Pricing Reciprocal Average Rate Forward

A reciprocal average rate forward is an forward contract whose matured payoff is the difference between the reciprocal of the arithmetic average of foreign exchange rates and the reciprocal of a strike. The average is calculated by those rates quoted with the market convention over a given set of observation dates. The forward strike is also quoted in the same convention. Clearly, a reciprocal average rate forward is a nonlinear forward contract, which means the contract matured payoff is a non-linear function of averaged rates.

The pricing model for the reciprocal average rate forward, in the view of a first order approximation, applies the same approach as in ordinary average rate forward. Pricing average rate related non-linear derivatives, a well-acceptable and effective with reasonable accuracy pricing model for the reciprocal average rate forward is not available yet.

We did some research of seeking a semi-close form solution for pricing non-linear average rate derivatives. This research has revealed satisfactory results for pricing linear reciprocal average rate derivatives (see https://finpricing.com/lib/EqAsian.html).

Let *A* be the arithmetic average of a number of correlated log-normal variables and *G* be the corresponding geometric average. Let *G* = *eB*, then *B* » N(*mB; vB*) or the density function of *G*, denoted by *fG*(¢), can be given as

$$f_G(y) = \frac{1}{\sqrt{2\pi v_B}y} \exp\left(-\frac{(\ln y - m_B)^2}{2v_B}\right) , \quad \forall y \in (0, \infty) .$$

Now, let *a* > 0 and *b* be two fixed real numbers. We try to calculate the following

$$\mathrm{E}\left[\frac{1}{aA+b}\right] ,$$

where *a* and *b* are such that the random variable [*aA* + *b*]¡1 has finite mean and variance. Since the distribution of *A* is generally unknown, this expectation may not be obtained easily. First, we have

$$
\begin{aligned}
\mathrm{E}\left[\frac{1}{aA+b}\right] &= \mathrm{E}\left[\mathrm{E}\left[\frac{1}{aA+b}\middle| G=y\right]\right] \\
&= \int_0^\infty \mathrm{E}\left[\frac{1}{aA+b}\middle| G=y\right]\cdot f_G(y)\,\mathrm{d}y \\
&= \int_0^\infty \mathrm{E}\left[\frac{1}{a(A-y)+(b+ay)}\middle| G=y\right]\cdot f_G(y)\,\mathrm{d}y\,.
\end{aligned}
$$

We assume that ln(*A*¡*y*) » N(*mA*(*y*)*; vA*(*y*)) where *mA*(*y*) and *vA*(*y*) > 0 are smooth functions with respect to *y 2* (0*;1*). Let *fAjy*(¢) be the conditional density function of *A* ¡ *y*, then we have

$$
f_{A|y}(x) = \frac{1}{\sqrt{2\pi v_A(y)}x}\exp\left(-\frac{(\ln x - m_A(y))^2}{2v_A(y)}\right)\ , \quad \forall x \in (0,\infty)\ .
$$

There is a trivial case where *A* = *G* surely. For example, it is the case when *A* is the arithmetic average of a single log-normal variable. In this case, (3) can be written as

$$
\mathrm{E}\left[\frac{1}{aA+b}\right] = \int_0^\infty \frac{1}{ay+b}\cdot f_G(y)\,\mathrm{d}y\ .
$$

Let us consider the non-trivial case where *A* > *G* almost surely. Thus,

$$
\begin{aligned}
&\mathrm{E}\left[\frac{1}{a(A-y)+(b+ay)}\middle| G=y\right] \\
=\ &\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{ax+ay+b}\exp\left[-\left(\frac{\ln x - m_A(y)}{\sqrt{2v_A(y)}}\right)^2\right]\mathrm{d}\left(\frac{\ln x - m_A(y)}{\sqrt{2v_A(y)}}\right)\ .
\end{aligned}
$$

Let us introduce

$$
H(x,y) = \frac{1}{a(x+y)+b}
$$

and

$$u_x = \frac{\ln x - m_A(y)}{\sqrt{2v_A(y)}} \quad \text{or} \quad x = x(u_x, y) = \exp\left[m_A(y) + \sqrt{2v_A(y)}u_x\right] .$$

After substituting (7) and (8) into (6), we have

$$\mathrm{E}\left[\frac{1}{a(A-y)+(b+ay)}\middle| G = y\right] = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_1(u_x, y)\cdot\exp(-u_x^2)\,\mathrm{d}u_x ,$$

where

$$H_1(u_x, y) = H(x, y)|_{x=x(u_x,y)} .$$

We may need that the function *H* is a well-behaved function with respect to (*ux; uy*) on R2 such as the double-integral exists and the inner integral in (11) is uniformly integrable with respect to *uy* on R. Then by using Gauss-Hermite quadrature, the integral (11) can be approximated with a high precision by