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High-order compact scheme finite difference discretization for Signorini's problem

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ABSTRACT

This paper brings out an analysis of the projection iterative algorithm for the numerical solution of the Signorini problem. The very closed connections with the switching method are highlighted. In addition, the relevance of higher-order discretization for Signorini problem is discussed. Thus, a specific iterative solver is developed to address the present fourth and sixth-order compact scheme discretizations. This method is based on a lower-order preconditioning method. Several numerical experiments have been performed to bring light to the accuracy of such method, despite the lack of smoothness at the Signorini boundary.

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1. Introduction

The design of efficient numerical methods for problems arising from mechanics is of primary importance. Applications are commonly encountered in fluid mechanics, deformable solid mechanics or heat transfer. Despite that the rapid growth of parallel computing allows computations on fine meshes with standard discretizations, the higher-order ones can be considered as an alternative to get accurate solutions while saving computational cost. Nevertheless, their implementation for complex mathematical models, like Signorini problems, remains an open issue because of complex algorithms due the non-linear equations. In the present paper, the projection iterative method [13,23] involved in solving the Signorini problem is first analysed. Then, the relevance of the higher-order finite difference discretizations for this kind of problem is shown, despite the limited differentiability of the solution at the boundaries.

Signorini problems arise in mathematical modeling of various physical and industrial problems such as beach percolation, contact problems or electropainting process. These non-linear problems are featured by two boundary inequalities on the solution and its normal derivative combined with a complementary equation. Typical approaches to get solutions of this problem are based on numerical approximations. Most of numerical methods involve a linearization strategy of the non-linear Signorini problem, and is combined with a discretization of the underlying elliptic problem. The linearization consists in determining the part of the Signorini boundary for which Dirichlet or Neumann/Robin boundary condition are prescribed. The decomposition-coordination method [18], the switching method [5], the optimization method [11,16] and the linear complementary method [24] have been proposed to this purpose.

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The Dirichlet/Neumann partition is computed in an iterative way with the switching method [5]. This partition is evaluated from inequalities that involve the solution and the normal derivative at the Signorini boundary. Thus, solving the Signorini problem is reduced to a sequence of elliptic equation for which the boundary condition kind switches from Dirichlet to Neumann until convergence of the iterative method. This specificity is shared with the semi-smooth Newton method for the Signorini problem [10]. In this approach, the use of the semi-smooth Newton method to address the Signorini non-linearity leads to an algorithm for which each non-linear iteration involve simultaneously Neumann and Robin boundary conditions. This partition of the Signorini boundary condition introduces active set nodes which define the Signorini boundary [10]. This method also introduces a shift solution computed in a preprocessing stage, and depends on a positive parameter c. Recently, a projection iterative method has been developed to address the Signorini problem [23]. The key point of this approach is an equivalence between the Signorini boundary condition and a fixed point equation. The resulting algorithms share some features with the switching [5] and the active set methods [10]. The Signorini boundary is defined as a convergent sequence of Dirichlet/Robin [23] or Neumann/Robin [13] boundary partition. As in [10] the projection iterative method, a parameter $c > 0$ is introduced and is closely associated to the Robin boundary condition. It should be mentioned that these methods have some similarities, as discussed in the context of contact mechanics of deformable bodies [2].

The linear problems resulting from these linearization methods have been discretized in several ways. Finite elements is the common discretization, although other approximations can be found in the literature. The Finite Differences [10], the Finite Elements [2], or the moving Kriging interpolation [13] and the recent Hybrid High-Order discretization [7] outline the diversity of the available discrete approximations. One can also note the recent contribution based on the element-free Galerkin method [14]. In [7], a fourth-order discretization has been developed and successfully tested with a smooth solution of a Signorini problem. According to our knowledge, this work seems to be the first one involving higher-order discretization for the Signorini problem. Higher-order discretizations may be considered as method for saving computational cost. Indeed, it is expected that less degrees of freedom are needed to approximate a solution in comparison with the low-order one. However, their implementation turns out to be a tricky problem. High-order compact scheme (HOCS) discretizations belong to this category of higher-order discretizations. They have a relative ease of use despite their implicit formulation. For instance, by using multigrid defect correction, we can build efficient linear solvers for HOCS discretizations of Poisson's equation [3]. According to our knowledge, there is no work reporting a such discretization for the Signorini problem.

The aim of this paper is twofold. First, we propose an analysis of the iterative projection method. We show that in the limit of large $c > 0$ parameter, the projection iterative method is practically the same as the switching method. Next, we present a fourth and a sixth-order accurate discretizations of the Signorini problem by making use of HOCS. Finally, several numerical experiments are performed to verify and to highlight the numerical properties which result from the present analysis.

2. Numerical method of solutions for the signorini problem

2.1. Problem description

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of boundary Γ and an outward unit normal $\mathbf{n}$. This boundary consists of three disjoint parts, a Dirichlet type boundary Γ_d, a Neumann type boundary Γ_n and a Signorini type boundary Γ_s. The Signorini problem is defined by:

$$-\nabla \cdot \nabla u = 0 \quad \text{in } \Omega \tag{1}$$

$$u = \bar{u} \quad \text{on } \Gamma_d \tag{2}$$

$$\partial_{\mathbf{n}} u = \bar{q} \quad \text{on } \Gamma_n \tag{3}$$

$$u \geq f, \partial_{\mathbf{n}} u \geq g, (u - f)(\partial_{\mathbf{n}} u - g) = 0 \quad \text{on } \Gamma_s \tag{4}$$

where $\bar{u}$, $\bar{q}$, f and g are known functions defined on Ω and Γ. We assume that the problem satisfies $\Gamma_d \neq \varnothing$ or $\int_{\Gamma_n} \bar{q}\mathrm{d}\Gamma + \int_{\Gamma_s} g\mathrm{d}\Gamma < 0$. Then according to the theory of variational inequalities, the Signorini problem has admits unique solution [8]. The non-linear boundary condition Equation (4) is an intrinsic part of the Signorini problem and consists in two inequalities for the unknown u and its derivative $q = \partial_{\mathbf{n}} u$ combined to a complementary equation. The problem Equations (1)–(4) results from the mathematical modelling of an elastic membrane submitted to gravity forces and such that the Signorini boundary moves freely up to with the obstacle modeled by f. In other hand, one have to determine the part of the Signorini boundary Γ_s on which Neumann boundary condition Γ_{sn} occurs and the remaining part where the Dirichlet boundary Γ_s/Γ_{sn} condition is prescribed. To this end, the non-linear boundary conditions Equation (4) necessitates a specific design [5,10,13,23]. The projection iterative methods [21–23] are such approaches. In the next section we provide an heuristic analysis to outline the connections with the switching method [5]. This analysis mainly consider the existence and uniqueness of the solution for the projection iterative method provided in [23].

2.2. An analysis of the projection iterative algorithm

The projection iterative algorithm was detailed in [23] to solve the Signorini problem Equations (1)–(4). The method relies on a way to linearize the non-linear boundary conditions Equation (4). This consists in subtly reformulating the Signorini boundary conditions in a fixed point equation, which is solved in an iterative way. Several similarities with the active set method or switching algorithm can be noted: it shares a strategy to find the correct subsets of Γ_s for which the Dirichlet, Neumann/Robin boundary conditions are prescribed [2,5]. In the following, starting from the projection iterative method, we we propose an iterative scheme which is free of penalization parameter/Then we discuss the connections with the switching method [5].

The projection iterative method is based on an equivalent form of the Signorini boundary conditions Equation (4). It is rewritten as a fixed point equation [13,23]:

$$\left(u - f\right) - \left[\left(u - f\right) - c\left(\partial_{\mathbf{n}} u - g\right)\right]_+ = 0 \tag{5}$$

with $c > 0$ and $[a]_+ = \max(0, a)$ denoting the projection operator. Thus, the solution of Equations (1)–(4) could be addressed with an iterative scheme using the Equation (5). This scheme for $k \geq 0$ is reported hereafter:

$$\left(u^{k+1} - f\right) - \left[\left(u^{k} - f\right) - c\left(\partial_{\mathbf{n}} u^{k+1} - g\right)\right]_+ = 0 \tag{6}$$

Thus, according to the iteration equation Equation (6), and identically to [10], the following algorithm denoted by (A1) has been derived in [23]:

$$\left\{\begin{array}{rcll} -\nabla \cdot \nabla u^{k+1} & = & s & \text{on } \Omega \\ u^{k+1} & = & f & \text{in } \bar{\Gamma}_{sn}^{k+1} \\ u^{k+1} + c\partial_{\mathbf{n}} u^{k+1} & = & u^{k} + cg & \text{in } \Gamma_{sn}^{k+1} \end{array}\right. \tag{7}$$

where, $\Gamma_{sn}^{k+1} = \{x \in \Gamma_s : \left(u^k - f\right) - c\left(\partial_{\mathbf{n}} u^k - g\right) > 0\}$ and $\bar{\Gamma}_{sn}^{k+1} = \Gamma_s/\Gamma_{sn}^{k+1}$. The convergence proof of the algorithm (A1) is detailed in [23]. Indeed, the mentioned algorithm (A1) introduces a constant $c > 0$ and affirms that the convergence proof holds for all $c > 0$. Therefore, the numerical experiments show that the best convergence results are obtained for large values of c [13,23]. Next, we can note that the boundary conditions in Equation (7) are derived from writing down Equation (6). However, the definition of the set Γ_{sn}^{k+1} in Equation (7) is not equivalent to Equation (6) since the term $\partial_{\mathbf{n}} u$ is evaluated at the iteration k. Thus, the present investigation is motivated by the following two remarks. First, we give an equivalent definition of the set Γ_{sn}^{k+1}. Indeed, from the definition $x \in \Gamma_{sn}^{k+1}$, the condition $\left(u^k - f\right) - c\left(\partial_{\mathbf{n}} u^k - g\right) > 0$ is fulfiled. Two different cases depending on the

status of the node x at the Signorini boundary for the previous iteration k have to be considered. If $x \in \Gamma_{sn}^k$ then $u^k - u^{k-1} = -c(\partial_{\mathbf{n}} u^k - g)$ because of the algorithm (A1) definition. Thus, the condition $\left(u^k - f\right) - c\left(\partial_{\mathbf{n}} u^k - g\right) > 0$ leads to $2u^k - u^{k-1} > f$. Next, if $x \in \bar{\Gamma}_{sn}^k$, then $u^k = f$ implies that the condition $\left(u^k - f\right) - c\left(\partial_{\mathbf{n}} u^k - g\right) > 0$ reduces to $\left(\partial_{\mathbf{n}} u^k - g\right) < 0$. One can summarize this with the following definition set of Γ_{sn}^{k+1}:

$$\Gamma_{sn}^{k+1} = \left\{x \in \Gamma_{sn}^k : 2u^k - u^{k-1} > f\right\} \cup \left\{x \in \bar{\Gamma}_{sn}^k : \left(\partial_{\mathbf{n}} u^k - g\right) < 0\right\} \tag{8}$$

From $x \in \bar{\Gamma}_{sn}^{k+1}$, the condition $\left(u^k - f\right) - c\left(\partial_{\mathbf{n}} u^k - g\right) < 0$ leads to:

$$\bar{\Gamma}_{sn}^{k+1} = \left\{x \in \Gamma_{sn}^k : 2u^k - u^{k-1} \leq f\right\} \cup \left\{x \in \bar{\Gamma}_{sn}^k : \left(\partial_{\mathbf{n}} u^k - g\right) \geq 0\right\} \tag{9}$$

Obviously, these two sets are a partition of Γ_s: $\Gamma_{sn}^{k+1} \cap \bar{\Gamma}_{sn}^{k+1} = \emptyset$ and $\Gamma_{sn}^{k+1} \cup \bar{\Gamma}_{sn}^{k+1} = \Gamma_s$. The present formulation of Γ_{sn}^{k+1} avoids the parameter $c > 0$ and could be summarized as follow:

- If the boundary condition was Dirichlet: Check if $\partial_{\mathbf{n}} u^k > g$. If true, retain the Dirichlet boundary condition for the next iteration. If false, switch to the Robin condition given in Equation (7).
- If the boundary condition was Robin: Check if $2u^k - u^{k-1} > f$. If true, retain the Robin boundary condition of Equation (7) for the next iteration. If false, switch to the Dirichlet condition.

This algorithm has important similarities with the switching method [5]. The only difference is the test $2u^k - u^{k-1} > f$ instead of $u^k > f$ for the switching method. It should be mentioned that the algorithm is initialized ($k = 0$) by prescribing Dirichlet boundary condition. The first inequality of the test involves only the normal derivative at the iteration $k = 0$. Indeed, the iterative sequence is well defined.

Next, the numerical experiments [23] show that the large values c lead to the best convergence rate of the iterative scheme Equation (7). Therefore the scheme Equation (7) reduces, under the limit of large $c > 0$, to:

$$\left\{\begin{array}{rcll} -\nabla \cdot \nabla u^{k+1} & = & s & \text{on } \Omega \\ u^{k+1} & = & f & \text{in } \bar{\Gamma}_{sn}^{k+1} \\ \partial_{\mathbf{n}} u^{k+1} & = & g & \text{in } \Gamma_{sn}^{k+1} \end{array}\right. \tag{10}$$

with, $\Gamma_{sn}^{k+1} = \{x \in \Gamma_{sn}^k : 2u^k - u^{k-1} > f\} \cup \{x \in \bar{\Gamma}_{sn}^k : \left(\partial_{\mathbf{n}} u^k - g\right) < 0\}$. This heuristic argumentation allows us to get a formulation which is free of parameter $c > 0$. This feature simplifies the implementation which consists in solving successive linear problems that the boundary condition kind evolves with respect to the iterations. In the following, this algorithm is denoted by (A2) and a more detailed version including the higher-order discretization is given in the Section 2.4.

2.3. Compact scheme discretization

Higher-order accuracy for the space discretization can be achieved by using compact finite difference schemes [12]. This discretization is retained hereafter while proposing a way to include it in the present non-linear solver described by the algorithm (A2). For sake of clarity, we detail the discretization for a 2d cartesian system coordinates using a cell-centred layout for the variable u. Thus, the first-derivatives are defined in staggered way. Let $0 \leq x, y \leq 1$ denotes the domain bound and $h = 1/N$ the uniform space step. If ξ stands for x or y, the centers and faces of the mesh cells are $\xi^c = \{\xi_i = (i - 1/2)h,\ 1 \leq i \leq N\}$ and $\xi^f = \{\xi_{i+1/2} = ih,\ 0 \leq i \leq N\}$ respectively. To take into account the staggered grid, *cell-to-face* and *face-to-cell* derivative and interpolation have to be defined. For instance, the compact scheme relations which define *face-to-cell* derivative and interpolation can

Table 1. Third-order compact scheme boundary relations.

$u'_0 + 23u'_1 = -25u_{1/2} + 26u_{3/2} - u_{5/2}$	$u_0 + 5u_1 = 15/4u_{1/2} + 5/2u_{3/2} - 1/4u_{5/2}$
$u'_{1/2} - u'_{3/2} = -u_0 + 2u_1 - u_2$	$u_{1/2} + u_{3/2} = 1/4u_0 + 3/2u_1 + 1/4u_2$

be written as [12]:

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = a\frac{u_{i+1/2} - u_{i-1/2}}{h} + b\frac{u_{i+3/2} - u_{i-3/2}}{2h} \tag{11}$$

and

$$\alpha u_{i-1} + u_i + \alpha u_{i+1} = a\frac{u_{i+1/2} + u_{i-1/2}}{2} + b\frac{u_{i+3/2} + u_{i-3/2}}{2} \tag{12}$$

for which the order of accuracy depends on the constants α, a and b. Hence, $(\alpha, a, b) = (1/22, 12/11, 0)$ or $(\alpha, a, b) = (9/62, 63/62, 17/62)$ lead to the fourth and sixth-order accuracy of the derivative, and $(\alpha, a, b) = (1/6, 2/3, 0)$ or $(\alpha, a, b) = (3/10, 3/4, 1/20)$ lead to the fourth and sixth-order approximation of the interpolation. The *cell-to-face* are written by using the shift form of Equations (11) and (12). The boundary relations are obtained by upwinding the discretization; they are detailed in the Table 1.

It is worth mentioning that from Equations (11) and (12), the computation of the derivatives involves the solution of tridiagonal linear systems. Discrete derivatives can be rewritten in a generic way as $MU' = BU$. Indeed, the discrete derivatives are formulated in an explicit way $U' = M^{-1}BU = \delta U$, which leads to a dense matrix operators δ. Hence, by using the tensorial notation, the compact finite difference discretization of the elliptic problem Equation (10) is rewritten as follow:

$$-\left(\delta_x^{fc}\delta_x^{cf} \otimes \mathbb{I}_y + \mathbb{I}_x \otimes \delta_y^{fc}\delta_y^{cf}\right) u = Lu = s \tag{13}$$

where $\mathbb{I}_\xi$ stands for the identity operator in the ξ-direction and L for the matrix which represents the linear system. Numerical solutions of Equation (13) is based on a regularity assumption due to the compact scheme derivation [12]. However, as experimented in [15,19], a local lack of regularity, as involved in solution of Signorini problems, can be accepted without compromising the overall accuracy of the solution. The successive diagonalizations method [3] is a way to get a solution for such dense linear systems. Nevertheless, the need to prescribe simultaneously Dirichlet and Neumann boundary conditions on Γ_s implies that the Equation (10) is not separable, prohibiting the diagonalization method. Moreover, despite the dense pattern of the matrix L, the computation of the residual $r = Lu - s$ involves only tridiagonal linear systems. Thus, a natural way to get the solutions of such linear systems are the iterative methods. This point is detailed in the next section.

2.4. Lower-order preconditioning

Within the framework of higher-order accurate discretizations of elliptic equations, the lower-order preconditioning method consists in building a preconditioner based on a lower-order accurate discretization of the same problem [9]. In case of the compact scheme discretizations Abide and Zeghmati [4] or Abide [1] shown that a preconditioner build on a second-order finite difference scheme can lead to an iterative solver that the number of iteration does not depend on the mesh size or variable coefficient. In other hand, the convergence rate of the residual behaves like multigrid method. Here, a variant is proposed by using Preconditioned Conjugate Residual instead of the Richardson method as used in [1,4]. This allows us to avoid the introduction of the optimal relation factor. The PCR method is detailed in [6] pp. 154. The two main steps of the PCR are the evaluation matrix vector product Lu and the evaluation of the preconditioning $Hz = r$, where H is a sparse linear system resulting in the second-order finite differences Equation (13). The matrix/vector product Lu, which

is the discrete evaluation of the Poisson's equation, is performed with a linear complexity for compact scheme [3]. The preconditioning step $Hz = r$ is solved with the semicoarsening multigrid [17]. In practice, the convergence of PCR is achieved by using only some multigrid iterations. In the following, the linear solver to compute the solution of Equation (10) is denoted by PCR(u^{k+1}, s, ϵ_i), where ϵ_i is the stopping criterion based on the $L2$-norm of the relative residual.

The full version of the algorithm (A2) to solve the Signorini problem is detailed hereafter:

Set $\Gamma_{sn}^0 = \emptyset$;
Compute the initial guess by solving Eq. (10) with PCR(u^1, s, ϵ_i);
Set $k = 1$;
while $\|u^k - u^{k-1}\| > \epsilon_o$ **do**
  Compute: $\Gamma_{sn}^{k+1} = \left\{x \in \Gamma_{sn}^k : 2u^k - u^{k-1} > f\right\} \cup \left\{x \in \bar{\Gamma}_{sn}^k : \left(\partial_{\mathbf{n}} u^k - g\right) < 0\right\}$;
  Solve Eq. (10) by using PCR(u^{k+1}, s, ϵ_i);
  $k = k + 1$;
end

Algorithm 1: Detailed algorithm of the present method (A2).

One can note that the algorithm A2 consists in an outer and inner loop associated to the non-linear and the linear iterations. The stopping criterion is based on this loop. The inner loop, associated to the PCR, is stopped when the $L2$-norm of the residual achieves an user defined threshold ϵ_i. The second criterion helps to monitor the convergence of the non-linear loop. Zhang et al. [23] retained $\|u^{k+1} - u^k\|_{\Gamma_S} < \epsilon_o$ as a stopping criterion, which is the difference between two successive iterations at the Signorini boundary. Here, a similar criterion is retained, however we extend it to the difference deined in the domain Ω, that is $\|u^{k+1} - u^k\|_{\Omega} < \epsilon_o$. Specifically, if a converged solution is obtained for each non-linear iteration by setting $\epsilon_i < 10^{-10}$ for instance, the convergence of the non-linear system holds for an integer k_0 that $\Gamma_{sn}^{k_0+1} = \Gamma_{sn}^{k_0}$.

An another strategy could be derived by requiring an approximate solution of the linear system by setting $\epsilon_i < 10^{-1}$ for example. In this case, the convergence holds when the non-linear stopping is encountered. This strategy is rather a merge of the inner and outer loops. This point is experimented in Section 3.3.

From a practical point of view, the high-order accurate solutions of the linearized elliptic problem are addressed using our research code devoted to the computational fluid dynamics [3]. This code is written in F90 and has been ported on several high performance computing centers. In this way, numerical experiments could benefit from parallel computing.

3. Numerical experiments

3.1. Manufactured smooth solutions

First, we check the accuracy of the space discretization. Indeed, if the solution of the problem Equation (1) is sufficiently differentiable, the fourth and sixth-order of accuracy are expected from the compact scheme definition Equation (11). To this end, we refer to the work of Wigley [20] and Cascavita [7], which concerns the design of a manufactured solution for the Signorini problem. Let us consider the following function expressed in cylindrical coordinates:

$$u(r, \theta) = -r^{(2k+1)/2} \sin\left((2k+1)/2\theta\right) \tag{14}$$

defined on a domain $\Omega =]-0.5, 0.5[\times]-1, 0[$ and with the source term $s = 0$. Following Cascavita [7], one can note that Equation (14) fulfils the Signorini boundary conditions at the top of the domain $\Gamma_s = [-0.5, 0.5] \times \{0\}$. The transition between the constraint $u = 0$ and $\partial_{\mathbf{n}} u = 0$ occurs at

Table 2. Order of accuracy: solution Equation (14) $k = 11$.

	2th-FD		4th-CS		6th-CS	
h	error	rate	error	rate	error	rate
0.0625	1.081e−02		5.802e−05		2.106e−07	
0.0312	3.054e−03	1.823	3.917e−06	3.889	3.978e−09	5.727
0.0156	8.089e−04	1.917	2.541e−07	3.946	8.496e−11	5.549
0.0078	2.080e−04	1.959	1.617e−08	3.974	1.850e−12	5.521

Table 3. Order of accuracy: solution Equation (14) $k = 7$.

	2th-FD		4th-CS		6th-CS	
h	error	rate	error	rate	error	rate
0.0625	3.396e−03	–	4.525e−06	–	1.494e−06	–
0.0312	9.679e−04	1.811	4.614e−07	3.294	1.317e−07	3.503
0.0156	2.630e−04	1.880	4.344e−08	3.409	1.173e−08	3.490
0.0078	6.947e−05	1.921	3.973e−09	3.450	1.040e−09	3.495

Table 4. Order of accuracy: solution Equation (14) $k = 4$.

	2th-FD		4th-CS		6th-CS	
h	error	rate	error	rate	error	rate
0.0625	4.729e−04	–	1.590e−04	–	1.841e−04	–
0.0312	1.730e−04	1.450	5.725e−05	1.473	6.796e−05	1.438
0.0156	6.276e−05	1.463	2.043e−05	1.487	2.463e−05	1.464
0.0078	2.247e−05	1.482	7.255e−06	1.493	8.818e−06	1.482

the node $(0, 0)$. Here, we consider three solutions obtained with $k = 3, 7$ and $k = 11$, leading to a set of solutions with increasing differentiability.

The $L2$-norm of the numerical error ϵ_h is computed for several meshes of size h. The order of accuracy is estimated by $p = \log(\epsilon_{2h}/\epsilon_h)/\log 2$. The results for $k = 11$ are reported in the Table 2 for the fourth and sixth-order compact schemes, and the centred second-order finite difference scheme. The order of accuracy matches the expected accuracy for the three discretizations. The same numerical experiments are performed with the exact solution Equation (14) $k = 7$ and $k = 3$. The numerical errors are reported in Tables 3 and 4. Obviously, the order of accuracy is worsen for a less smooth solution. Hence, only 3.5 order is noted for the fourth and the six-order schemes. The exact solution Equation (14) $k = 3$ leads to a stall of the accuracy to 1.5 for the three discretizations. This feature has been noted in the work of Cascavita [7], for which a similar Signorini problem is addressed with the Hybrid High-Order discretization.

3.2. The Signorini problem in a semi-annular domain

Additional numerical experiments are performed to assess the equivalence between the projection iterative method and the present formulation for large c coefficients. Thus, the Signorini problem in a semi-annular domain [13,18] is considered. This problem admits an exact solution with a non-homogeneous Signorini boundary condition, which also contributes to a more general validation. The notations of this Signorini problem are:

$$\begin{cases} -\nabla \cdot \nabla u = 0 & \text{in } \Omega \\ u = u_{ex} & \text{on } \Gamma_d \\ u \geq \phi, \partial_{\mathbf{n}} u \geq \varphi \quad (u - \phi)(\partial_{\mathbf{n}} u - \varphi) = 0 & \text{on } \Gamma_s \end{cases} \tag{15}$$

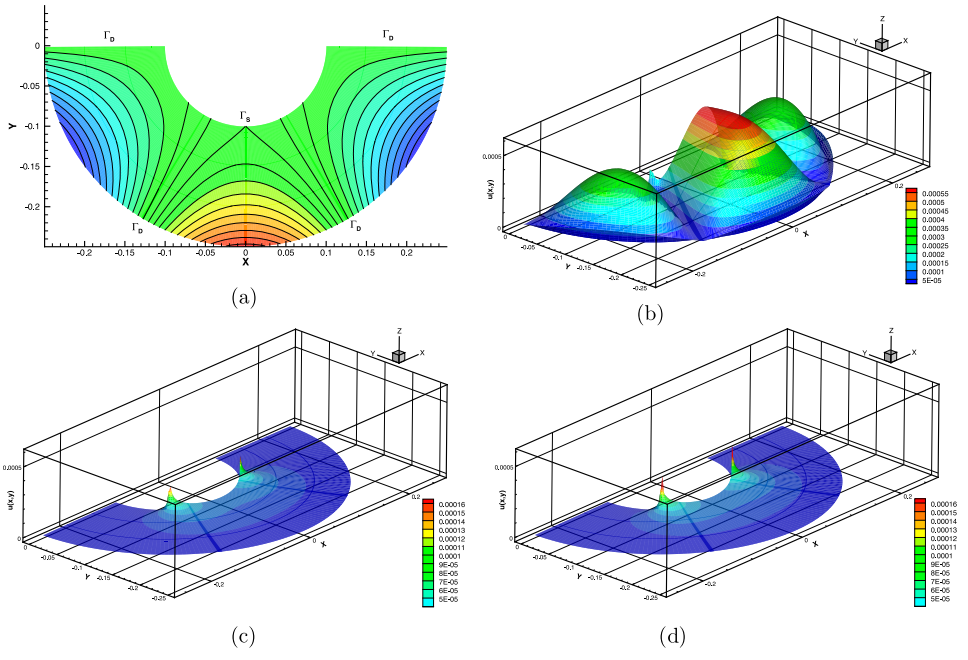

Figure 1. The Signorini problem in a semi-annular domain: solution and numerical errors. (a) Exact solution, (b) Second-order, (c) Fourth-order, (d) Sixth-order.

with Ω the semi-annular domain of inner radius 0.1 and outer radius 0.25 [see Figure 1(a)]. The Signorini boundary conditions are prescribed at the inner radius boundary Γ_s, and the Dirichlet at the remaining boundary Γ_d. The expressions for the solution u, the boundary values φ and φ are detailed in [23] and are not reported here. Figure 1(a) presents the solution on the semi-annular domain. The switching points that feature the transition between the conditions $u = \phi$ and $\partial_{\mathbf{n}} u = \varphi$ are located at the angles $\pi/4$ and $3\pi/4$. Figure 1(b) shows the numerical error of the second-order discretization with a mesh of size 64. We can note that the maximum error is located inside the computational domain. At the switching points, two peaks of the numerical error can be observed. This former as a lower level than the discretization error inside the domain. Figures 1(c) and 1(d) present the numerical error resulting from the fourth and the sixth-order discretizations. The numerical error is of high value at the switching points with a local peak. Inside the domain, the high-accuracy of the compact scheme discretizations provides a low level of numerical error, in comparison with second-order finite differences. The numerical error for a similar test case is reported in [14]. It can be noted that the present fourth and six-order discretizations compares favourably with the element-free Galerkin method[14]. In the following, we focus on the history of the convergence of the sequence Equation (10). The L2-norm of the difference between two iterations is monitored for the projection iterative method, the switching method and the present algorithm (A2). The tests are carried out on a mesh of size 64, using the second-order FD scheme and with the threshold of the inner loop set to $\epsilon_i = 10^{-10}$. Such a threshold means that the linear problem is solved at each non-linear iteration. The iteration history is presented in the Figure 2. As previously noted by Zhang [23], we find that the number of iterations used with the projection iterative method decreases for the largest values of c. It is noted that the convergence curve for the proposed formulation Equation (10) is perfectly aligned with those of the projection iterative method with $c = 10^7$ and with the switching method. These numerical experiments support the equivalence between the projection iterative method and the present algorithm, which avoids the parameter $c > 0$.

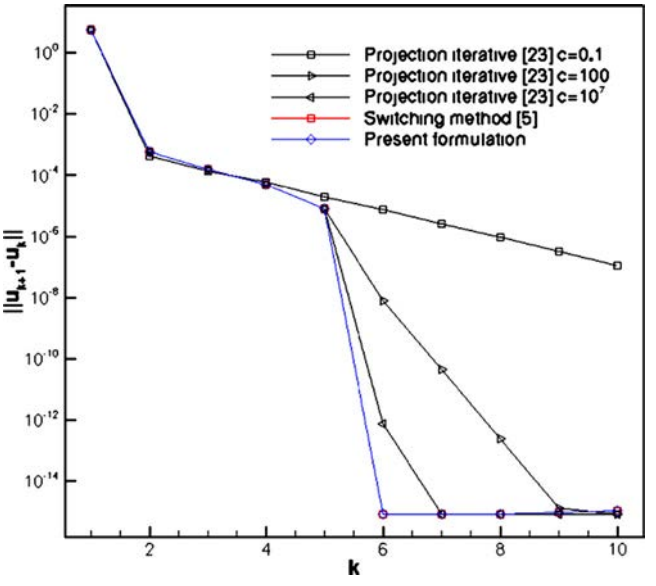

Figure 2. Iteration history of the relative difference $\delta_k = \|u^k - u^{k-1}\|_2$.

3.3. The electropainting problem

Additional numerical experiments are performed to discuss the iterative strategy. Indeed, as mentioned in the Section 2.4 one can look at the meaning of merging the linear with the non-linear loop. To this purpose, the electropainting problem is considered. This Signorini problem [13] is defined as:

$$\begin{cases} -\nabla \cdot \nabla u = 0 & \text{in } \Omega = (0, 0.5) \times (0, 1) \\ u = 1 & \text{on } \Gamma_d = [0, 0.5] \times \{0\} \\ \partial_{\mathbf{n}} u = 0 & \text{on } \Gamma_n = \{0\} \times [0, 1] \\ u \geq 0,\ \partial_{\mathbf{n}} u \geq \epsilon,\ u(\partial_{\mathbf{n}} u + \epsilon) = 0 & \text{on } \Gamma_s = [0, 1] \times \{1\} \cup \{0.5\} \times [0, 1] \end{cases} \tag{16}$$

The computational domain and the boundary conditions are plotted in Figure 3(a).

The solution is first computed on a uniform mesh of size 128×64 using the fourth-order accurate discretization and the present algorithm (A2). The solution is obtained by setting the parameter $\epsilon = 0.55$. The value of u at the Signorini boundary is reported in Figure 3(b), using the arc-length s as abscissa. The results are compared with the solution computed with the Meshless Projection Iterative Method (MPIM) [13]. Using this specific value of ϵ, the solution exhibits a plateau at the vicinity to the top-right corner $s = 1.5$. The results agree the recent literature [13], and give an additional argument for the validation of the present method. In the remaining part of the section, we address the effect of the stopping criterion ϵ_i to the efficiency of the algorithm (A2). The efficiency is measured by the total number of PCR iterations n_{pcr}^{tot} to achieve the non-linear stopping criterion $\epsilon_o = \|\delta u\| < 10^{-12}$. The Figure 4 represents n_{pcr}^{tot} with respect to the stopping criterion ϵ_{inner}. According to Figure 4, a decrease in ϵ_i value leads to an increase of n_{pcr}^{tot}, and consequently to the computational cost. Specifically, it is observed that the lower values of ϵ_i minimize the total number of PCR iterations. In this case, only one PCR iteration per non-linear iteration is necessary to achieve the convergence, which leads to 12, 24 and 29 PCR iterations per second for the fourth and sixth-order discretization. The rise of the iterations number with respect to the order of accuracy can be explained by the lower-order preconditioning approach [1]. This shows that the linear and the non-linear loops can be merged. It is should be mentioned that this property contributes to the computational efficiency of the overall method since exact solution of the linearized problem is not required.

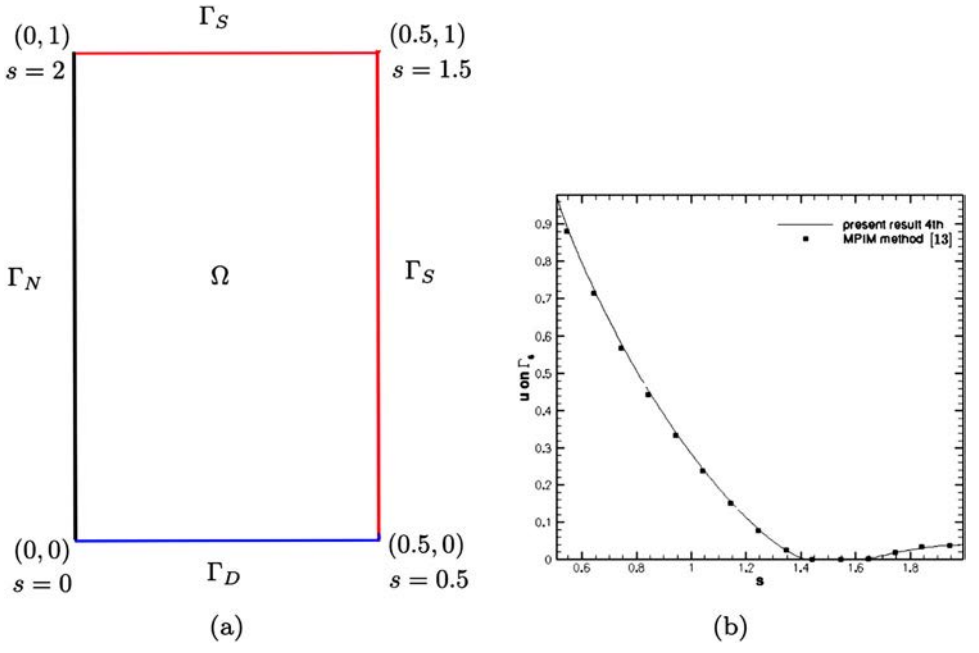

Figure 3. The electropainting problem: sketch of the configuration (a), solution u on Γ_s for $\epsilon = 0.55$ (b).

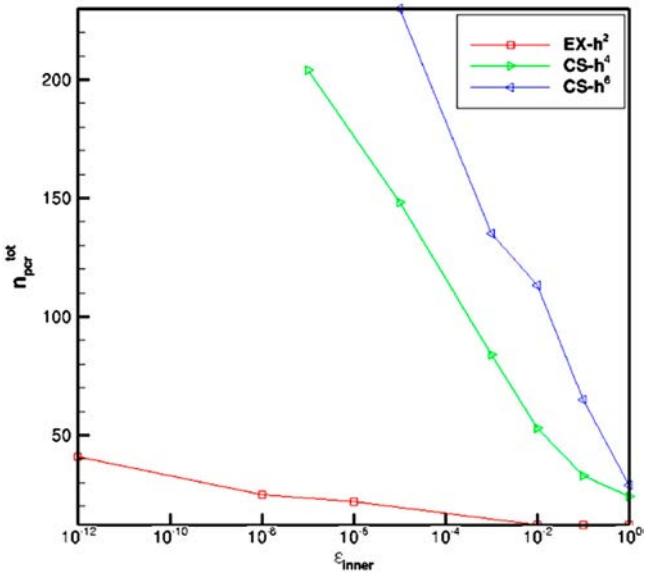

Figure 4. Total number of PCR iterations versus the inner loop stopping criterion ϵ_i.

4. Conclusion

A new methodology for the solutions of Signorini problems has been provided. Specifically two points have been discussed. First, a heuristic argumentation of the equivalence between the projection iterative method and the switching method is proposed. Indeed, the switching method can be viewed as a limit case of the projection iterative method. Nevertheless, a slight discrepancy with the definition of the active set nodes has been outlined. Then, the fourth and sixth-order compact scheme finite

difference solvers have been deviced to deal with the non-linear Signorini problem. The difficulties associated to the implicit formalism inherent to compact scheme finite differences has been overcome by making use of a lower-order preconditioning method.

Several numerical experiments are performed to validate the proposed method and to outline its efficiency. The fourth and sixth-order accuracy have been demonstrated on a Signorini problem having an exact solution while selecting the differentiability. The effective accuracy deterioration associated to the solution regularity is clearly pointed out. Also, it is shown that the accuracy deterioration is associated to the Dirichlet/Neumann singularity which occurs with the Signorini boundary conditions. The numerical experiments also focus on the convergence rate of the proposed method. Thus, numerical solutions of the electropaint problem has allowed a detailed investigation of the iterative method convergence. It has been shown that the linear loop can be merged with the outer non-linear loop, which leads us to conclude that the proposed method is efficient. This should allows us to consider an extension of this to three-dimensional configurations.

Because of the loss of accuracy remains confined to the Dirichlet/Neumann singularity, the present work opens up some perspectives on high-order discretization for Signorini problems. For instance, approaches based on singularity removing could be investigated to fully benefit from the high-accuracy of the compact scheme discretizations.

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Disclosure statement

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