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Anisotropic (p, q)-equations with superlinear reaction

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Abstract

In this paper, we consider a Dirichlet problem driven by the anisotropic (p, q)-Laplacian and a superlinear reaction which need not satisfy the Ambrosetti–Robinowitz condition. By using variational tools together with truncation and comparison techniques and critical groups, we show the existence of at least five nontrivial smooth solutions, all with sign information: two positive, two negative and a nodal (sign-changing).

Keywords Anisotropic regularity · Extremal constant sign solutions · Nodal solution · Critical point theory · Critical group

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1 Introduction

In this paper we study the following anisotropic (p, q)-equation

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = f(z, u(z)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. Given $r \in C(\bar{\Omega})$ with $r_- = \min_{\bar{\Omega}} r > 1$, by $\Delta_{r(z)}$ we denote the $r(z)$-Laplace differential operator with variable exponent $r(\cdot)$ defined by

$$\Delta_{r(z)}u = \operatorname{div}(|Du|^{r(z)-2}Du) \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

This operator, in contrast to the isotropic r-Laplacian, is not homogeneous and this is a source of difficulties in the study of anisotropic equations. In problem (1), we have the sum of two such operators with different exponents. So, even in the isotropic case (that is, $p(\cdot)$ and $q(\cdot)$ are constants), the differential operator is not homogeneous as well. The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable in Ω, and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). Moreover, we assume that for a.a. $z \in \Omega$, $f(z, \cdot)$ is $(p_+ - 1)$-superlinear (here $p_+ = \max_{\bar{\Omega}} p$), but need not satisfy the Ambrosetti–Robinowitz condition (the AR-condition, for short), which is common in the literature when studying "superlinear" problems. In the present work, instead, we employ a less restrictive condition, which incorporates in our framework also nonlinearities with "slower" growth near $\pm\infty$, and which fail to satisfy the AR-condition.

Using variational tools from the critical point theory together with truncation and comparison techniques as well as critical groups, we prove a multiplicity theorem for problem (1), producing five nontrivial smooth solutions all with sign information, namely, two positive solutions, two negative solutions and a nodal (sign-changing) solution. Our work here extends those of Gasiński and Papageorgiou [7] and of Tan and Fang [20]. In both these works, the authors considered equations driven by the anisotropic p-Laplacian and with a superlinear reaction which need not satisfy the AR-condition. They proved multiplicity theorems producing three nontrivial solutions, but do not obtain nodal solutions (see Theorem 4.4 of [7] and Theorems 1.2 and 1.3 of [20]). Moreover, in [20] the hypotheses on the reaction $f(z, x)$ are more restrictive.

The study of variational problems and partial differential equations with nonstandard growth conditions, was motivated by various applications. There are materials whose study requires such a more general theory. Electrorheological fluids are the fluids whose viscosity depends on the electric field in the fluid (for example, lithium polymetachrylate). The viscosity is inversely proportional to the strength of the electric field. The study of such fluids requires the use of the equations with nonstandard growth conditions. In the context of continuum mechanics, these fluids are treated as non-Newtonian fluids and have been used in robotics and space technology. For details, we refer to the book of Ruzicka [18]. On the other hand, boundary value problems involving a combination of several differential operators of different nature (such as (p, q)-equations), arise in many mathematical models of physical processes. We mention the works of Benct et al. [2] (models of elementary particles and soliton-type solutions), Cherfils and Ilyasov [3] (stationary reaction-diffusion systems), Zhikov [23] (problems in nonlinear elasticity theory). Recently such problems were examined with variable exponents. We refer to the works of Papageorgiou et al. [9], Papageorgiou et al. [13], Papageorgiou and Vetro [14], Rădulescu [15], Rădulescu and Repovš [16], Ragusa and Tachikawa [17], Vetro [21], Vetro and Vetro [22].

2 Mathematical background and hypotheses

The analysis of problem (1) requires the use of Lebesgue and Sobolev spaces with variable exponents. These are particular instance of Musielak-Orlicz spaces and a comprehensive treatment of these spaces can be found in the books of and Cruz-Uribe and Fiorenza [4] and Diening et al. [5].

Let $L^0(\Omega)$ be the vector space of all measurable functions $u\colon \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a Lebesgue null set. Also, let $E_1 = \{r \in C(\bar{\Omega}) \mid 1 < r_- = \min_{\bar{\Omega}} r\}$ (in the sequel for any $r \in C(\bar{\Omega})$, we set $r_- = \min_{\bar{\Omega}}$ and $r_+ = \max_{\bar{\Omega}} r$). Then given $r \in E_1$, we define the anisotropic Lebesgue space $L^{r(z)}(\Omega)$ by

$$L^{r(z)}(\Omega) := \left\{ u \in L^0(\Omega) \mid \int_\Omega |u(z)|^{r(z)} dz < \infty \right\}.$$

This space is equipped with the so-called "Luxemburg norm" defined by

$$\|u\|_{r(z)} = \inf \left[\lambda > 0 \mid \int_\Omega \left(\frac{|u(z)|}{\lambda} \right)^{r(z)} dz \le 1 \right].$$

With this norm $L^{r(z)}(\Omega)$ is a Banach space, which is separable and reflexive (in fact uniformly convex). If $r' \in E_1$ is defined by $r'(z) = \frac{r(z)}{r(z)-1}$ for all $z \in \bar{\Omega}$ (that is, $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$ for all $z \in \bar{\Omega}$), then we have

$$L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega).$$

Moreover, the following variant of Hölder's inequality holds,

$$\int_\Omega |uv| dz \le \left[\frac{1}{r_-} + \frac{1}{r'_-} \right] \|u\|_{r(z)} \|v\|_{r'(z)}$$

for all $u \in L^{r(z)}(\Omega)$ and all $v \in L^{r'(z)}(\Omega)$. Also, if $r, r_0 \in E_1$ and $r(z) \le r_0(z)$ for all $z \in \bar{\Omega}$, then the embeddings

$$L^{r_0(z)}(\Omega) \hookrightarrow L^{r(z)}(\Omega) \hookrightarrow L^1(\Omega) \text{ are all continuous.}$$

In addition to the norm $\|\cdot\|_{r(z)}$, we consider also the modular function $\rho_r : L^{r(z)}(\Omega) \to \mathbb{R}_+ := [0, +\infty)$ defined by

$$\rho_r(u) := \int_\Omega |u(z)|^{r(z)}\, dz \text{ for all } u \in L^{r(z)}(\Omega).$$

The Luxemburg norm and this modular function are closely related.

Proposition 1 *If $r \in E_1$, $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$ and $u \in L^{r(z)}(\Omega)$, then we have*

(a) $\|u\|_{r(z)} = \lambda \Leftrightarrow \rho_r(\frac{u}{\lambda}) = 1$;
(b) $\|u\|_{r(z)} < 1 \Rightarrow$ *(resp.* $= 1$ *and* > 1) $\Leftrightarrow \rho_r(u) < 1$(*resp.* $= 1$, *and* > 1);
(c) $\|u\|_{r(z)} \leq 1 \Rightarrow \|u\|_{r(z)}^{r+} \leq \rho_r(u) \leq ||u||_{r(z)}^{r-}$, *and* $\|u\|_{r(z)} \geq 1 \Rightarrow \|u\|_{r(z)}^{r-} \leq \rho_r(u) \leq \|u\|_{r(z)}^{r+}$;
(d) $\|u_n\|_{r(z)} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho_r(u_n) \to 0$(*resp.* $\to +\infty$);
(e) $\|u_n - u\|_{r(z)} \to 0 \Leftrightarrow \rho_r(u_n - u) \to 0$.

Using the variable exponent Lebesgue spaces, we can define the variable exponent Sobolev spaces. So, let $r \in E_1$. We define

$$W^{1,r(z)}(\Omega) := \left\{u \in L^{r(z)}(\Omega) \mid |Du| \in L^{r(z)}(\Omega)\right\}.$$

We equip this space with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)} \text{ for all} u \in W^{1,r(z)}(\Omega),$$

where $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$. Suppose that $r \in E_1 \cap C^{0,1}(\bar{\Omega})$ (that is $r \in E_1$ is Lipschitz continuous). We define

$$W_0^{1,r(z)}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

The spaces $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are both Banach spaces which are separable and reflexive (in fact uniformly convex). For the space $W_0^{1,r(z)}(\Omega)$, we know that the Poincaré inequality holds, namely,

$$\|u\|_{r(z)} \leq \hat{c}\|Du\|_{r(z)} \text{ for all } u \in W_0^{1,r(z)}(\Omega)$$

for some $\hat{c} > 0$.

Given $r \in E_1 \cap C^{0,1}(\bar{\Omega})$, we consider $r^*(\cdot)$ the critical Sobolev exponent corresponding to $r(\cdot)$, defined by

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if } r(z) < N \\ +\infty & \text{if } N \leq r(z) \end{cases} \text{ for all } z \in \bar{\Omega}.$$

Consider $q \in C(\bar{\Omega})$ and assume that $1 \leq q_- \leq q(z) \leq r^*(z)$ (resp. $1 \leq q_- \leq q(z) \leq q_+ < r^*(z)$) for all $z \in \bar{\Omega}$. Set $X = W^{1,r(z)}(\Omega)$ or $X = W_0^{1,r(z)}(\Omega)$. Then we have

$$X \hookrightarrow L^{q(z)}(\Omega) \text{ continuously (resp.} X \hookrightarrow L^{q(z)}(\Omega) \text{ compactly).}$$

This is the so-called "anisotropic Sobolev embedding theorem". Moreover, if $r \in E_1 \cap C^{0,1}(\bar{\Omega})$, then

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

Let us introduce the nonlinear operator $A_{r(z)}\colon W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ defined by

$$\langle A_{r(z)}(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in W_0^{1,r(z)}(\Omega).$$

The next proposition summarizes the main properties of this operator (see Gasiński and Parpagerogiou [7] (Proposition 2.5) and Rădulescu and Repovš [16] (p.40)).

Proposition 2 *The operator $A_{r(z)}(\cdot)$ is bounded (that is, it maps bounded sets in $W_0^{1,r(z)}(\Omega)$ to bounded sets in $W^{-1,r'(z)}(\Omega)$), continuous, strictly monotone (thus, maximal monotone too) and type $(S)_+$, i.e.,*

$$\text{if } u_n \xrightarrow{w} u \text{ in } W_0^{1,r(z)}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A_{r(z)}(u_n), u_n - u\rangle \leq 0,$$
$$\text{then } u_n \to u \text{ in } W_0^{1,r(z)}(\Omega).$$

On account of the anisotropic regularity theory, we will also use the Banach space

$$C_0^1(\bar{\Omega}) := \left\{u \in C^1(\bar{\Omega}) \mid u|_{\partial\Omega} = 0\right\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ := \left\{u \in C_0^1(\bar{\Omega}) \mid u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ := \left\{u \in C_+ \mid u(z) > 0 \text{ for all } z \in \Omega \text{ with } \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Suppose $u, v\colon \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \leq v(z)$ for a.a. $z \in \Omega$. We define the following order interval in $W_0^{1,r(z)}(\Omega)$,

$$[u, v] := \left\{h \in W_0^{1,r(z)}(\Omega) \mid u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\right\}.$$

By $\mathrm{int}_{C_0^1(\bar{\Omega})}[u, v]$ we denote the interior in $C_0^1(\bar{\Omega})$ of $[u, v] \cap C_0^1(\bar{\Omega})$. For every measurable function $u : \Omega \to \mathbb{R}$, we define $u^{\pm}(z) = \max\{\pm u(z), 0\}$ for all $z \in \Omega$. We have $u = u^+ - u^-$, $|u| = u^+ + u^-$, and if $u \in W_0^{1,r(z)}(\Omega)$, then $u^{\pm} \in W_0^{1,r(z)}(\Omega)$.

Let X be a Banach space and $\varphi \in C^1(X)$. We set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi).$$

Besides, we say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

- if $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, and

$$(1 + \|u_n\|_X)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

  then $\{u_n\}_{n\in\mathbb{N}}$ admits a strongly convergent subsequence.

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Let $\hat{u} \in K_\varphi$ be isolated and $c = \varphi(\hat{u})$. Then, the critical groups of φ at $\hat{u}$ are defined by

$$C_K(\varphi, \hat{u}) := H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{\hat{u}\}) \text{ for all } k \in \mathbb{N}_0,$$

with $\varphi^c = \{u \in X \mid \varphi(u) \le c\}$ and U a neighborhood of $\hat{u}$ such that $K_\varphi \cap \varphi^c \cap U = \{\hat{u}\}$. The excision property of singular homology, implies that this definition of critical groups is independent of the choice of the isolating neighborhood U.

We end the section by introducing the hypotheses of the data of problem (1).

H_0: $p, q \in C^{0,1}(\bar{\Omega})$ are such that $1 < q_- \le q_+ < p_- \le p_+ < +\infty$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \le \alpha(z)\left[1 + |x|^{r(z)-1}\right]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)$, $r \in E_1$ and $p_+ < r(z) < p^*(z)$ for all $z \in \bar{\Omega}$;

(ii) there exist $a, c > 0$ such that

$$f(z, a) \le -\hat{\lambda} < 0 < \lambda_0 \le f(z, -c) \text{ for a.a. } z \in \Omega;$$

(iii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\lim_{x\to\pm\infty} \frac{F(z, x)}{|x|^{p_+}} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exists $\tau \in C(\bar{\Omega})$ such that

$$\tau(z) \in \left((r_+ - p_-) \max\left\{\frac{N}{p_-}, 1\right\}, p^*(z)\right) \text{ for all } z \in \bar{\Omega},$$

and

$$0 < \beta_0 \leq \liminf_{x \to \pm\infty} \frac{f(z,x)x - p(z)F(z,x)}{|x|^{\tau(z)}} \text{ uniformly for a.a. } z \in \Omega;$$

(v) there exist $\theta, \mu \in E_1$ and $\delta \in (0, \min\{a, c\})$ such that

$$\begin{aligned} &1 < \theta(z) \leq \mu(z) < q_- \text{ for all } z \in \bar{\Omega}, \\ &c_0|x|^{\mu(z)} \leq f(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta, \\ &\limsup_{x \to 0} \frac{f(z,x)}{|x|^{\theta(z)-2}x} \leq c_* \text{uniformly for a.a. } z \in \Omega; \end{aligned}$$

(vi) there exists $\hat{\varepsilon} > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z,x) + \hat{\varepsilon}|x|^{p(z)-2}x$$

is nondecreasing on $[-m_0, m_0]$ with $m_0 = \max\{a, c\}$.

Remark 3 Hypotheses H_1(ii)–(iii) imply that for a.a. $z \in \Omega$, $f(z, \cdot)$ is $(p_+ - 1)$-superlinear, but without satisfying the AR-condition (see Ambrosetti–Rabinowitz [1]). For example the function

$$f(z,x) = \begin{cases} |x|^{\mu(z)-2}x - 2|x|^{\theta(z)-2} & \text{if } |x| \leq 1 \\ |x|^{p_+-2}x \ln|x| - |x|^{p(z)-2} & \text{if } 1 < |x| \end{cases},$$

satisfies hypotheses H_1 but fails to satisfy the AR-condition. Hypotheses H_1(ii) and (iv) dictate an oscillatory behavior near zero for $f(z, \cdot)$.

In what follows by $\|\cdot\|$ we denote the norm of the anisotropic Sobolev space $W_0^{1,p(z)}(\Omega)$. On account of the Poincaré inequality $\|u\| = \|Du\|_{p(z)}$ for all $u \in W_0^{1,p(z)}(\Omega)$.

3 Constant sign solutions

In this section, we search for constant sign solutions. We show the existence of four such smooth solutions (two positive and two negative). We also obtain the existence of extremal constant sign solutions, that is, we prove the existence of a smallest positive solution and the existence of a biggest negative solution. These extremal constant sign solutions will be used in Sect. 4, to generate a nodal solution.

For the purpose we introduce the energy (Euler) functional $\varphi \colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ for problem (1) defined by

$$\varphi(u) := \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega F(z,u)\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$. It is not difficult to see that $\varphi \in C^1(W_0^{1,p(z)}(\Omega))$ (see, for example, [16]). Also, in order to produce constant sign solutions, we introduce the positive and negative truncations of $\varphi(\cdot)$, respectively. More precisely, we consider the C^1- functionals $\varphi_\pm\colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) := \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega F(z, \pm u^\pm)\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

First we produce two positive solutions.

Proposition 4 *If hypotheses H_0 and H_1 hold, then problem (1) has at least two positive solutions,*

$$u_0, \hat{u} \in \mathrm{int}C_+ \text{ and } u_0 \neq \hat{u}.$$

Proof We introduce the Carathéodory function $\hat{f}_+\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}_+(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq a \\ f(z,a) & \text{if } a < x. \end{cases} \tag{2}$$

Set $\widehat{F}_+(z,x) := \int_0^x \hat{f}_+(z,s)\,ds$ and consider the C^1-functional $\hat{\psi}_+\colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) := \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega \widehat{F}_+(z,u)\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$. From (2), it is clear that $\hat{\psi}_+(\cdot)$ is coercive. Also, using the anisotropic Sobolev embedding theorem (see Sect. 2), we see that $\hat{\psi}_+(\cdot)$ is sequentially weakly lower semicontinuous. Hence, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\psi}_+(u_0) = \min\left\{\hat{\psi}_+(u) \;\middle|\; u \in W_0^{1,p(z)}(\Omega)\right\}. \tag{3}$$

Let $u \in \mathrm{int}C_+$ and $\delta > 0$ be as postulated by hypotheses H_1(v). We choose $t \in (0,1)$ small such that $0 \leq tu(z) \leq \delta$ for all $z \in \bar{\Omega}$. Because of $t \in (0,1)$ and $q_- < p_+$, then, we have

$$\hat{\psi}_+(tu) \leq \frac{t^{p_-}}{p_-}\rho_p(Du) + \frac{t^{q_-}}{q_-}\rho_q(Du) - \frac{t^{\mu_+}}{\mu_+}c_0\rho_\mu(u) \leq \widetilde{c}_0 t^{p_-} + c_1 t^{q_-} - c_2 t^{\mu_+}$$

for some $\widetilde{c}_0, c_1, c_2 > 0$. By hypothesis H_1(v), we have $\mu_+ < q_- < p_-$. So choosing $t \in (0,1)$ even smaller if necessary, we see that

$$\hat{\psi}_+(tu) < 0 \quad \Rightarrow \quad \hat{\psi}_+(u_0) < 0 = \hat{\psi}_+(0) \text{ (see (3))} \quad \Rightarrow \quad u_0 \neq 0.$$

From (3), we have $\hat{\psi}'_+(u_0) = 0$, i.e.,

$$\langle A_{p(z)}(u_0), h\rangle + \langle A_{q(z)}(u_0), h\rangle = \int_\Omega \hat{f}_+(z, u_0)h\, dz \text{ for all } h \in W_0^{1,p(z)}(\Omega). \quad (4)$$

In (4) first we choose $h = -u_0^- \in W_0^{1,p(z)}(\Omega)$. From (2), Proposition 1 and the assumption $f(z, 0) = 0$ for a.a. $z \in \Omega$, we have

$$\rho_p(Du_0^-) + \rho_q(Du_0^-) = 0 \quad \Rightarrow \quad u_0 \geq 0, u_0 \neq 0 \text{ (see Proposition 1)}.$$

Next in (4) we use the test function $h = [u_0 - a]^+ \in W_0^{1,p(z)}(\Omega)$. Then, we have

$$\langle A_{p(z)}(u_0), (u_0 - a)^+\rangle + \langle A_{q(z)}(u_0), (u_0 - a)^+\rangle = \int_\Omega f(z, a)(u_0 - a)^+\, dz \leq 0,$$

so, it holds $u_0 \leq a$, where we have used (2) and hypothesis H_1(ii). So, we have proved that

$$u_0 \in [0, a] \text{ and } u_0 \neq 0. \quad (5)$$

From Theorem 1.2 of Fan [6] (see also [20, Corollary 3.3] and Lieberman [8] (p. 320), for the isotropic case), we have that $u_0 \in C_+ \setminus \{0\}$. Moreover, from Proposition 4 of Papageorgiou et al. [9], we have that $u_0 \in [0, a] \cap \operatorname{int} C_+$. Let $\hat{\varepsilon} > 0$ be as postulated by hypothesis H_1(vi). Using (5) and H_1(vi), we have

$$\begin{aligned} -\Delta_{p(z)}u_0 - \Delta_{q(z)}u_0 + \hat{\varepsilon}u_0^{p(z)-1} &= f(z, u_0) + \hat{\varepsilon}u_0^{p(z)-1} \\ &\leq f(z, a) + \hat{\varepsilon}a^{p(z)-1} \text{ (see (5)) and hypothesis } H_1 \text{ (vi))} \\ &\leq -\hat{\lambda} + \hat{\varepsilon}a^{p(z)-1} < -\Delta_{p(z)}a - \Delta_{q(z)}a + \hat{\varepsilon}a^{p(z)-1}. \end{aligned}$$

Since $\hat{\lambda} > 0$ and $a > 0$, from Proposition 2.4 of Papageorgiou et al. [13], we infer that

$$u_0(z) < a \text{ for all } z \in \bar{\Omega} \quad \Rightarrow \quad u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0, a]. \quad (6)$$

From (2) it is clear that

$$\varphi_+|_{[0,a]} = \hat{\psi}_+|_{[0,a]}.$$

Thus, from (3) and (6), it follows that

$$u_0 \text{ is a local } C^1(\bar{\Omega}) - minimizer\ of\ \varphi_+(\cdot).$$

Hence,

$$u_0 \text{ is a local } W_0^{1,p(z)}(\Omega) - minimizer\ of\ \varphi_+(\cdot) (see\ [7, 20]). \quad (7)$$

Using the definition of $\varphi_+(\cdot)$, the anisotropic regularity theory and the anisotropic maximum principle, we have that $K_{\varphi_+} \setminus \{0\} \subseteq \mathrm{int}C_+$. So, we may assume K_{φ_+} is finite or otherwise we already have an infinity of positive smooth solutions of (1), and so we are done. So, $u_0 \in K_{\varphi_+}$ is isolated and this together with (7) and Theorem 5.7.6, p.449, of Papageorgiou et al. [12], imply that we can find $\eta \in (0, 1)$ small such that

$$\varphi_+(u_0) < \inf \{\varphi_+(u) \mid \|u - u_0\| = \eta\} = m_+. \tag{8}$$

If $u \in \mathrm{int}C_+$, then an account of hypothesis H_1(iii), we have

$$\varphi_+(tu) \to -\infty \text{ as } t \to +\infty. \tag{9}$$

Finally reasoning as in the proof of Proposition 4.1 in Gasiński and Papageorgiou [7], we infer that

$$\varphi_+(\cdot) \text{ satisfies the C-condition.} \tag{10}$$

Then (8)–(10) permit the use of the mountain pass theorem. So, we obtain $\hat{u} \in K_{\varphi_+} \subseteq C_+$ such that

$$\varphi_+(u_0) < m_+ \leq \varphi_+(\hat{u}) \text{ (see (1))} \quad \Rightarrow \quad \hat{u} \neq u_0.$$

Since $\hat{u}$ is a critical point of φ_+ of mountain pass type from Theorem 6.5.8, p. 527, of Papageorgiou et al. [12], we have

$$C_1(\varphi_+, \hat{u}) \neq 0. \tag{11}$$

On the other hand, hypothesis H_1(v) and Proposition 3.7 of Papagerogiou and Rădulescu [10] imply that

$$C_k(\varphi, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{12}$$

Note that $|F(z, x)| \leq c_3 \left[|x|^{\theta(z)} + |x|^{r(z)}\right]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ and some $c_3 > 0$. In addition, we may assume that K_φ is finite or otherwise we already have an infinity of smooth solution of (1) and so we are done. The C^1-continuity of critical groups (see [12], Theorem 6.3.4, p. 503) implies that $C_k(\varphi_+, 0) = C_k(\varphi, 0) = 0$ for all $k \in \mathbb{N}_0$ (see (12)). Then comparing this with (11), we infer that $\hat{u} \neq 0$. So, $\hat{u} \in \mathrm{int}C_+$ is the second positive solution of (1) distinct from u_0. □

Similarly truncating this time $f(z, \cdot)$ at $-c < 0$ and using the functional $\varphi_-(\cdot)$ we obtain a similar multiplicity result for negative solutions.

Proposition 5 *If hypotheses H_0 and H_1 hold, then problem* (1) *has at least two negative solutions*

$$v_0, \hat{v} \in \mathrm{int}C_+ \textit{ and } v_0 \neq \hat{v}.$$

Furthermore, we show the existence of extremal constant sign solutions. These solutions will be used in Sect. 4 to generate a nodal solution. On account of hypotheses H_1(i) and (v), we have

$$f(z, x)x \geq c_0|x|^{\mu(z)} - c_4|x|^{r(z)} \tag{13}$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, some $c_4 > 0$. This unilateral growth restriction on $f(z, \cdot)$, leads to the following auxiliary anisotropic Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)}u - \Delta_{q(z)}u = c_0|u|^{\mu(z)-2}u - c_4|u|^{r(z)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{14}$$

Proposition 6 *If hypotheses H_0 hold, then problem* (14) *has a unique positive solution $\bar{u} \in \operatorname{int}C_+$, and since the problem is odd $\bar{v} = -\bar{u} \in \operatorname{int}C_+$ is the unique negative solution of* (14).

Proof First we show the existence of a positive solution for problem (14). To this end, we introduce the C^1-function $\sigma_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + c_4 \int_\Omega \frac{1}{r(z)}(u^+)^{r(z)}\,dz \\ - c_0 \int_\Omega \frac{1}{\mu(z)}(u^+)^{\mu(z)}\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$. Then, it holds

$$\sigma_+(u) \geq \frac{1}{p_+}\rho_p(Du) + \frac{c_4}{r_+}\rho_r(u^+) - \frac{c_0}{\mu_-}\rho_\mu(u^+).$$

Recall that $\mu(z) < p(z) < r(z)$ for all $z \in \bar{\Omega}$. So, it follows that $\sigma_+(\cdot)$ is coercive (see Proposition 1). Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma_+(\bar{u}) = \inf\left\{\sigma_+(u) \mid u \in W_0^{1,p(z)}(\Omega)\right\}. \tag{15}$$

Let $u \in \operatorname{int}C_+$ and $t \in (0, 1)$. Keeping in mind that $t \in (0, 1)$ and $q_- < p_- < r_-$, it yields

$$\sigma_+(tu) \leq \frac{t^{p_-}}{p_-}\rho_p(Du) + \frac{t^{q_-}}{q_-}\rho_q(Du) + \frac{c_4 t^{r_-}}{r_-}\rho_r(u) - \frac{c_0 t^{\mu_+}}{\mu_+}\rho_\mu(u) \leq c_5 t^{q_-} - c_6 t^{\mu_+}$$

for some $c_5, c_6 > 0$. Since $t \in (0, 1)$ and $\mu_+ < q_-$ (see hypothesis H_1(v)), choosing $t \in (0, 1)$ small, it finds

$$\sigma_+(tu) < 0 \quad \Rightarrow \quad \sigma_+(\bar{u}) < 0 = \sigma_+(0) \text{ (see (15))} \quad \Rightarrow \quad \bar{u} \neq 0.$$

From (15), we have $\sigma'_+(\bar{u}) = 0$. This means that

$$\langle A_{p(z)}(\bar{u}), h\rangle + \langle A_{q(z)}(\bar{u}), h\rangle = \int_\Omega c_0(\bar{u}^+)^{\mu(z)-1} h\, dz - \int_\Omega c_4(\bar{u}^+)^{r(z)-1} h\, dz \quad (16)$$

for all $h \in W_0^{1,p(z)}(\Omega)$. In (16), we use the test function $h = -\bar{u}^- \in W_0^{1,p(z)}(\Omega)$ and obtain that $\bar{u} \geq 0$ and $\bar{u} \neq 0$. Therefore, $\bar{u}$ is a positive solution of (14) and then the anisotropic regularity theory and the anisotropic maximum principle, as before, imply that $\bar{u} \in \mathrm{int}C_+$.

Now we show the uniqueness of this positive solution. For this purpose, we introduce the integral functional $j\colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)} |Du^{\frac{1}{q_-}}|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du^{\frac{1}{q_-}}|^{q(z)}\, dz, & \text{if } u \geq 0,\, u^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise .} \end{cases}$$

From Theorem 2.2 of Takáč and Giacomoni [19], we have that the functional $j(\cdot)$ is convex. Let $\mathrm{dom} j = \{u \in L^1(\Omega) \mid j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). Suppose that $\bar{v} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of (14). Again we have $\bar{v} \in \mathrm{int}C_+$. On account of Proposition 4.1.22, p. 274, of Papagerogiou et al. [12], we have

$$\frac{\bar{u}}{\bar{v}}, \frac{\bar{v}}{\bar{u}} \in L^\infty(\Omega).$$

Hence, if $h = \bar{u}^{q_-} - \bar{v}^{q_-}$, then for $|t| < 1$ small enough, we have

$$\bar{u}^{q_-} + th \in \mathrm{dom} j \text{ and } \bar{v}^{q_-} + th \in \mathrm{dom} j.$$

Then, the convexity of $j(\cdot)$ implies the Gâteaux differentiability at $\bar{u}^{q_-}$ and at $\bar{v}^{q_-}$ in the direction h, respectively. Moreover, the chain rule and Green's theorem imply that

$$\begin{aligned} j'(\bar{u}^{q_-})(h) &= \frac{1}{q_-} \int_\Omega \frac{-\Delta_{p(z)}\bar{u} - \Delta_{q(z)}\bar{u}}{\bar{u}^{q_- -1}} h\, dz \\ &= \frac{1}{q_-} \int_\Omega \left[c_0 \bar{u}^{\mu(z)-q_-} - c_4 \bar{u}^{r(z)-q_-} \right] h\, dz, \end{aligned}$$

and

$$\begin{aligned} j'(\bar{v}^{q_-})(h) &= \frac{1}{q_-} \int_\Omega \frac{-\Delta_{p(z)}\bar{v} - \Delta_{q(z)}\bar{v}}{\bar{v}^{q_- -1}} h\, dz \\ &= \frac{1}{q_-} \int_\Omega \left[c_0 \bar{v}^{\mu(z)-q_-} - c_4 \bar{v}^{r(z)-q_-} \right] h\, dz. \end{aligned}$$

Whereas, the convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Therefore, it gives

$$0 \leq \int_\Omega c_0[\bar{u}^{\mu(z)-q_-} - \bar{v}^{\mu(z)-q_-}](\bar{u}_{q_-} - \bar{v}_{q_-})\, dz \\ - \int_\Omega c_4[\bar{u}^{r(z)-q_-} - \bar{v}^{r(z)-q_-}](\bar{u}_{q_-} - \bar{v}_{q_-}) dz \leq 0,$$

so, $\bar{u} = \bar{v}$.

This proves the uniqueness of the positive solution $\bar{u} \in \mathrm{int}C_+$ of (14). Since the equation is odd, $\bar{v} = -\bar{u} \in -\mathrm{int}C_+$ is the unique negative solution of (14). □

These solutions provide bounds for the constant sign solutions of problem (1). We introduce the following two sets:

$$S_+ = \text{ set of positive solutions of problem(1)}, \\ S_- = \text{ set of negative solutions of problem(1)}.$$

From Propositions 4 and 5, we have

$$\emptyset \neq S_+ \subseteq \mathrm{int}C_+ \text{ and } \emptyset \neq S_- \subseteq -\mathrm{int}C_+.$$

Proposition 7 *If hypotheses H_0 and H_1 hold, then the statements hold true:*

(a) $\bar{u} \leq u$ *for all* $u \in S_+$*;*
(b) $v \leq \bar{v}$ *for all* $v \in S_-$.

Proof (a) Let $u \in S_+$ and consider the Carathéodory function $k_+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_+(z, x) := \begin{cases} c_0(x^+)^{\mu(z)-1} - c_4(x^+)^{r(z)-1} & \text{if } x \leq u(z) \\ c_0 u(z)^{\mu(z)-1} - c_4 u(z)^{r(z)-1} & \text{if } u(z) < x \end{cases}. \tag{17}$$

We set $K_+(z, x) := \int_0^x k_+(z, s)\, ds$ and consider the C^1-functional $\gamma_+ \colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) := \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega K_+(z, u)\, dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

From (17) it is clear that $\gamma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\gamma_+(\tilde{u}) = \inf \left\{ \gamma_+(u) \;\middle|\; u \in W_0^{1,p(z)}(\Omega) \right\}. \tag{18}$$

Let $v \in \mathrm{int}C_+$. Since $u \in S_+ \subseteq \mathrm{int}C_+$, we can find $t \in (0, 1)$ small enough such that $tv < u$ (see [12], Proposition 4.1.22, p. 274). Since $\mu_+ < q_-$, as in the proof of Proposition 6, we have $\gamma_+(tv) < 0$. Hence

$$\gamma_+(\tilde{u}) < 0 = \gamma_+(0) \text{ (see (18))} \quad \Rightarrow \quad \tilde{u} \neq 0. \tag{19}$$

Using (17), we easily show that

$$K_{\gamma_+} \subseteq [0, u] \cap C_+ \quad \Rightarrow \quad \tilde{u} \in [0, u] \cap C_+ \text{ and } \tilde{u} \neq 0 \quad \Rightarrow \quad \tilde{u} = \bar{u} \in \mathrm{int} C_+,$$

where we have applied (17)–(19) and Proposition 6. So, we conclude that $\bar{u} \leq u$ for all $u \in S_+$.

(b) Similarly, we show that $v \leq \bar{v}$ for all $v \in S_-$. □

Using these bounds, we can produce the extremal constant sign solutions for problem (1).

Proposition 8 *If hypotheses H_0 and H_1 hold, then we can find $u^* \in S_+ \subseteq \mathrm{int}C_+$ and $v^* \in S_- \subseteq -\mathrm{int}C_+$ such that*

$$u^* \leq u \; for \; all \; u \in S_+ \; and \; v \leq v^* \; for \; all \; v \in S_-.$$

Proof From Papageorgiou et al. [11] (see the proof Proposition 7), we know that the solution set S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$ and $u \leq u_2$). Also, from the proof of Proposition 4, we can restrict ourselves to the set $S_+ \cap [0, a]$. Then, we can find a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq S_+ \cap [0, a]$ such that

$$\inf S_+ = \inf_{n \in \mathbb{N}} u_n.$$

For each $n \in \mathbb{N}$, we have

$$\langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle = \int_\Omega f(z, u_n) h \, dz \tag{20}$$

for all $h \in W_0^{1,p(z)}(\Omega)$. Choosing $h = u_n \in W_0^{1,p(z)}(\Omega)$ and recalling that $u_n \in [0, a]$ for all $n \in \mathbb{N}$, we obtain

$$\rho_p(Du_n) \leq c_7 \|u_n\| \text{ for some} c_7 > 0 \text{all} n \in \mathbb{N},$$
$$\Rightarrow \{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded (see Proposition 1).}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } u_n \to u^* \text{ in } L^{p(z)}(\Omega). \tag{21}$$

In (20) we use the test function $h = u_n - u^* \in W_0^{1,p(z)}(\Omega)$, pass to the limits as $n \to \infty$ and apply (21). Then, we have

$$\lim_{n \to \infty} [\langle A_{p(z)}(u_n), u_n - u^*) \rangle + \langle A_{q(z)}(u_n), u_n - u^*) \rangle] = 0.$$

But, the monotonicity of $A_{q(z)}(\cdot)$ reveals that

$$\limsup_{n\to\infty}[\langle A_{p(z)}(u_n), u_n - u^*\rangle\rangle + \langle A_{q(z)}(u^*), u_n - u^*\rangle\rangle] \leq 0.$$

The latter combined with the $(S)_+$-property of A and (21) imply that

$$u_n \to u^* \text{ in } W_0^{1,p(z)}(\Omega). \tag{22}$$

If in (20) we pass to the limit as $n \to \infty$ and use (22), then we have

$$\langle A_{p(z)}(u^*), h\rangle + \langle A_{q(z)}(u^*), h\rangle = \int_\Omega f(z, u^*)h\, dz \tag{23}$$

for all $h \in W_0^{1,p(z)}(\Omega)$. From Proposition 7 and (22)–(23), we have

$$\bar{u} \leq u_n \text{ for all } n \in \mathbb{N} \quad \Rightarrow \quad \bar{u} \leq u^* \quad \Rightarrow \quad u^* \in S_+ \subseteq \mathrm{int}C_+ \text{ and } u^* = \inf S_+.$$

Similarly, we produce $v^* \in S_-$ and $v^* = \sup S_-$. We mention that the set S_- is upward directed (that is, if $v_1, v_2 \in S_-$, we can find $v \in S_-$ such that $v_1 \leq v$ and $v_2 \leq v$). Also on account of the proof of Proposition 4, we can restrict ourselves to $S_- \cap [-c, 0]$. □

4 Nodal solutions

In this section we show the existence of a nodal solution to problem (1). To obtain such a solution, we will focus on the order interval $[v^*, u^*]$ by using truncations. More precisely, observe that any nontrivial solution of problem (1) in this order interval distinct from u^* and v^*, is nodal on account of the extremality of u^* and v^*.

So, we introduce the Carathéodory function $\hat{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{g}(z, x) = \begin{cases} f(z, v^*(z)) & \text{if } x < v^*(z) \\ f(z, x) & \text{if } v^*(z) \leq x \leq u^*(z) \\ f(z, u^*(z)) & \text{if } u^*(z) < x \end{cases}. \tag{24}$$

Also we consider the positive and negative truncations of $\hat{g}(z, \cdot)$, namely, the Carathéodory functions

$$\hat{g}_\pm(z, x) = \hat{g}(z, \pm x^\pm). \tag{25}$$

Then we set $\widehat{G}(z, x) = \int_0^x \hat{g}(z, s)\, ds$ and $\widehat{G}_\pm(z, x) = \int_0^x \hat{g}_\pm(z, s)\, ds$ and introduce the C^1-functionals $\hat{\beta}, \hat{\beta}_\pm\colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\beta}(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\, dz - \int_\Omega \widehat{G}(z, u)\, dz,$$

and

$$\hat{\beta}_{\pm}(u) = \int_{\Omega} \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_{\Omega} \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_{\Omega} \widehat{G}_{\pm}(z, u)\, dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Using (24)–(25), the anisotropic regularity theory and the extremality of u^* and v^*, we obtain the following result.

Proposition 9 *If hypotheses H_0 and H_1 hold, then $K_{\hat{\beta}} \subseteq [v^*, u^*] \cap C_0^1(\bar{\Omega})$, $K_{\hat{\beta}_+} = \{0, u^*\}$, $K_{\hat{\beta}_-} = \{0, v^*\}$.*

The next observation, will allow the use of the mountain pass theorem.

Proposition 10 *If hypotheses H_0 and H_1 hold, then $u^* \in \mathrm{int} C_+$ and $v^* \in -\mathrm{int} C_+$ are local minimizers of the functional $\hat{\beta}(\cdot)$.*

Proof From (24) and (25) it is clear that $\hat{\beta}_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}^* \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\beta}_+(\tilde{u}^*) = \inf \left\{ \hat{\beta}_+(u) \mid u \in W_0^{1,p(z)}(\Omega) \right\}. \tag{26}$$

Since $u^* \in \mathrm{int} C_+$, if $u \in \mathrm{int} C_+$ for $t \in (0, 1)$ small we will have that $tu \leq \min\{u^*, \delta\}$ (see hypothesis H_1(v) and [12], p. 274). Therefore as in the proof of Proposition 6, since $\mu_+ < q_-$, we have

$$\hat{\beta}_+(tu) < 0 \quad \Rightarrow \quad \hat{\beta}_+(\tilde{u}^*) < 0 = \hat{\beta}_+(0) \text{ (see (26))} \quad \Rightarrow \quad \tilde{u}^* \neq 0. \tag{27}$$

From (26) and (27) we have $\tilde{u}^* \in K_{\hat{\beta}_+} \setminus \{0\}$, hence $\tilde{u}^* = u^* \in \mathrm{int} C_+$ (see Proposition 9). Notice that

$$\hat{\beta}|_{C_+} = \hat{\beta}_+|_{C_+} \quad \Rightarrow \quad u^* \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \hat{\beta}(\cdot).$$

The latter together with [7] and [20] implies that

$$u^* \text{ is a local} W_0^{1,p(z)}(\bar{\Omega})\text{-minimizer of } \hat{\beta}(\cdot). \tag{28}$$

Similarly for $v^* \in -\mathrm{int} C_+$ we can obtain the desired conclusion by using the functional $\hat{\beta}_-(\cdot)$. □

Now we are ready to produce a nodal solution to problem (1).

Proposition 11 *If hypotheses H_0 and H_1 hold, then problem (1) admits a nodal solution $y_0 \in [v^*, u^*] \cap C_0^1(\bar{\Omega})$.*

Proof On account of Proposition 9, we may assume that $K_{\hat{\beta}}$ is finite. Otherwise we already have an infinity of smooth nodal solutions (see Proposition 9) and so we are done. So, we may assume that

$$\hat{\beta}(v^*) \leq \hat{\beta}(u^*).$$

The analysis is similar if the opposite inequality holds. Since $u^* \in \mathrm{int}C_+$ is a local minimizer of $\hat{\beta}(\cdot)$ (see Proposition 10) and $K_{\hat{\beta}}$ is finite, using [12,Theorem 5.7.6, p. 449], we can find $\eta \in (0, 1)$ small such that

$$\hat{\beta}(v^*) \leq \hat{\beta}(u^*) < \inf\left\{\hat{\beta}(u) \mid \|u - u^*\| = \eta\right\} = \hat{m} \text{ and } \|v^* - u^*\| > \eta. \quad (29)$$

Recall that $\hat{\beta}(\cdot)$ is coercive (see (24)). So, from [12], Proposition 5.1.15, p. 369, we have that

$$\hat{\beta}(\cdot) \text{ satisfies the C-condition.} \quad (30)$$

Then (28)–(30) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\left\{\begin{array}{l} y_0 \in K_{\hat{\beta}} \subseteq [v^*,\ u^*] \cap C_0^1(\bar{\Omega}) \text{ see Proposition 9,} \\ y_0 \notin \{v^*, u^*\},\ C_1(\hat{\beta}, y_0) \neq 0 \text{ see [11], p. 527} \end{array}\right\}. \quad (31)$$

Since $0 \in \inf_{C_0^1(\bar{\Omega})}[v^*, u^*]$, through a standard homotopy invariance argument, we infer that

$$C_k(\hat{\beta}, 0) = C_k(\varphi, 0) = 0, \text{ for all } k \in \mathbb{N}_0 \text{ (see [[9]]).} \quad (32)$$

From (30) and (32), we infer that

$$\begin{aligned} & y_0 \notin \{0, u^*, v^*\},\ y_0 \in [v^*, u^*] \cap C_0^1(\bar{\Omega}), \\ & \Rightarrow y_0 \in C_0^1(\bar{\Omega}) \text{ is a nodal solution of problem (1).} \end{aligned}$$

This completes the proof of the proposition. □

So, summarizing, we can state the following multiplicity theorem for the problem (1), which provides the exact sign information for all the solutions produced.

Theorem 12 *If hypotheses H_0 and H_1 hold, then problem* (1) *has at least five nontrivial solutions*

$$u_0, \hat{u} \in \mathrm{int}C_+ \textit{ with } u_0 \neq \hat{u}, \quad v_0, \hat{v} \in -\mathrm{int}C_+ \textit{ with } v_0 \neq \hat{v} \textit{ and } y_0 \in C_0^1(\bar{\Omega}) \textit{ nodal}.$$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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