Capped Accumulated Return Call with Volatility Surface

A pricing model for capped-accumulated-return-call (CARC) with volatility surface is presented. Proprietary approaches to interpreting volatility surface are employed during pricing. To accelerate the convergence when low discrepancy sequences are used in Monte Carlo simulation (Quasi-Monte Carlo simulation), the Brownian Bridge Path Construction has been employed in some CARC transactions.

Let $S(t)$ be a price process of a given underlying asset, $\{t_0 < t_1 < \cdots < t_n\}$ be a set of reset dates and $T \geq t_n$ be a payoff settlement date. The capped-accumulated-return call (CARC) with the underlying *S* is a European type derivative security whose matured payoff at the settlement date is given by

$$N \times \max\left\{0, \prod_{i=1}^{n}(1+\min(R_{cap}, R_i)) - (1+R_f)\right\} \tag{1}$$

where R_{cap} is the capped-return-rate for each period, R_f is the global floor of the return rate, *N* is the notional principal, and

$$R_i = \frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}, i = 1, \cdots, n. \tag{2}$$

Let *t* be the current value date, then the current value of this CARC can be written as

$$df(t,T) \times N \times E_t\left[\max\left\{0, \prod_{i=1}^{n}(1+\min(R_{cap}, R_i)) - (1+R_f)\right\}\right] \quad (3)$$

where $df(t,T)$ is the discounting factor at the value date. The above formula is in a world that is forward risk-neutral with respect to a specific currency C_p. As a result, the notional principal *N* is measured in the currency C_p, and the discounting factor should be calculated by a C_p zero curve (ref. https://finpricing.com/lib/IrBasisCurve.html) given at the value date. If the underlying asset is measured in another currency C_U, the governing price dynamics of the underlying asset in the risk-neutral world of C_p should be written as

$$dS_t = (r_U - q - \rho\sigma_x\sigma_s)S_t dt + \sigma_s S_t dW_t \quad (4)$$

where r_U is the short rate of C_U, q is the dividend yield of the asset, σ_s is the volatility of the asset price, ρ is the correlation coefficient between the asset price and the cross-currency exchange rate, σ_x is the volatility of the exchange price, and W_t is the Wiener process. All these parameters are assumed deterministic. Monte Carlo simulation can be used to price CARC.

As is known, quasi-Monte Carlo methods provides a way to improve the accuracy and reliability of Monte Carlo simulation by using deterministic sequences known as quasi-random sequences. This results in better convergence and deterministic error bounds (Joy, Boyle and Tan, 1996). There are a few techniques aimed at speeding up quasi-Monte Carlo, and the Brownian bridge path construction is one of them. It attempts to use the best coordinates of each point to determine most of the structure of a path.

Relying on properties of Gaussian Markov processes, particularly the following property

$$Cov(W(t_i), W(t_j)) = \min(t_i, t_j) \quad (5)$$

where $W(t)$ is the Wiener process, t_i and t_j are any two time-steps, for a quasi-Monte Carlo problem that involves $N > 1$ time steps, the Brownian bridge path construction first generates W_T, then using this value, and $W_0 = 0$, it generates $W_{T/2}$. It generates $W_{T/4}$ using W_0 and $W_{T/2}$, and it generates $W_{3T/4}$ using $W_{T/2}$ and W_T. The construction proceeds recursively filling in the mid points of the subintervals. Thus, the discretely sampled Brownian path is generated by determining its values at $T, T/2, T/4, 3T/4, \cdots, (d-1)T/d$ according to

$$
\begin{aligned}
W_T &= \sqrt{T} z_1 \\
W_{T/2} &= \frac{1}{2} W_T + \frac{\sqrt{T}}{2} z_2 \\
W_{T/4} &= \frac{1}{2} W_{T/2} + \frac{\sqrt{2T}}{4} z_3 \\
W_{3T/4} &= \frac{1}{2}(W_{T/2} + W_T) + \frac{\sqrt{2T}}{4} z_4 \\
&\vdots \\
W_{(d-1)T/d} &= \frac{1}{2}(W_{(d-2)T/d} + W_T) + \sqrt{\frac{T}{2d}}\, z_d, \qquad (6)
\end{aligned}
$$

where z_i, $i = 1, \cdots, d$ are standard normal random numbers. The Brownian bridge can be generalized to include unequal length intervals. For $t_{j+1} = t_j + \Delta t, j = 0, \cdots, d-1$, $\Delta t = T/d$, we can simulate a future value $W_{t_k}, k > j$, (given the value W_{t_j}) according to

$$W_{t_k} = W_{t_j} + \sqrt{(k-j)\Delta t}\, z. \qquad (7)$$

We can simulate W_{t_i} at any intermediate point $t_j < t_i < t_k$ (given values of W_{t_j} and W_{t_k}) according to the Brownian bridge formula

$$W_{t_i} = (1-\gamma)W_{t_j} + \gamma W_{t_k} + \sqrt{\gamma(1-\gamma)(k-j)\Delta t}z \quad (8)$$

where $\gamma = (i-j)/(k-j)$.

A new representation of the volatility skew is provided. To use this new representation, the user must input "STRIKE_REPRESENTATION PERCENTAGE" in the token file. Then, the ATM strike is always assumed to be 100. The first quantity in the volatility skew still signifies the stock price when the volatility skew was built.

The model will interpret this skew as follows. At the time 1 year from the date when the volatility skew was built, the 10% ITM volatility is 0.4, with the strike level being 90*1050/100=945. The ATM volatility is 0.38 with the strike level being 1050*100/100=1050. The 10% OTM volatility is 0.36 with the strike level being 110*1050/100=1155. At the time 2.0 years, a similar interpretation can be obtained.

As to the interpolation of the volatility surface in pricing CARC, for each reset period, the model needs vol_spot and vol_strike to determine the volatility to use for that period. For the current period, the vol_spot = current_stock_price at the value date, vol_strike = (1.0+cap) *l ast_reset_price. For all future periods, the vol_spo t= base_spot of the volatility skew, and the vol_strike = (1.0+cap) * base_spot. The base_spot is the stock price when the volatility skew was built, such as 1050 in the above example. Based on the value of vol_strike/vol_spot for each reset period, the volatility for that period is obtained by linear interpolating (or flat extrapolating) time in the volatility skew representation, followed by moneyness interpolation (or flat extrapolation).

Beside the above conventional interpretation / interpolation, we also use a proprietary approach to interpreting volatility skew for instruments with capped returns, such as CARC.