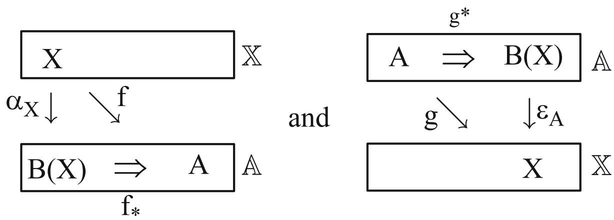
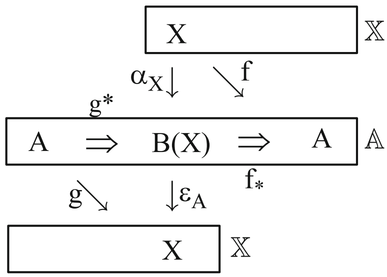


Fig. 9 Left and right representation diagrams for the brain functor $B : \mathbb{X} \to \mathbb{A}$

Fig. 10 Butterfly diagram combining two representations at the common $B(X)$



the brain functor $B(X)$. If we think of the diagram for a representation as right triangle, then the adjunctive square diagram is obtained by gluing two triangles together on the hypotenuses, and the diagram for the brain functor is obtained by gluing two triangles together at the right angle vertices to form the *butterfly diagram* (Fig. 10).

If both the triangular "wings" could be filled-out as adjunctive squares, then the brain functor would have left and right adjoints. Thus all functors with both left and right adjoints are brain functors (although not vice-versa). The previous example of the diagonal functor $\Delta : Sets \to Sets \times Sets$ is a brain functor since the product functor $\times(a, b) = a \times b$ is the right adjoint, and the coproduct or disjoint union functor $\biguplus(a, b) = a \biguplus b$ is the left adjoint. The underlying set functor (see "Appendix") that takes a group G to its underlying set $U(G)$ is a rather trivial example of a brain functor that does not arise from having both a left and right adjoint. It has a left adjoint (the free group functor) so U provides a right representation for the set-to-group maps or hets $X \to G$. Also it trivially provides a left representation for the hets $G \to X$ but has no right adjoint.

In the butterfly diagram below, we have labelled the diagram for the brain as the language faculty for understanding and producing speech (Fig. 11).

Wilhelm von Humboldt recognized the symmetry between the speaker and listener, which in the same person is abstractly represented as the dual functions of the "selfsame power" of the language faculty in the above butterfly diagram.

> Nothing can be present in the mind (Seele) that has not originated from one's own activity. Moreover understanding and speaking are but different effects of the selfsame power of speech. Speaking is never comparable to the

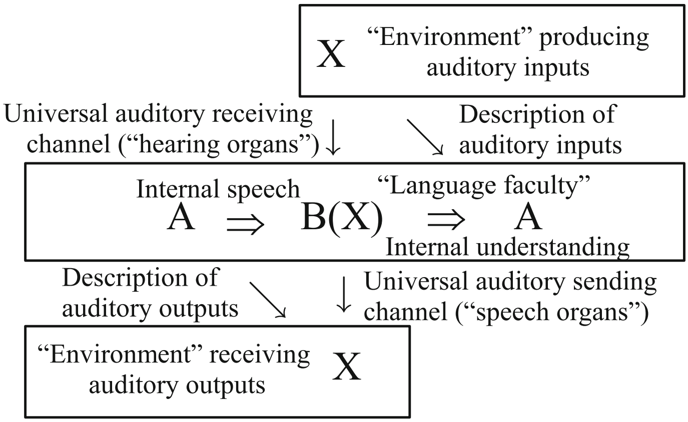


Fig. 11 Brain functor butterfly diagram interpreted as language faculty

> transmission of mere matter (Stoff). In the person comprehending as well as in the speaker, the subject matter must be developed by the individual's own innate power. What the listener receives is merely the harmonious vocal stimulus. (von Humboldt 1997, p. 102)

7 A Mathematical Example of a Brain Functor

A non-trivial mathematical example of a brain functor is provided by the functor taking a finite set of vector spaces $\{V_i\}_{i=1,\ldots,n}$ over the same field (or R-modules over a ring R) to the product $\prod_i V_i$ of the vector spaces. Such a product is also the coproduct $\sum_i V_i$ (Hungerford 1974, p. 173) and that space may be written as the biproduct:

$$V_1 \oplus \cdots \oplus V_n \cong \prod_i V_i \cong \sum_i V_i.$$

The het from a set of spaces $\{V_i\}$ to a single space V is a *cocone* of vector space maps $\{V_i \Rightarrow V\}$ and the canonical such het is the set of canonical injections $\{V_i \Rightarrow V_1 \oplus \cdots \oplus V_n\}$ (taking the "brain" as a coproduct) with the "brain" at the point of the cocone. This brain functor captures the *integrative* aspects of both perception and action.

The perception left representation then might be taken as conceptually representing the function of the brain as integrating multiple sensory inputs into an interpreted perception (Fig. 12).[6]

Dually, a het from single space V to a set of vector spaces $\{V_i\}$ is a *cone* $\{V \Rightarrow V_i\}$ with the single space V at the point of the cone, and the canonical het is the set of canonical projections (taking the "brain" as a product) with the "brain" as the point of the cone: $\{V_1 \oplus \cdots \oplus V_n \Rightarrow Vi\}$. The action right representation then

[6] The cocones and cones are represented in the diagrams using cone shapes.

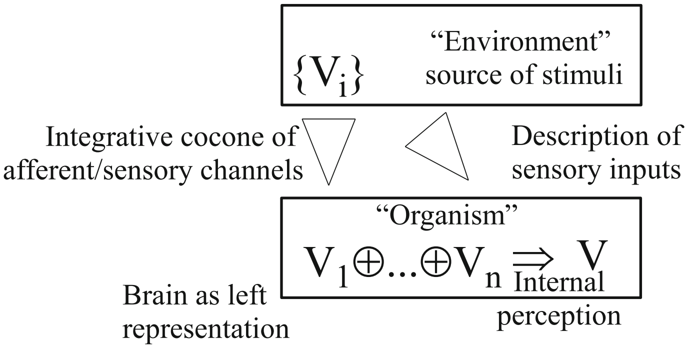


Fig. 12 Brain as integrating sensory inputs into a perception

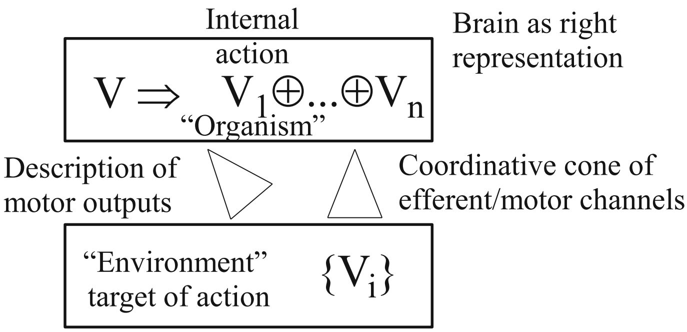


Fig. 13 Brain as coordinating motor outputs into an action

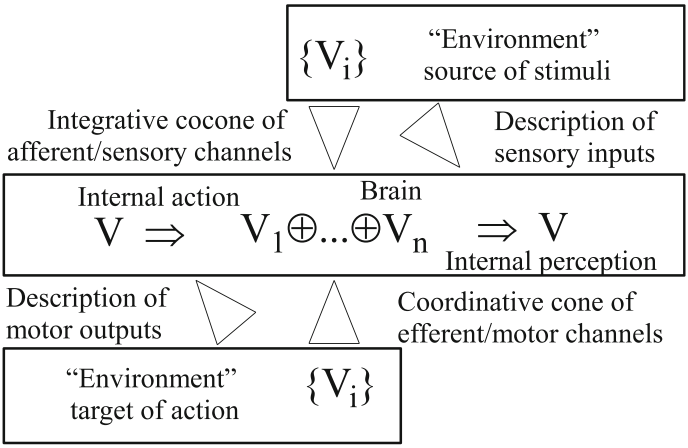


Fig. 14 Conceptual model of a perceiving and acting brain

might be taken as conceptually representing the function of the brain as integrating or coordinating multiple motor outputs in the performance of an action (Fig. 13).

Putting the two representations together gives the butterfly diagram for a brain (Fig. 14).

This gives a conceptual model of a single organ that integrates sensory inputs into a perception and coordinates motor outputs of an action, i.e., a brain.

8 Conclusion

In view of the success of category theory in modern mathematics, it is perfectly natural to try to apply it in the life and cognitive sciences. Many different approaches need to be tried to see which ones, if any, will find "where theory lives" (and will be something more than just applying biological names to bits of pure math). The approach developed here differs from other approaches in several ways, but the most basic difference is the use of heteromorphisms to represent interactions between quite different entities (i.e., objects in different categories).

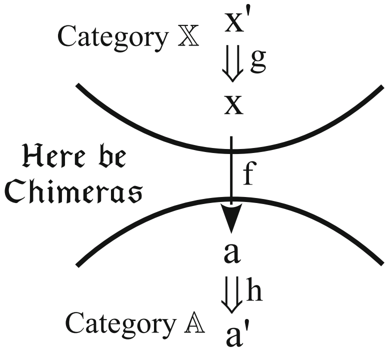
Heteromorphisms also provide the natural setting to formulate universal mapping problems and their solutions as left or right representations of hets. In spite of abounding in the wilds of mathematical practice, hets are not recognized in the orthodox presentations of category theory. One consequence is that the notion of an adjunction appears as one atomic concept that cannot be factored into separate parts. But that is only a artifact of the homs-only treatment. The heteromorphic treatment shows that an adjunction factors naturally into a left and right representation of the hets going from one category to another–where, in general, one representation might exist without the other.

One benefit of this heteromorphic factorization is that the two atomic concepts of left and right representations can then be recombined in a new way to form the cognate recombinant concept of a brain functor. The main conclusion of the paper is that this concept of a brain functor seems to fit very well as an abstract and conceptual but non-trivial description of the dual universal functions of a brain, perception (using the sensory or afferent systems) and action (using the motor or efferent systems).

Mathematical Appendix: Are Hets Really Necessary in Category Theory?

Since the concept of a brain functor requires hets for its formulation, it is important to consider the role of hets in category theory. The homomorphisms or homs between the objects of a category $\mathbb{X}$ are given by a hom bifunctor $\mathrm{Hom}_{\mathbb{X}} : \mathbb{X}^{op} \times \mathbb{X} \to Sets$. In the same manner, the heteromorphisms or hets from

Fig. 15 Composition of a het with a hom on either end



the objects of a category $\mathbb{X}$ to the objects of a category $\mathbb{A}$ are given by a het bifunctor $\text{Het} : \mathbb{X}^{op} \times \mathbb{A} \to Sets$.[7]

The Het-bifunctor gives the rigorous way to handle the composition of a het $f : x \to a$ in $\text{Het}(x, a)$ [thin arrows $\to$ for hets] with a homomorphism or hom $g : x' \Longrightarrow x$ in X [thick Arrows $\Longrightarrow$ for homs] and a hom $h : a \Longrightarrow a'$ in A. For instance, the composition $x' \overset{g}{\Longrightarrow} x \overset{f}{\to} a$ is the het that is the image of f under the map: $\text{Het}(g, a) : \text{Het}(x, a) \to \text{Het}(x', a)$. Similarly, the composition $x \overset{f}{\to} a \overset{h}{\Longrightarrow} a'$ is the het that is the image of f under the map: $\text{Het}(x, h) : \text{Het}(x, a) \to \text{Het}(x, a')$ (Fig. 15).[8]

This is all perfectly analogous to the use of Hom-functors to define the composition of homs. Since both homs and hets (e.g., injection of generators into a group) are common morphisms used in mathematical practice, both types of bifunctors formalize standard mathematical machinery.

Chimeras in the Wilds of Mathematical Practice

The homs-only orientation may go back to the original conception of category theory "as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings." (Eilenberg and MacLane 1945, p. 237) While chimeras do not appear in the orthodox "ontological zoo" of category theory, they abound in the wilds of mathematical practice. In spite of the reference to "Working Mathematician" in the title of MacLane's text (MacLane 1971), one might seriously doubt that any working mathematician would give, say, the universal mapping property of free groups using the "device" of the underlying set functor U instead of the traditional description given in the left representation diagram (which does not even mention

[7] Although often with a somewhat different interpretation, the *Sets*-valued *profunctors* (Kelly 1982), *distributors* (Benabou 1973), or *correspondences* (Lurie 2009, p. 96) are formally the same as het bifunctors.

[8] The definition of a bifunctor also insures the associativity of composition so that schematically: $\text{hom} \circ (\text{het} \circ \text{hom}) = (\text{hom} \circ \text{het}) \circ \text{hom}$.

U) as can be seen in most any non-category-theoretic text that treats free groups. For instance, consider the following description in Nathan Jacobson's text (Jacobson 1985, p. 69).

> To summarize: given the set $X = \{x_1, \ldots, x_r\}$ we have obtained a map $x_i \rightarrow \bar{x}_i$ of X into a group $FG^{(r)}$ such that if G is any group and $x_i \rightarrow a_i$, $1 \leq i \leq r$ is any map of X into G then we have a unique homomorphism of $FG^{(r)}$ into G, making the following diagram commutative:

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & . \\ FG^{(r)} & \Longrightarrow & G \end{array}$$

In Jacobson's diagram, only the $FG^{(r)} \Longrightarrow G$ morphism is a group homomorphism; the vertical and diagonal arrows are called "maps" and are set-to-group hets so it is the diagram for a left representation.[9]

Hets as "Homs" in a Collage Category

The notion of a homomorphism is so general that hets can always be recast as "homs" in a larger category variously called a *directly connected category* (Pareigis 1970, p. 58) (since Pareigis calls the het bifunctor a "connection"), a *cograph* category (Shulman 2011), or, more colloquially, a *collage* category (since it combines quite different types of objects and morphisms into one category in total disregard of any connection to the Erlangen Program). The *collage category* of a het bifunctor $\mathrm{Het} : \mathbb{X}^{op} \times \mathbb{A} \rightarrow Sets$, denoted $\mathbb{X} \bigstar^{\mathrm{Het}} \mathbb{A}$ (Lurie 2009, p. 96), has as objects the disjoint union of the objects of $\mathbb{X}$ and $\mathbb{A}$. The *homs* of the collage category are defined differently according to the two types of objects. For x and x' objects in $\mathbb{X}$, the homs $x \Rightarrow x'$ are the elements of $\mathrm{Hom}_{\mathbb{X}}(x, x')$, the hom bifunctor for $\mathbb{X}$, and similarly for objects a and a' in $\mathbb{A}$, the homs $a \Rightarrow a'$ are the elements of $\mathrm{Hom}_{\mathbb{A}}(a, a')$. For the different types of objects such as x from $\mathbb{X}$ and a from $\mathbb{A}$, the "homs" $x \Rightarrow a$ are the elements of $\mathrm{Het}(x, a)$ and there are no homs $a \Rightarrow x$ in the other direction in the collage category.

Does the collage category construction show that "hets" are unnecessary in category theory and that homs suffice? Since all the information given in the het bifunctor has been repackaged in the collage category, any use of hets can always be repackaged as a use of the "$\mathbb{X}$-to-$\mathbb{A}$ homs" in the collage category $\mathbb{X} \bigstar^{\mathrm{Het}} \mathbb{A}$. In any application, like the previous example of the universal mapping property (UMP) of the free-group functor as a left representation, one must distinguish between the two types of objects and the three types of "homs" in the collage category.

Suppose in Jacobson's example, one wanted to "avoid" having the different "maps" and group homomorphisms by formulating the left representation in the

[9] We modified Jacobson's diagram according to our het-hom convention for the arrows. Similar examples of hets can be found in the MacLane–Birkhoff's text (MacLane and Birkhoff 1988).

collage category formed from the category of *Sets*, the category of groups *Grps*, and the het bifunctor, $\text{Het} : Sets^{op} \times Grps \rightarrow Sets$, for set-to-group maps. Since the UMP does not hold for arbitrary objects and homs in the collage category, $Sets \bigstar^{\text{Het}} Grps$, one would have to differentiate between the "set-type objects" X and the "group-type objects" G as well as between the "mixed-type homs" in $\text{Hom}(X, G)$ and the "pure-type homs" in $\text{Hom}\left(FG^{(r)}, G\right)$. Then the left representation UMP of the free-group functor could be formulated in the het-free collage category $Sets \bigstar^{\text{Het}} Grps$ as follows.

> For every set-type object X, there is a group-type object $F(X)$ and a mixed-type hom $\eta_X : X \Rightarrow F(X)$ such that for any mixed-type hom $f : X \Rightarrow G$ from the set-type object X to any group-type object G, there is a unique pure-type hom $f_* : F(X) \Rightarrow G$ such that $f = f_* \eta_X$.

Thus the answer to the question "Are hets really necessary?" is "No!"–since one can always use sufficient circumlocutions with the *different* types of "homs" in a collage category. Jokes aside, the collage category formulation is essentially only a reformulation of the left representation UMP using clumsy circumlocutions. Working mathematicians use phrases like "mappings" or "morphisms" to refer to hets in contrast to homomorphisms–and "mixed-type homs" does not seem to be improved phraseology for hets.

There is, however, a more substantive point, i.e., the general UMPs of left or right representations show that the hets between objects of different categories can be represented by homs *within* the codomain category or *within* the domain category, respectively. If one conflates the hets and homs in a collage category, then the point of the representation is rather obscured (since it is then one set of "homs" in a collage category being represented by another set of homs in the same category).

What About the Homs-Only UMPs in Adjunctions?

There is another het-avoidance device afoot in the homs-only treatment of adjunctions. For instance, the left-representation UMP of the free-group functor can, for each $X \in Sets$, be formulated as the natural isomorphism: $\text{Hom}_{Grps}(F(X), G) \cong \text{Het}(X, G)$. But if we fix G and use the underlying set functor $U : Grps \rightarrow Sets$, then there is trivially the right representation: $\text{Het}(X, G) \cong \text{Hom}_{Sets}(X, U(G))$. Putting the two representations together, we have the heteromorphic treatment of an adjunction first formulated by Pareigis (1970):

$$\text{Hom}_{Grps}(F(X), G) \cong \text{Het}(X, G) \cong \text{Hom}_{Sets}(X, U(G)).$$

If we delete the het middle term, then we have the usual homs-only formulation of the free-group adjunction,

$$\text{Hom}_{Grps}(F(X), G) \cong \text{Hom}_{Sets}(X, U(G)),$$

without any mention of hets. Moreover, the het-avoidance device of the underlying

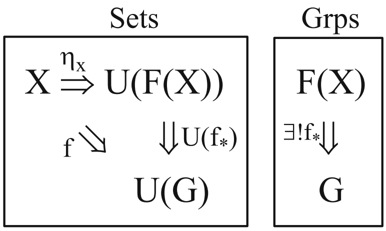


Fig. 16 Over-and-back diagram for free group adjunction

set functor U allows the UMP of the free group functor to be reformulated with sufficient circumlocutions to avoid mentioning hets.

> For each set X, there is a group $F(X)$ and a set hom $\eta_X : X \Rightarrow U(F(X))$ such that for any set hom $f : X \Rightarrow U(G)$ from the set X to the underlying set $U(G)$ of any group G, there is a unique group hom $f_* : F(X) \Rightarrow G$ over in the other category such that the set hom image $U(f_*)$ of the group hom f_* back in the original category satisfies $f = U(f_*)\eta_X$(Fig. 16).[10]

Such het-avoidance circumlocutions have no structural significance since there is a general adjunction representation theorem (Ellerman 2006, p. 147) that *all* adjoints can be represented, up to isomorphism, as arising from the left and right representations of a het bifunctor.

Are all UMPs Part of Adjunctions?

Even though the homs-only formulation of an adjunction only ignores the underlying hets (due to the adjunction representation theorem), is that formulation sufficient to give all UMPs? Or are there important universal constructions that are not either left or right adjoints?

Probably the most important example is the tensor product. The universal mapping property of the tensor product is particularly interesting since it is a case where the heteromorphic treatment of the UMP is forced (under one disguise or another). The tensor product functor $\otimes : \langle A, B\rangle \longmapsto A \otimes B$ is *not* a left adjoint so the usual device of using the other functor (e.g., a forgetful or diagonal functor) to avoid mentioning hets is not available.

For A, B, C modules (over some commutative ring R), one category is the product category $Mod_R \times Mod_R$ where the objects are ordered pairs $\langle A, B\rangle$ of R-modules and the other category is just the category Mod_R of R-modules. The values of the Het-bifunctor $\mathrm{Het}(\langle A, B\rangle, C)$ are the bilinear functions $A \times B \rightarrow C$. Then the tensor product functor $\otimes : Mod_R \times Mod_R \rightarrow Mod_R$ given by $\langle A, B\rangle \longmapsto A \otimes B$ gives a left representation:

$$\mathrm{Hom}_{Mod_R}(A \otimes B, C) \cong \mathrm{Het}(\langle A, B\rangle, C)$$

that characterizes the tensor product. The canonical het $\eta_{\langle A,B\rangle} : A \times B \rightarrow A \otimes B$ is

[10] Even the "over-and-back" formulation using two different categories could be avoided by using the further circumlocutions of the only pure-type homs in the single collage category.

the image under the left-representation isomorphism of the identity hom $1_{A \otimes B}$ obtained by taking $C = A \otimes B$, so we have:

$$\begin{array}{ccc} \langle A, B \rangle & & \\ \eta_{\langle A,B \rangle} \downarrow & \searrow^{f} & \\ A \otimes B & \underset{\exists! \mathrm{f}_*}{\Longrightarrow} & C \end{array}$$

Left representation diagram to characterize tensor products

where the single arrows are the bilinear hets and the thick Arrow is a module homomorphism within the category Mod_R.

For instance, in MacLane and Birkhoff's *Algebra* textbook (MacLane and Birkhoff 1988), they explicitly use hets (bilinear functions) starting with the special case of an R-module A (for a commutative ring R) and then stating the universal mapping property of the tensor product $A \otimes R \cong A$ using the left representation diagram (MacLane and Birkhoff 1988, p. 318)–like any other working mathematicians. For any R-module A, there is an R-module $A \otimes R$ and a canonical bilinear het $h_0 : A \times R \to A \otimes R$ such that given any bilinear het $h : A \times R \to C$ to an R-module C, there is a unique R-module hom $t : A \otimes R \Longrightarrow C$ such that the following diagram commutes.

$$\begin{array}{ccc} A \times R & & \\ h_0 \downarrow & \searrow^{h} & \\ A \otimes R & \overset{\exists! t}{\Longrightarrow} & C \end{array}$$

Left representation diagram of special case of tensor product.

References

Awodey S (2006) Category theory. Clarendon Press, Oxford

Benabou J (1973) Les distributeurs, vol 33, Institut de Mathmatique Pure et Applique

Boolos G (1971) The iterative conception of set. J Philos 68(April 22):215–231

Ehresmann AC, Vanbremeersch JP (2007) Memory evolutive systems: hierarchy, emergence, cognition. Elsevier, Amsterdam

Eilenberg S, MacLane S (1945) General theory of natural equivalences. Trans Am Math Soc 58(2):231–294

Ellerman D (1988) Category theory and concrete universals. Erkenntnis 28:409–429

Ellerman D (2006) A theory of adjoint functors with some thoughts on their philosophical significance. In: Sica G (ed) What is category theory?. Polimetrica, Milan, pp 127–183

Ellerman D (2007) Adjoints and emergence: applications of a new theory of adjoint functors. Axiomathes 17(1 March):19–39

Goldblatt R (2006) Topoi: the categorical analysis of logic (revised ed.). Dover, Mineola

Halford GS, Wilson WH (1980) A category theory approach to cognitive development. Cogn Psychol 12(3):356–411

Hungerford TW (1974) Algebra. Springer, New York

Jacobson N (1985) Basic algebra I, 2nd edn. W.H. Freeman, New York

Kainen PC (2009) On the Ehresmann–Vanbremeersch theory and mathematical biology. Axiomathes 19:225–244

Kan D (1958) Adjoint functors. Trans Am Math Soc 87(2):294–329

Kelly M (1982) Basic concepts of enriched category theory. Cambridge University Press, Cambridge
Lambek J (1981) The influence of Heraclitus on modern mathematics. In: Agassi J, Cohen RS (eds) Scientific philosophy today: essays in honor of Mario Bunge. D. Reidel, Boston, pp 111–121
Lawvere FW (1969) Adjointness in foundations. Dialectica 23:281–295
Louie AH (1985) Categorical system theory. In: Rosen R (ed) Theoretical biology and complexity: three essays on the natural philosophy of complex systems. Academic Press, Orlando, pp 68–163
Louie AH, Poli R (2011) The spread of hierarchical cycles. Int J Gen Syst 40(3 April):237–261
Lurie J (2009) Higher topos theory. Princeton University Press, Princeton
MacLane S (1948) Groups, categories, and duality. Proc Nat Acad Sci USA 34(6):263–267
MacLane S (1971) Categories for the working mathematician. Springer, New York
MacLane S, Birkhoff G (1988) Algebra, 3rd edn. Chelsea, New York
Magnan F, Reyes GE (1994) Category theory as a conceptual tool in the study of cognition. In: Macnamara J, Reyes GE (eds) The logical foundations of cognition. Oxford University Press, New York, pp 57–90
Makkai M (1999) Structuralism in mathematics. In: Jackendoff R, Bloom P, Wynn K (eds) Language, logic, and concepts: essays in memory of John Macnamara. MIT Press (A Bradford Book), Cambridge, pp 43–66
Pareigis B (1970) Categories and functors. Academic Press, New York
Philips S (2014) Analogy, cognitive architecture and universal construction: a tale of two systematicities. PLOS One 9(2):1–9
Philips S, Wilson WH (2014) Chapter 9: a category theory explanation for systematicity: universal constructions. In: Calvo P, Symons J (eds) Systematicity and cognitive architecture. MIT Press, Cambridge, pp 227–249
Rosen R (1958) The representation of biological systems from the standpoint of the theory of categories. Bull Math Biophys 20(4):317–342
Rosen R (2012) Anticipatory systems: philosophical, mathematical, and methodological foundations, 2nd edn. Springer, New York
Russell B (2010) Principles of mathematics. Routledge Classics, London
Samuel P (1948) On universal mappings and free topological groups. Bull Am Math Soc 54(6):591–598
Shulman M (2011) Cograph of a profunctor
Taylor P (1999) Practical foundations of mathematics. Cambridge University Press, Cambridge
von Humboldt W (1997) The nature and conformation of language. In: Mueller-Vollmer K (ed) The hermeneutics reader. Continuum, New York, pp 99–105
Wood RJ (2004) Ordered sets via adjunctions. In: Pedicchio MC, Tholen W (eds) Categorical foundations. Encyclopedia of mathematics and its applications, vol 97. Cambridge University Press, Cambridge, pp 5–47
Zafiris E (2012) Rosen's modelling relations via categorical adjunctions. Int J Gen Syst 41(5):439–474