Supplementary Information for manuscript "Parametric separation of phase-locked and non-phase-locked activity"

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S1 Probability distribution estimation of A_r and B_r

We have shown that with the model described in equation(3), it not possible to calculate A_r, B_r at the trial level. But we were able to calculate first and second order moments. Here, in this section we present estimates of higher order moments for A_r and B_r.

A_r, B_r **higher order moment estimates:** Substituting A_r from equation(14) into (15), we have

$$B_r sin(\phi_r - \alpha) = R_r cos(\alpha) - I_r sin(\alpha). \tag{S1.1}$$

Raising the equation S1.1 to the j^{th} power, we have

$$B_r^j sin^j(\phi_r - \alpha) = (R_r cos(\alpha) - I_r sin(\alpha))^j. \tag{S1.2}$$

Since ϕ_r ranges from $-\pi$ to π, $\langle sin^j(\phi_r - \alpha)\rangle = 0$, $\frac{2^j}{{}^jC_{j/2}}$ for j is odd, even respectively. Averaging S1.2 for j is even gives

$$\langle B_r^j \rangle = \frac{2^j}{{}^jC_{j/2}} \langle (R_r cos(\alpha) - I_r sin(\alpha))^j \rangle. \tag{S1.3}$$

The above equation gives us even moments for B_r, for odd moments, we separate the trials for which $R_r cos(\alpha) - I_r sin(\alpha) > 0$. Refering to equation S1.2, this means that the left-hand-side should also be greater positive. Which translates to $sin^j(\phi_r - \alpha)$ being positive. Therefore, $\langle sin^j(\phi_r - \alpha_r)\rangle$ becomes $\frac{1}{2^{j-1}\pi} \sum_{k=0}^{j} \frac{(-1)^{(j-k)} {}^jC_k}{j-2k}$ and j^{th} odd moment for B_r can be written as

$$\langle B_r^j \rangle = \frac{2^{j-1}\pi}{\sum_{k=0}^{j} \frac{(-1)^{(j-k)j}C_k}{j-2k}} \langle (R_r cos(\alpha) - I_r sin(\alpha))^j \rangle \quad \forall\, r\ where\ (R_r cos(\alpha) - I_r sin(\alpha)) > 0. \tag{S1.4}$$

Equations (34) and (35) together gives us the estimates of odd and even moments for B_r. To estimate the moments of A_r, we raise equation (14) to n^{th} power, which using binomial expansion can be written as

$$\sum_j {}^nC_j (A_r sin(\alpha))^{n-j} (B_r sin(\phi_r))^j = R_r^n \tag{S1.5}$$

Taking trial average of the above equation, and considering $\langle sin^j(\phi_r) \rangle = 0$ (when j is odd), we have

$$\sum_{j=0,even}^{n} {}^nC_j \langle (A_r sin(\alpha))^{n-j} \rangle \langle (B_r sin(\phi_r))^j \rangle = \langle R_r^n \rangle. \tag{S1.6}$$

Substituting for the moments of B_r, we have

$$\sum_{j=0,even}^{n} {}^nC_j \langle A_r^{n-j} \rangle sin(\alpha)^{n-j} \langle (R_r cos(\alpha - I_r sin(\alpha)))^j \rangle = \langle R_r^n \rangle. \tag{S1.7}$$

The above equation relate the A_r moments with the averages of the observed R_r fourier coefficients. Equation S1.7 can be used to calculate A_r moments. But we don't need to explicitly calculate A_r moments to fit to a probability distribution. Equation S1.7 can be used directly to fit probablity distribution fuctions.

Estimating the emperical PL and NPL probability distribution In theory, the moments can be mapped to the probability density function of a random variable. But the proposed higher order moments are estimates and not the true moments of underlying random variable and have error in their estimation. Therefore, inverting the moments to get the discrete probability distribution gives errornous results. Here, we choose to fit the estimated moments with moments generated from gaussian, gamma, exponential distribution. Fitting the various probability distributions we can select the distribution, which best fits the estimated A_r, B_r moments, thereby best represent the underlying probability disrbution. We apply the above developed analysis to audio steady-state response EEG data. We fitted three different probability distribution functions (PDF), i.e., gaussian, gamma, and exponential distribution and calculated the fit error in each case. PDF was fitted for phase-locked and non-phase-locked activity at 40 Hz for both pre-stimulus and stimulus (audio stimulus) conditions.Table **S1** shows the result for different PDF fits. We can clearly see that gamma PDF gave minimum fitting error for all cases except for PL amplitude in pre-stimulus condition. We don't expect phase-locked activity

	Gaussian PDF (Mean,Deviation)	**Gamma PDF** (shape and scale parameter)	Exponential PDF (rate paramter)
Pre-stimulus Ar	$1.6x10^{-11}$, $1.7x10^{-16}$	$5.5x10^{-6}$, $2.4x10^{-6}$	16384
Error	0.1	297.3	0.1
Pre-stimulus Br	0.39 0.63,	3.2, 0.2	2.0
Error	0.0.39	**0.032**	0.350
Audio-stimulus Ar	$6.72x10^{-2}$, $6.80x10^{-16}$	100, $6.72x10^{-6}$	15.8
Error	139.5	**139.5**	144.5
Audio-stimulus Br	0.57, 0.97	2.47, 0.35	3.12
Error	0.11	**0.05**	0.55

Table S1: Estimated paramters and error to the fit for different probability density functions, for ASSR data (calculated for 40Hz at PO7 electrode).

during pre-stimuls condition, which is confirmed by near zero mean for both gamma and gaussian distribution and very large rate parameter for exponential distribution. So, we conclude that both A_r, B_r follows gamma distribution in experimental EEG data.

S2 Trial varying phase-locked activity (zitters in α)

It has been shown in numerous studies that the ERP i.e., phase-locked activity is not exactly time-locked but has delay zitters. This trial varying time zitters in ERP corresponds to time latencies in the ERP components e.g. latencies in P300. The time latencies in the ERP maps to the phase zitters in frequency domain i.e., the phase α of the PL activity is not exactly constant but has some variance across trials. Therefore here we consider α to be trial varing. So, our model equation of sum of phase-locked and non-phase-locked ativity is written as

$$S_r(t) = A_r sin(\omega_0 t + \alpha_r) + B_r sin(\omega_0 t + \phi_r). \tag{S2.8}$$

In fourier domain above equation transform to

$$A_r cos(\alpha_r) + B_r cos(\phi_r) = I_r \tag{S2.9}$$
$$A_r sin(\alpha_r) + B_r sin(\phi_r) = R_r. \tag{S2.10}$$

System of equations S2.9 and S2.10 forms an underdetermined system. So, we assume α_r to follow some distribution. Here, for the sake of simplistic analytic calculations we choose α_r to be distributed unifromly around a mean (μ) and have deviation θ. Assumption, probability distribution function for α_r:

$$p(\alpha_r) = \frac{1}{2\theta} \quad \forall \alpha_r \in [\mu - \theta, \mu + \theta]. \tag{S2.11}$$

Averaging equations S2.9 and S2.10 across trials, under the above assumption, we have

$$\langle A_r \rangle \frac{sin(\theta)}{\theta} sin(\mu) = \langle I_r \rangle \tag{S2.12}$$
$$\langle A_r \rangle \frac{sin(\theta)}{\theta} cos(\mu) = \langle R_r \rangle. \tag{S2.13}$$

Solving for μ, we have

$$\mu = tan^{-1} \langle R_r \rangle / \langle I_r \rangle. \tag{S2.14}$$

And for A_r, θ we have

$$\langle A_r \rangle \frac{sin(\theta)}{\theta} = \frac{\langle R_r \rangle}{sin(\mu)}. \tag{S2.15}$$

Multiplying equation S2.10 and S2.9 and averaging, we get

$$\langle A_r^2 \rangle \frac{sin(2\theta)}{2\theta} = \frac{2\langle R_r I_r \rangle}{sin(2\mu)}. \tag{S2.16}$$

Similarly evaluating average of $(3R_r I_r^2 - R_r^3)$ with equation S2.10 and S2.9, we get

$$\langle A_r^3 \rangle \frac{sin(3\theta)}{3\theta} = \frac{\langle 3R_r I_r^2 - R_r^3 \rangle}{sin(3\mu)}. \tag{S2.17}$$

We have shown in the last section that A_r and B_r occurs as gamma distributed in the EEG data. A gamma distribution is characterized by two parameters, shape parameter k and scale paramter Θ, with probability distribution function given by euation S2.18.

$$p(A_r) = \frac{1}{\Gamma(k)\Theta^k} A_r^{k-1} e^{-A_r/\Theta}. \tag{S2.18}$$

The first three moments of gamma distribution are given by Θk, $\Theta^2 k(k+1)$, $\Theta^3 k(k+1)(k+2)$. Assuming A_r is gamma distributed and substituting for A_r moments in equations S2.15-S2.17, we have

$$\Theta k \frac{sin(\theta)}{\theta} = \frac{\langle R_r \rangle}{sin(\mu)} \tag{S2.19}$$
$$\Theta^2 k(k+1) \frac{sin(2\theta)}{2\theta} = \frac{2\langle R_r I_r \rangle}{sin(2\mu)} \tag{S2.20}$$
$$\Theta^3 k(k+1)(k+2) \frac{sin(3\theta)}{3\theta} = \frac{\langle 3R_r I_r^2 - R_r^3 \rangle}{sin(3\mu)}. \tag{S2.21}$$

Equations S2.19,S2.20,S2.21 can be solved for A_r distribution parameters i.e.,Θ, k and θ the deviation in α. Next, assuming B_r also follows gamma distribution, we can compute the distribution parameters by equating the distribution moments with the calculated moments. The second and fourth moment for B_r can be calculated as

$$\langle B_r^2 \rangle = \langle R_r^2 + I_r^2 \rangle - \langle A_r^2 \rangle \tag{S2.22}$$

$$\langle B_r^4 \rangle = \langle (R_r^2 + I_r^2)^2 \rangle - \langle A_r^4 \rangle - 4 \langle A_r^2 \rangle \langle B_r^2 \rangle. \tag{S2.23}$$

We can use these estimates to calculate probability distribution paramters for B_r.

S3 Derivation of phase-locked value from Concurrent phaser model

In this section, we calculate the PLV given that the signal takes the form given in equation (3). The signal for trial r can be represented as

$$S_r(t) = R_r cos(\omega t) + I_r sin(\omega t) \tag{S3.24}$$

where, $R_r = A_r sin(\alpha) + B_r sin(\phi_r)$ and $I_r = A_r cos(\alpha) + B_r cos(\phi_r)$. The hilbert transform of the above signal is given by

$$H\{S_r(t)\} = R_r sin(\omega t) - I_r cos(\omega t). \tag{S3.25}$$

The analytic signal is given by $S_r(t) + iH\{S_r(t)\}$. From this, the instantaneous phase can be written as

$$\theta_r(t) = tan^{-1} \frac{R_r sin(\omega t) - I_r cos(\omega t)}{R_r cos(\omega t) + I_r sin(\omega t)}. \tag{S3.26}$$

PLV is defined as

$$PLV(t) = \left| \sum_r e^{i\theta_r(t)} \right| \tag{S3.27}$$

For instantaneous phase given in S3.26, plv can be written as

$$PLV(t) = |a + ib| = \sqrt{a^2 + b^2} \tag{S3.28}$$

where

$$a = 1/N \sum_r cos(\theta_r(t)) = 1/N \sum_r \frac{R_r cos(\omega t) + I_r sin(\omega t)}{\sqrt{R_r^2 + I_r^2}}, \tag{S3.29}$$

$$b = 1/N \sum_r sin(\theta_r(t)) = 1/N \sum_r \frac{R_r sin(\omega t) - I_r cos(\omega t)}{\sqrt{R_r^2 + I_r^2}}. \tag{S3.30}$$

We can further write $'a'$ as

$$a = cos(\omega t)\sum R + sin(\omega t)\sum I \tag{S3.31}$$

$$b = sin(\omega t)\sum R - cos(\omega t)\sum I. \tag{S3.32}$$

where, $\sum R = 1/N \sum R_r/\sqrt{R_r^2 + I_r^2}$ and $\sum I = 1/N \sum I_r/\sqrt{R_r^2 + I_r^2}$.

To solve for $\sum R$ and $\sum I$, putting back R_r and I_r in terms of A_r and B_r gives,

$$\sum R = 1/N \sum_r \frac{A_r sin\alpha + B_r sin\phi_r}{\sqrt{A_r^2 + B_r^2 + A_r B_r cos(\phi_r - \alpha)}}, \tag{S3.33}$$

$$= 1/N \sum_r \left(\frac{(A_r/B_r)sin\alpha}{\sqrt{1 + (A_r/B_r)^2 + 2(A_r/B_r)cos(\phi_r - \alpha)}} + \frac{sin\phi_r}{\sqrt{1 + (A_r/B_r)^2 + 2(A_r/B_r)cos(\phi_r - \alpha)}} \right) \tag{S3.34}$$

Taking the denominator to numerator and using bionomial expansion and consider the ratio A_r/B_r to be small (observed emperically Table S1) and keeping first oder and neglecting higher order terms of A_r/B_r. $\sum R$ approximates to,

$$\sum R \approx 1/N \sum_r (A_r/B_r) sin\alpha(1 - (A_r/B_r cos\phi_r)) + sin\phi_r(1 - (A_r/B_r cos\phi_r)) \tag{S3.35}$$

As $1/N \sum_r cos\phi_r$ and $1/N \sum_r sin\phi_r = 0$, $\sum R$ is evaluated to be

$$\sum R \approx \left\langle \frac{A_r}{B_r} \right\rangle sin\alpha. \tag{S3.36}$$

$$\tag{S3.37}$$

Similiarly $\sum I$ is evaluated to be

$$\sum I \approx \left\langle \frac{A_r}{B_r} \right\rangle cos\alpha - 1/2 \left\langle \frac{A_r}{B_r} \right\rangle. \tag{S3.38}$$

Putting above values back in equation (S3.31), (S3.32) and substituting for a, b in equation(S3.28), gives

$$PLV \approx \left\langle \frac{A_r}{B_r} \right\rangle \sqrt{5/4 - cos\alpha}. \tag{S3.39}$$

Though above equation gives a simple expression for PLV from concurrent-phaser model. But the calculation of $\left\langle \frac{A_r}{B_r} \right\rangle$ is not straight forward. Following steps shows the steps and assumption to calculate $\left\langle \frac{A_r}{B_r} \right\rangle$. Since A_r and B_r are independent,

$$\left\langle \frac{A_r}{B_r} \right\rangle = \langle A_r \rangle \left\langle \frac{1}{B_r} \right\rangle. \tag{S3.40}$$

The $\langle A_r \rangle$ has been calculated is section(2.3.2). Estimating $\left\langle \frac{1}{B_r} \right\rangle$ needs assumption regarding the probability distribution of B_r. It has been in section (5.1) that B_r folllows a gamma distribution in the emperical data. Under this assumption, $\left\langle \frac{1}{B_r} \right\rangle$ can be calculated as

$$\left\langle \frac{1}{B_r} \right\rangle = \frac{(Var(B_r))^4}{\langle B_r \rangle^4 (2\langle B_r \rangle^2 - \langle B_r^2 \rangle)}. \tag{S3.41}$$

The above equation represents $\left\langle \frac{1}{B_r} \right\rangle$ in terms of known estimates presented in section(2.3.2).

S4 Phase-locked nature of ASSR

The CPM estimates for PL and NPL power for ASSR experiment shows that only the PL power increase during stimulus presentation whereas NPL power remains at the pre-stimulus levels, as can be seen in the figure (S4).

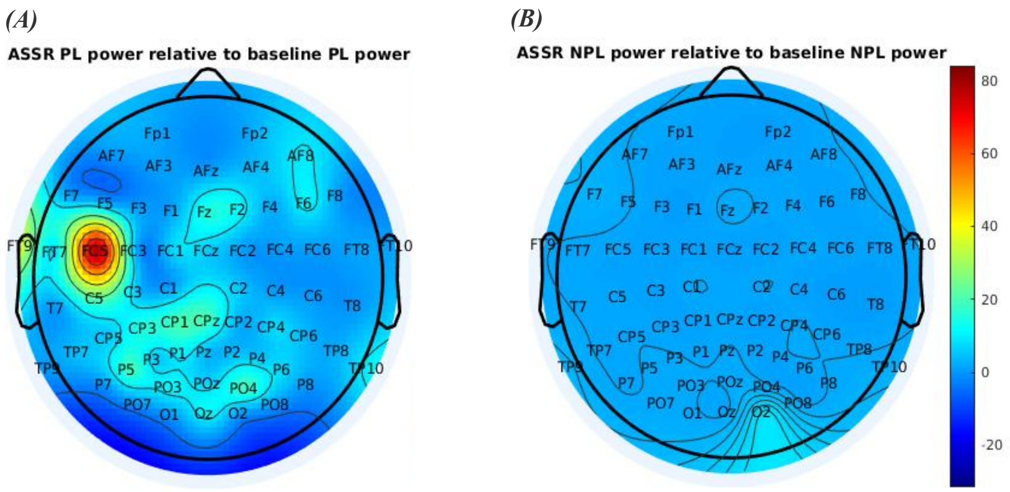

Figure S4: ASSR (A) PL and (B) NPL power relative to baseline PL, NPL power calculated via CPM method. (A) shows large increase (80 folds) in PL power for ASSR whereas (B) NPL power remains at baseline level.

S5 Constant phase-α estimate

Taking trial average on both sides of equations (14) and (15) and assuming that B_r and ϕ_r are independent of each other gives

$$\begin{aligned}\langle A_r\rangle cos(\alpha) + \langle B_r\rangle\langle cos(\phi_r\rangle =&\langle I_r\rangle \\ \langle A_r\rangle sin(\alpha) + \langle B_r\rangle\langle sin(\phi_r\rangle =&\langle R_r\rangle.\end{aligned} \tag{S5.42}$$

Since ϕ_r is expected to be uniformly distributed from $-\pi$ to π, $\langle cos(\phi_r)\rangle, \langle sin(\phi_r)\rangle$ can be approximated to be zeros. This gives

$$\langle A_r\rangle cos(\alpha) =\langle I_r\rangle \tag{S5.43}$$

$$\langle A_r\rangle sin(\alpha) =\langle R_r\rangle. \tag{S5.44}$$

Dividing equations (S5.43) by (S5.44) gives

$$\begin{aligned}tan(\alpha) =&\langle R_r\rangle/\langle I_r\rangle \\ \alpha =&tan^{-1}(\langle R_r\rangle/\langle I_r\rangle).\end{aligned} \tag{S5.45}$$

S6 The power operation

Here we define $Power\{.\}$ as an operation that we used in equation (5). $Power\{.\}$ is energy per unit time of the signal that it is operated on. It is given by sum of square of fourier coeffecient. Signal at right-hand side of equation(5) can be expanded as

$$\begin{aligned}(A_r - \langle A_r\rangle)sin(\omega_0 t + \alpha) + B_r sin(\omega_0 t + \phi_r) = \\ ((A_r - \langle(A_r\rangle))cos\alpha + B_r cos\phi_r)sin\omega_0 t + ((A_r - \langle(A_r\rangle))sin\alpha + B_r sin\phi_r)cos\omega_0 t\end{aligned} \tag{S6.46}$$

Operating power operation on both sides of S6.47, we get

$$Power\{(A_r - \langle A_r\rangle)sin(\omega_0 t + \alpha) + B_r sin(\omega_0 t + \phi_r)\}$$

$$= ((A_r - \langle(A_r\rangle))cos\alpha + B_r cos\phi_r)^2 + ((A_r - \langle(A_r\rangle))sin\alpha + B_r sin\phi_r)^2 \tag{S6.47}$$

$$= B_r^2 + (A_r - \langle A_r\rangle)^2 + 2(A_r - \langle A_r\rangle)B_r cos(\phi_r - \alpha) \tag{S6.48}$$

S7 Analytical expression for Evoked Potential

Taking trial-averages on both the sides of equation 2, we can express

$$\langle S_r(t)\rangle = \langle A_r sin(\omega_0 t + \alpha) + B_r sin(\omega_0 t + \phi_r)\rangle \tag{S7.49}$$

Since, $\omega_0 t$ and α are not changing across trials we can write,

$$\langle S_r(t)\rangle = \langle A_r\rangle sin(\omega_0 t + \alpha) + \langle B_r\rangle\langle sin(\omega_0 t + \phi_r)\rangle \tag{S7.50}$$

Now, since ϕ_r is symmetric, i.e., takes values between $[-\pi, \pi]$, the trial average of odd function $\langle sin(\omega_0 t + \phi_r)\rangle = 0$ (assumption of uniform variation of phase). Hence,

$$\langle S_r(t)\rangle = \langle A_r\rangle sin(\omega_0 t + \alpha) \tag{S7.51}$$