Credit Pricing Model Calibration

A calibration procedures of the Default Correlation model is presented. There are two principal modifications. The first is to change the manner in which asset correlations are converted into default correlations, the second is a small change in the algorithm by which the probability equations of the model are solved. These changes are considered appropriate, and are necessary for the model to be considered robust enough to underpin the structuring and trading of complex credit contingent instruments.

The first modification involves the conversion of asset correlations into default correlations. Conversion of asset correlations into default correlations in the original model is carried out by equating the joint default probability of the bivariate normal copula between two names at a specific time horizon, i.e., if $F_{ij}^{(A)}(T,T,\rho_{ij}^{A})$ is the joint default probability for time horizon T with asset correlation ρ_{ij}^{A} and $F_{ij}(T,T,\rho_{ij}^{D})$ is the joint probability of default for this model with default correlation ρ_{ij}^{D} then given that

$$F_{ij}^{(A)}(t_1,t_2,\rho_{ij}^{A}) = \Phi\left(\Phi^{-1}(F_i(t_1)),\Phi^{-1}(F_j(t_2)),\rho_{ij}^{A}\right) \quad (1)$$

and

$$F_{ij}(t_1,t_2,\rho_{ij}^{D}) = 1 - S_i(t_1) - S_j(t_2) + S_i(t_1)S_j(t_2)\exp\left\{\int_0^{\min(t_1,t_2)} h_{ij}(s)ds\right\} \quad (2)$$

where

$$S_i(t) = e^{-\int_0^t h_i(s)ds} \quad (3)$$

we can solve for

$$\rho_{ij}^{D} = \frac{-k}{1+k}$$

where

$$k = \frac{\log\left(\dfrac{F_{12}^{(A)}(t_1, t_2, \rho_{ij}^{A}) - 1 + S_i(t_1) + S_j(t_2)}{S_i(t_1)S_j(t_2)}\right)}{\log\left(S_i(\min(t_1, t_2))S_j(\min(t_1, t_2))\right)}. \tag{4}$$

Previously, this conversion was carried out at a fixed time horizon T even though the credit curves $h_i(t)$ can vary with time. Default correlations consistent with the model must lie in the range $0 \le \rho_{ij}^{D}(t) \le \hat{\rho}_{ij}^{D}(t)$, where

$$\hat{\rho}_{ij}^{D}(t) = \frac{\min\left(h_i(t), h_j(t)\right)}{\max\left(h_i(t), h_j(t)\right)} \tag{5}$$

Default correlations generated by this conversion procedure can easily lie outside the allowed range for $t > T$.

The modified conversion procedure simply re-converts the (constant) asset correlation into a default correlation each time there is a change in the value of one of the credit curves. This assures that the default correlations are always within the allowed range. It also has the benefit of ensuring that the asset correlation is held constant over time.

The second modification involves a modification of the procedure used for solving the probability equations of the model. The solution to these equations can be written

$$\begin{aligned} \lambda_i &= h_i \sum_{k=1}^{i-1} \lambda_k p_{ki} \\ p_{ij} &= \frac{1}{\lambda_i}\left(h_{ij} - \sum_{k=1}^{i-1} \lambda_k p_{ki} p_{kj}\right) \qquad i < j \\ p_{ij} &= 0 \qquad i > j \end{aligned} \tag{6}$$

and in the original model, any computed $\lambda_i < 0$, or $p_{ij} > 1$ or $p_{ij} < 0$ caused a fatal error. The new procedure for computing according to equation (6) has been dubbed the "ostrich" algorithm. This is so named because we will simply pretend that the problem does not exist. In the course of constructing the triangular solution using equation (6), we adjust the results as follows:

If $\lambda_i < 0$ then $\lambda_i = 0$.

If $p_{ij} < 0$ then $p_{ij} = 0$

If $p_{ij} > 1$ then $p_{ij} = 1$

Then continue constructing the solution using the modified values for the remainder of the computation.

This simple procedure has the effect of normalizing the solution, allowing the solution to recover. This modification is accompanied by an error-checking condition. If the solution obtained by the ostrich algorithm does not match the inputs to within a specified tolerance, a failure is reported.

The testing process was carried out in two stages. First, the independent implementation of the model which was constructed for the testing of the original model outlined in was modified.

The second phase of the process was carried out using the previous version of the model independently modified to conform to the new specifications by the author to make further tests. These tests were conducted with both the new implementation of the model and the previous version of the model.

In order to further investigate the differences between the modified model and the original model, we have constructed our own modified version of the model. This version is equipped with an error reporting facility, which reports the error between the inputs and the recomputation of those inputs from the solution to the probability equations. The error is computed as the Frobenius norm of the differences in the inputs and the inputs as reconstructed from the solution as given by equation (6) with the ostrich algorithm.

Several test cases were developed. All involved the pricing of a first to default basket of ten names (see https://finpricing.com/lib/EqBarrier.html. The ten credit curves were all created from the following simple formula. The credit spread s from which the credit curves are built are created from the formula

$$s(t_i) = \alpha\left(\beta s(t_{i-1}) + s(t_0)\right) \tag{7}$$

thus α is an overall multiplicative factor, and β is a slope parameter. Changing α and β thus allows us to create a wide variety of curves from a single set of $s(t_0)$ values.

The inputs to the model consist of a set of possibly time dependent hazard rates h_i which can be considered the unconditional intensities of Poisson processes which govern the arrival of default events which involve the $i-$th defaultable entity, which we will refer to as names. Thus we can write that

$$h_i = \sum_{i \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} \tag{1}$$

where the notation $i \in \Omega \subseteq N$ indicates that the sum is to be taken over all subsets Ω of N which contain the name i. For the moment we leave $h_{\Omega, \overline{N-\Omega}}$ unspecified.

We can also define in a similar manner a quantity h_{ij} which is the unconditional intensity of a Poisson process which governs the arrival rate of default events involving simultaneous default of name i and name j, given as

$$h_{ij} = \sum_{i,j \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} \tag{2}$$

where in this case the notation $i, j \in \Omega \subseteq N$ indicates that the sum is to be taken over all subsets Ω of N which include both name i and name j.

A third quantity which we will define is k_{ij} , which is the unconditional intensity of a Poisson process which governs the arrival rate of default events involving name i, name j or simultaneously names i and j. This will be given as

$$k_{ij} = \sum_{i, \not{j} \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} + \sum_{j, \not{i} \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} + \sum_{i, j \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} \tag{3}$$

where the notation $i, \not{j} \in \Omega \subseteq N$ indicates summation over all subsets Ω of N which include name i but do not include name j. Clearly k_{ij} can be rewritten

$$k_{ij} = \sum_{i \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} + \sum_{j \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} - \sum_{i,j \in \Omega \subseteq N} h_{\Omega, \overline{N-\Omega}} \tag{4}$$

or more simply

$$k_{ij} = h_i + h_j - h_{ij}. \tag{4b}$$

We are now in a position to define default correlation as the probability for both name i and name j to default if either of them defaults, which will be given by the conditional probability

$$\rho_{ij} = \frac{h_{ij}}{k_{ij}} \tag{5}$$

or

$$\rho_{ij} = \frac{h_{ij}}{h_i + h_j - h_{ij}}. \tag{6}$$

This definition of ρ_{ij} then implies that

$$h_{ij} = \frac{\rho_{ij}}{1 + \rho_{ij}} \left(h_i + h_j \right). \tag{7}$$

Given this definition of default correlation ρ, the h_{ij} are then given in terms of the inputs h_i, and the ρ_{ij}, and we have a system of equations which we can then solve for the $h_{\Omega,\overline{N-\Omega}}$, providing a means to model credit contingent structures.

Working directly in terms of the intensities of the Poisson processes for the default of all subsets Ω of N entities may be unwieldy. Therefore we seek to find a way to reduce the complexity of the problem in some way. This can be done by introducing the idea of primitive and joint events. We therefore define a set of primitive event arrival rates λ_i and a set of joint event arrival rates p_{ij}, which are related to $h_{\Omega,\overline{N-\Omega}}$ as follows:

$$h_{\Omega,\overline{N-\Omega}} = \sum_{i \in \Omega} \left\{ \lambda_i \prod_{j \in \Omega - i} p_{ij} \prod_{k \in N-\Omega} \left(1 - p_{ik} \right) \right\} \tag{8}$$

which determines the interpretation that λ_i is the intensity of a Poisson process that governs the arrival rate of default events involving name i, and that p_{ij} is the probability of simultaneous default of name j conditional on name j not having already defaulted.

The introduction of λ_i, p_{ij} according to equation (8) can be shown to lead to the following equations which we must solve:

$$h_i = \sum_{j=1}^{N} \lambda_j p_{ji} \tag{9}$$

and

$$h_{ij} = \frac{\rho_{ij}}{1+\rho_{ij}}\left(h_i + h_j\right) = \sum_{k=1}^{N} \lambda_k p_{ki} p_{kj} \qquad i \neq j \tag{10}$$

where $p_{ii} = 1$, and where the $h_i(t)$ and the ρ_{ij} are the inputs to the model.

We must solve this system of equations subject to the constraints that $\lambda_i \geq 0$ and $0 \leq p_{ij} \leq 1$ for $i \neq j$. Equation (9) provides N constraints. From the structure of equation (10), we see that the symmetry of $h_{ij} = h_{ji}$ does not by itself impose any constraints on the p_{ij}, and thus we have an additional $N(N-1)/2$ constraints.

On the other hand, we are searching for N unknown quantities λ_i, and $N(N-1)$ unknowns $p_{ij}, i \neq j$. Therefore , we have $N(N+1)/2$ equations and N^2 unknowns, and therefore thesystem of equations is underdetermined. In order to have the number of equations equal the number of unknowns, we find that we must impose an additional $N(N-1)/2$ constraints.

In the existing model, the "triangular" ansatz, setting $p_{ij} = 0$ for $i > j$ is made, which permits solving for the λ_i and the p_{ij} explicitly. The triangular ansatz imposes the additional $N(N-1)/2$ constraints by setting $p_{ij} = 0$ for $i > j$. Note, however, that in the preceding analysis we have not taken into account the built-in constraints on the ranges of the solution. These represent an additional $N(N+1)/2$ constraints from $0 \leq p_{ij} \leq 1$ for $i \neq j$. The fact that $\lambda_i \leq h_i$, can be seen to follow from equation (9) when the ranges constraints on p_{ij} are applied, which leaves us to impose an additional N constraints that $\lambda_i \geq 0$.

On the face of it, it might seem that we are free to impose any default correlation we choose, and if we had defined default correlation in a manner analogous to statistical correlation, that would no doubt be the case. However, the assumptions and definitions of the present model will not permit this, as we will see. Recall from equation (6) the definition of default correlation ρ_{ij}:

$$\rho_{ij} = \frac{h_{ij}}{h_i + h_j - h_{ij}}$$

Suppose now that we have two names i and j with their respective hazard rates. Assume also that $h_i > h_j$, and let us write $h_j = \alpha h_i$ for some $0 \le \alpha \le 1$. Now, we can approximate $h_{ij} \approx \min(h_i, h_j)$ and then we have

$$\rho_{ij} \approx \frac{h_j}{h_i + h_j - h_j} = \frac{h_j}{h_i} = \alpha \,. \tag{6b}$$

We can see then that if the specified inputs are inconsistent with this limit, which is a consequence of the definition of default correlation in this model, we will not be able to find an appropriate solution to equations (9) and (10).

It is thus important therefore, when calibrating the model to maintain the best possible consistency across time, with the possibility that the maximal default correlation between two curves can vary with time.