Quantum Waves, Entropy Density and Conservation?

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Quantum free particles are represented by exp(ipx) which we argue is a dynamical probability which maps into a static probability exp(-ipx)exp(ipx) i.e. the usual spatial density. Shannon's entropy = - Sum over i P(i) ln(P(i)) is often associated with the static probability P(x)=W*(x)W(x) (1), but in (2) we argue that one may introduce a different probability C1 Real(W(x)). In this note we consider both in different examples. In particular we argue that the two different probability definitions are ultimately linked with momentum conservation. The form W*(x)W(x) introduces local nonconstant spatial densities if there are mixed exp(ip1x) and exp(ip2x)'s within W(x). If there are not, this form W*(x)W(x) does not distinguish space i.e. exp(ipx)exp(ipx+ib) is still a constant. Nevertheless interference may still occur which may be linked with entropy.

Thus we first examine the case of exp(ipx) + exp(ipx + ib) where b is a constant phase shift. W*(x)W(x)=constant which does not lead to a structure different from exp(-ipx)exp(ipx), yet W(x)=C2{exp(ipx) + exp(ipx+ib)} interfere. On the other hand |cos(px)| and | cos(px)(1+cos(b)) - sin(px)sin(b) | are not the same distributions (one is a parity eigenstate, the other is not) and so we argue there is a different local entropy density. Thus introducing extra information i.e. b changes local density and entropy density if one adjusts the definition of probability used.

We next consider W(x)= exp(ip1x)+exp(ip2x). One may also use the probability C3 Real|W(x)|, but here we are interested in an association with conservation of momentum. For a given exp(ipx), exp(-ipx)exp(ipx)=1. We argue that this represents a steady state picture i.e. a kind of equilibrium in which all x points are treated the same. W*(x)W(x) breaks this equilibrium scenario by introducing a term 2 cos((p1-p2)x). Thus both local spatial density and local entropy (based on W*W) are different and do not represent the "equilibrium" scenario. We further argue that momentum conservation may be linked with this equilibrium type of density and entropy density at least in the case of the transition amplitude <p1 V(x) p2>. V(x)exp(ipx) yields a sum of exp(ipx)'s, but only exp(ip1x) survives the integral over dx. The others create spatial densities which are not constant and so represent local entropy densities which do not treat all x points the same i.e .is not an equilibrium one.

Thus we try to use Shannon's entropy with two different probability definitions to analyse different quantum interference scenarios and argue that momentum conservation (and possibly others) may be linked with local entropy density in some situations.

Shannon's Entropy

Shannon's spatial entropy S = - Sum over i P(i) ln(P(i)) where P(i) is a probability is sometimes applied to quantum spatial density (1) to define a spatial entropy density:

Sx density = - W*(x)W(x) ln{ W*(x)W(x)]   ((1a))

Given a momentum eigenstate exp(ipx)/sqrt(L) this yields a constant density and entropy density independent of p.

We argue, however, that p is still an important part of entropy density in the following manner. exp(ipx) represents a wavelength of hbar/p which represents a region of space in which there is uncertainty of the particle's position even though its momentum is exactly defined. In a previous note (2) we suggested that in some problems it may be more useful to consider an entropy density based on Real(W(x)) i.e.

Sx density new = - | Real(W(x))| ln { | Real(W(x))| } ((1b))

We use both of these forms in this note.

exp(ipx) and exp(ipx + ib) Phase Shifts with the Same Momentum

In this example we wish to investigate how extra information, namely a constant phase shift b, manifests itself in a change in entropy density using |Real(W(x))| as the probability in Shannon's entropy expression.

Consider two waves exp(ipx +ib) and exp(ipx) i.e. the second is phase shifted. Each by itself has the same constant spatial density, but waves represent probabilities and add at the wave level. (exp(ipx) we argue is a dynamic probability.) Given that there is a phase shift there is extra information in the system and if b is not an integer multiple of 2*3.14 the two spatial uncertainty regions do not overlap leading to more spatial uncertainty in the location of the particle. This all exists at the wave level. To try to quantity using ((1)) one may first write:

W*(x)W(x) = C1 (exp(ipx) + exp(ipx+ib))* (exp(ipx) + exp(ipx+ib))

= C1 2 (1+cos(b)) ((2))

((2)) is thus still a constant even though there is an interference pattern in space dependent on b.

Why is W*(x)W(x) still constant? We argue that W*(x)W(x) is linked to momentum conservation. If W(x) only contains exp(ipx) then exp(-ipx)exp(ipx) leads to no spatial density, while mixed p values exp(-ip1x)exp(ip2x) do. Thus if one has two beams with the same p value, then W*(x)W(x) is not the probability to use if one wishes to examine changes in space. It is possible for an exp(ipx) to interfere with an exp(ipx+ib) and there should be an entropy density value to quantify this locally.

One may, however, analyse this at a wave level using Probability = |Real(W(x))| i.e. cos(px) and cos(px+b). cos(px) is periodic (with period 2*3.14) in space as is cos(px+b). Thus for either examined by itself one has spatial uncertainty in repeated hbar/p regions.
B=cos(px)+cos(px+b)= cos(px)[1+cos(b)] - sin(px)sin(b) has a more complicated spatial structure and is of mixed parity. The zeros of cos(px) are at x=3.14/2, 3*3.14/2 etc while the zeros of B are at:

$[1+\cos(b)] / \sin(b) = \sin(px)/\cos(px)$ ((3))

Given the nature of tan(px) if xo is a solution of ((3)), then pxo+n2*3.14 is also a solution, i.e. one has periodic units of 2*3.14 just as with cos(px), but the distribution itself within these units is different if one only considers |Real{ W(x)}| and not W*(x)W(x). Thus:

Sx density new = Real(W(x)) ln | Real(W(x))| ((4))

Then exp(ipx) and exp(ipx)+exp(ipx+ib) yield different entropy densities. We argue this is not a problem because the two systems have different information i.e. the later contains the extra b value. Thus we associated interference with a change in entropy density due to extra possible information entering the picture which may lead to a different uncertainty.

For example a coin has only two pieces of possible information (heads or tails), but a die has 6. The extra possible information leads to lower probabilities i.e. 1/6 and greater entropy. In other words possible information may lead to a change in entropy. We argue that a similar situation appears here. The change in entropy density based on Real(W(x)) is linked to additional information which affects uncertainty in the particle's position even though its momentum is known exactly.

Interference, Entropy and Conservation

Next we consider the case of exp(ip1 x) and exp(ip2 x) i.e. two different momenta. These have different wavelengths hbar/p1 and hbar/p2. If combined in a OR situation one has:

W(x) =C1 { exp(ip1x) + exp(ip2x) } ((5))

In this case one may compare cos(p1x) and cos(p2x) with cos((p1+p2) x) to see that the combined spatial absolute value is different. One may also create a "wave based" entropy density.

In this particular example, however, ((2)) may be of use.

W*(x)W(x) = C1*C1 { 2 + 2 cos((p1-p2) x) } ((6))

Thus instead of having a constant spatial density at all x, one has a more complicated local pattern indicative of a different local entropy density. This follows for all different p1,p2 values. If p1=p2, one obtains a constant spatial density and constant entropy density which treats all x points as the same (i.e. produces a global equilibrium type of scenario). This we suggest is linked to conservation of momentum i.e. one has p and -p. To see this more clearly, consider the transition amplitude:

<p1 | V(x) | p2> ((7))

V(x) combines with exp(ip2x) to create various exp(i pn x) values, but only the value exp(ip1 x) yields a nonzero value for ((7)) when integrated over all x. This leads to a constant “spatial type” density which treats all x points the same. This is again like an equilibrium type scenario (usually associated with maximum entropy). There is no preference for one x point over another and this is identical to conserved momentum. Thus we argue that a spatial density and its associated entropy density may indicate a lack of momentum conservation. Only the special case of a constant density and entropy density in this example yields an equilibrium/steady state result.

We have argued in previous notes that the existence of a wave associated with a quantum particle (or photon) may be linked to some kind of conservation law. Here we try to link this further with entropy density. In this case the conservation property is linked with orthonormality which in turn requires function*(x) multiplied by function(x) or a W*(x)W(x) spatial density instead of considering only |Real(W(x))| as a kind of density. The conservation is associated with a constant density (and entropy density) in space treating all x points the same.

Conclusion

In conclusion we argue that interference in quantum mechanics, which is linked to different spatial distributions, may be further linked with entropy. This may, however, require different definitions of probability in Shannon’s entropy expression - Sum over i P(i) ln{P(i)}. For example, if one consider exp(ipx)+exp(ipx+ib), then the usual quantum spatial density W*(x)W(x) is a constant just as it is for exp(ipx). Yet exp(ipx) and exp(ipx + ib) interfere leading to a different spatial distribution i.e. |cos(px)| versus | cos(px)(1+sinb) - sin(px)cosb | with different parity scenarios. Thus we argue that the extra information b leads to a different local entropy (using |Real(W(x))| as probability).

As a second example, we consider the case of W(x)= C1{ exp(ip1x) + exp(ip2x) }. This may be analyzed using the probability |Real(W(x))| as well, but in this case we wish to link entropy with conservation, namely that of momentum, so we use the form W*(x)W(x) because different exp(ipx)s are orthonormal. We argue that if one has a single momentum, the density is exp(-ipx)exp(ipx) = constant which treats all x points the same i.e. represents a kind of steady state or equilibrium scenario associated with a single p value i.e. a conserved p.

For p1 and p2 (i.e. no conservation) one has an extra density term 2cos((p1-p2)x) which suggests a different density and local entropy density which does not seem to be the equilibrium one. One may apply these ideas to the transition amplitude: <p1 V(x) p2>. V(x) exp(ip2x) leads to a sum of exp(ipx)s, but all but exp(ip1 x) lead to densities and hence entropy densities which do not represent an equilibrium scenario. Thus conservation of momentum (i.e. picking out p1) seems to be linked to an equilibrium type of entropy density. Thus we stress the importance of considering the entropy associated with interference even if this quantity needs to be evaluated using different probabilities in different examples.

References

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