FX Fade Option Model

So-called "fade option" can be more precisely named as "point-barrier option". The fade option is a vanilla option that exists or dies if a barrier is breached on a single preset date, which is prior or equal to the contract maturity. Therefore, the barrier type can be up-cross or down-cross; the exercise type can be knock-in or knock-out; and the underlying vanilla can be call or put.

Let 0 < *T*1 · *T*2 · *T*, *K* > 0 and *h* ¸ 0. Let us define the barrier knock type index as

$$\text{knock} = \begin{cases} 1\,, & \text{up cross}\,, \\ -1\,, & \text{down cross}\,, \end{cases}$$

and the exercise type index as

$$\text{exType} = \begin{cases} 1\,, & \text{knock in}\,, \\ 0\,, & \text{knock out}\,, \end{cases}$$

as well as the underlying vanilla option type index as

$$\beta = \begin{cases} 1\,, & \text{call option}\,, \\ -1\,, & \text{put option}\,. \end{cases}$$

Finally, define a function of *H* on (0*;1*) with any given (knock*;* exType*; h*) as

$$H(S; \text{knock}, \text{exType}, h) = \begin{cases} \text{exType}\,, & \text{if knock} \cdot S \geq \text{knock} \cdot h\,, \\ 1 - \text{exType}\,, & \text{otherwise}\,. \end{cases}$$

The point-barrier option can be defined as follows. Let *fSt* : *t* ¸ 0*g* be the (strictly positive) price process of the underlying asset, *T*1 be the point barrier time, *h* be the single barrier, *K* be the underlying vanilla option strike, *T*2 be the option maturity, and *T* be the option payoff time. Then the matured payoff at *T* of the option, denoted by *VT* , can be given by

$$V_T = H(S_{T_1}; \text{knock}, \text{exType}, h) \times [\beta(S_{T_2} - K)]^+ \ .$$

It should be noticed that for *ST*1 with continuous distribution, the probability of an event *fST*1 = *sg* is always zero for any *s* > 0. Hence, in the definition of the function *H*, the inequality (¸) and the strict inequality (>) can be inter-changeable with no effect on the initial value of the option.

There are four types of single point-barrier options and they are *Up-In*, *Up-Out*, *Down-In*, and *Down-Out*. A double point-barrier option can be easily composed of two single point-barrier options. Let 0 · *h*L <*h*U · *1*. First, we have

$$[h_\mathrm{L}, h_\mathrm{U}]\text{-Knock-In} = (0, h_\mathrm{L}) \cup (h_\mathrm{U}, \infty)\text{-Knock-Out}$$

$$(h_\mathrm{L}, h_\mathrm{U})\text{-Knock-Out} = (0, h_\mathrm{L}] \cup [h_\mathrm{U}, \infty)\text{-Knock-In} \ .$$

Let I*f¢¢¢ g* be the indicate function. For [*h*L*; h*U]-Knock-In, we have

$$\begin{aligned} \mathbb{I}_{\{h_\mathrm{L} \le S_{T_1} \le h_\mathrm{U}\}} &= \mathbb{I}_{\{S_{T_1} \le h_\mathrm{U}\} \setminus \{S_{T_1} < h_\mathrm{L}\}} = \mathbb{I}_{\{S_{T_1} \ge h_\mathrm{L}\} \setminus \{S_{T_1} > h_\mathrm{U}\}} \\ &= \mathbb{I}_{\{S_{T_1} \le h_\mathrm{U}\}} - \mathbb{I}_{\{S_{T_1} < h_\mathrm{L}\}} = \mathbb{I}_{\{S_{T_1} \ge h_\mathrm{L}\}} - \mathbb{I}_{\{S_{T_1} > h_\mathrm{U}\}} \ , \end{aligned}$$

i.e., it is equal to one Down-In minus another Down-In or one Up-In minus another Up-In. Similarly, for (*h*L*; h*U)-Knock-Out, we have

$$1 - \mathbb{I}_{\{h_\mathrm{L} < S_{T_1} < h_\mathrm{U}\}} = \mathbb{I}_{\{S_{T_1} \le h_\mathrm{L}\}} + \mathbb{I}_{\{h_\mathrm{U} \le S_{T_1}\}} \ ,$$

i.e., it is equal to one Down-In plus one Up-In. Therefore, it suffices to consider single point-barrier options.

We assume that the zero-coupon bond price (see https://finpricing.com/lib/FiBond.html)

$$N_t = \mathrm{df}(t, T) \ , \qquad t \le T$$

can be used as the numeraire for the market such that, under the equivalent martingale measure N with respect to the numeraire *N*, the dynamics of the underlying price can be given by

$$S_t = S_s \frac{F_0(t)}{F_0(s)} \exp\left[-\frac{1}{2}\int_s^t \sigma^2 \,\mathrm{d}\tau + \int_s^t \sigma \,\mathrm{d}W_\tau^{\mathbb{N}}\right] , \qquad 0 \le s < t ,$$

where *F*0(¢) is the initial forward price of the underlying asset, *W*N is the standard Wiener process under N and ¾ is the volatility of *S* which can be implied by market term volatilities of vanilla options of the underlying asset, i.e., we have

$$\bar{\sigma}_t^2 = \frac{1}{t}\int_0^t \sigma^2 \,\mathrm{d}\tau , \qquad t > 0 .$$

Then the initial value of the option, denoted by *V*0, can be given as

$$\begin{aligned} V_0 &= \mathrm{df}(0,T) \times \mathrm{E}_0^{\mathbb{N}}[V_T] \\ &= \mathrm{df}(0,T) \times \mathrm{E}_0^{\mathbb{N}}\left[H(S_{T_1};\mathrm{knock},\mathrm{exType},h) \times \mathrm{E}_{T_1}^{\mathbb{N}}\left[[\beta(S_{T_2}-K)]^+\right]\right] . \end{aligned}$$