CENTRAL LIMIT THEOREM FOR M-DEPENDENT RANDOM VARIABLES

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Abstract: In this article, we proved central limit theorem for m-dependent random variables.

Keywords: Central limit theorem; m-dependent random variables

Theorem. Let $\{X_{n,i}\}$ be a triangular array of mean zero random variables. For each $n=1,2,...$ let $d=d_n$, $m=m_n$ and suppose $X_{n,1},...,X_{n,d}$ is an m-dependent sequence of random variables. Define

$$B^2_{n,k,a}=Var\left(\sum_{i=a}^{a+k-1}X_{n,i}\right),$$

$$B^2_n\equiv B_{n,d,1}\equiv Var\left(\sum_{i=1}^{d}X_{n,i}\right).$$

Assume the following conditions hold. For some $\delta>0$ and some $-1\le\gamma<1$:

$E\left|X_{n,i}\right|^{2+\delta}\le\Delta_n$ for all i,

(1)

$B^2_{n,k,a}/(k^{1+\gamma})\le K_n$ for all a and for all $k\ge m$,

(2)

$B^2_n/(dm^{\gamma+1})\ge L_n$,

(3)

$K_n/L_n=O(1)$,

(4)

$\Delta_n/L_n^{(2+\delta)/2}=O(1)$

(5)

$m^{1+(1-\gamma)(1+2/\delta)}/d\to 0$

(6)

Then, $B_n^{-1}(X_{n,1}+...+X_{n,d})\Rightarrow N(0,1)$

Proof of Theorem . In the proof we will need a result for bounding moments of m-dependent sequences. We will state it as a corollary of the following lemma, which implicitly is given in Chow and Teicher (1978) and deals with independent sequences.

Lemma A.1. Let $\{Y_i\}$ be an independent sequence of mean zero random variables. Assume $E|Y_i|^q \le \Delta$ for some $q \ge 2$ and all i.

Then, $E\left|\sum_{i=1}^{n} Y_i\right|^q \le C_q^q \Delta n^{q/2}$

Where C_q is a positive constant depending only upon q.

Proof. See Theorem 2 and Corollary 2 in Section 10:3 of Chow and Teicher (1978).

Corollary A.1. Let $\{X_i\}$ be an m-dependent sequence of mean zero random variables. Assume $E|X_i|^q \le \Delta$ for some $q \ge 2$ and all i.

Then, for all $n \ge 2m$,

$$E\left|\sum_{i=1}^{n} X_i\right|^q \le C_q^q \Delta (4mn)^{q/2}$$

where C_q is a positive constant depending only upon q.

Proof. Define $t = [n/m]$ where $[\cdot]$ denotes the integer part. Now split $X_1 + ... + X_n$ into t blocks of size m and a remainder block: $X_1 + ... + X_n \equiv A_1 + ... + A_t + A_{t+1}$ Due to m-dependence, the odd-numbered blocks are independent of each other, as are the even-numbered blocks. This allows us to apply Lemma A.1:

$$\left\|\sum_{i=1}^{n} X_i\right\|_q \le \left\|\sum_{i \text{ odd}} A_i\right\|_q + \left\|\sum_{i \text{ even}} A_i\right\|_q \quad \text{(by Minkowski)}$$

$$\le 2C_q m(\Delta)^{1/q}(t/2+1)^{1/2} \quad \text{(by Lemma A:1 and Minkowski):}$$

But, this is equivalent to

$$E\left|\sum_{i=1}^{n} X_i\right|^q \le C_q^q 2^q m^q \Delta (t/2+1)^{q/2}$$

$$\le C_q^q 2^q m^q \Delta (t)^{q/2} \le C_q^q 2^q \Delta (mn)^{q/2} = C_q^q \Delta (4mn)^{q/2}.$$

We are now able to prove the theorem. The main idea of the proof follows Berk (1973), but we need some modifications, since our theorem is more general.

For each n, we choose an integer $p = p_n > 2m$ so that

$$\lim_{n\to\infty} m/p = 0, \qquad \lim_{n\to\infty} p^{1+(1-\gamma)(1+2/\delta)}/d = 0.$$

(7)

This can be done, for example, by remembering assumption (6) and choosing p to be the smallest integer greater than $2m$ and greater than $m^{1/2}d^{1/2\xi}$, where ξ is equal to $1+(1-\gamma)(1+2/\delta)$. Next, define integers $t=t_n$ and $q=q_n$ by $d=pt+q$, $0\le q<p$. The main idea of the proof is to split the sum $X_{n,1}+...+X_{n,d}$ into alternate blocks of length $p-m$ (the big blocks) and m (the little blocks). This is a common approach to proving central limit theorems for dependent random variables, and is attributed to Markov in Bernstein (1927). Let

$$U_{n,i}=X_{n,(i-1)p+1}+...+X_{n,ip-m},\ 1\le i\le t,$$

$$V_{n,i}=X_{n,ip-m+1}+...+X_{n,ip},\quad 1\le i\le t$$

$$U_{n,t+i}=X_{n,tp+1}+...+X_{n,d}.$$

By definition, $X_{n,1}+...+X_{n,d}=\sum_{i=1}^{t+1}U_{n,i}+\sum_{i=1}^{t}V_{n,i}$. Since the $X_{n,i}$ are m-dependent and $p>2m$, $\{U_{n,i}\}$ and $\{V_{n,i}\}$ are each independent sequences. It is easily seen that the difference between $B_n^{-1}(X_{n,1}+...+X_{n,d})$ and has variance approaching zero. Indeed,

$$Var\left(B_n^{-1}\sum_{i=1}^{t+1}V_{n,i}\right)=B_n^{-2}\sum_{i=1}^{l}Var(V_{n,i})$$

$$\le B_n^{-2}t\left[\sup_i\ Var(V_{n,i})\right]\le B_n^{-2}tK_nm^{1+\gamma}\qquad \text{(by assumption (2))}$$

$$\le B_n^{-2}(d/p)K_nm^{1+\gamma}$$

$$\le\frac{K_n}{L_n}\frac{m}{n}\to 0\qquad \text{(by assumption (3) and (4)).}$$

Hence, provided they exist, the asymptotic distributions of the two quantities $B_n^{-1}\sum_{i=1}^{t+1}U_{n,i}$ and $B_n^{-1}\sum_{i=1}^{d}X_{n,i}$ are the same, and the goal now is to show that

$$B_n^{-1}\sum_{i=1}^{t+1}U_{n,i}\Rightarrow N\left(0,1\right).$$

In order to apply assumption (3) again, we will first establish that $B_n^{-2}Var\left(\sum_{i=1}^{t+1}U_{n,i}\right)$

tends to one, or, equivalently, $B_n^{-2}Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)$ tends to zero. Note first that $Cov\left(U_{n,i},V_{n,i}\right)=0$ unless $j=i$ or $i-1$. Furthermore,

$$\left|Cov\left(U_{n,i},V_{n,i}\right)\right|=\left|E\left(U_{n,i},V_{n,i}\right)\right|\le\left[Var(U_{n,i})Var(V_{n,i})\right]^{1/2}$$

$$\le K_n(mp)^{(1+\gamma)/2}\quad \text{(by assumption (2)).}$$

Combining these two facts, we obtain $\left|Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)\right|\le 2K_n(mp)^{(1+\gamma)/2}$

and finally,

$$B_n^{-2}Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)\le 2\frac{K_n}{L_n}\frac{t}{dm^{\gamma}}(mp)^{(1+\gamma)/2}$$

$$\le 2\frac{K_n}{L_n}\frac{1}{pm^{\gamma}}(mp)^{(1+\gamma)/2}$$

$$=2\frac{K_n}{L_n}\left(\frac{m}{p}\right)^{(1-\gamma)/2}\to 0 \text{ (by assumption (4) and since } \gamma<1).$$

By Lyapounov's theorem, it will now suffice to verify that $\sum_{i=1}^{t+1}E\left|U_{n,i}\right|^{2+\delta}/B_n^{2+\delta}$ tends to zero. By Corollary A.1,

$$E\left|U_{n,i}\right|^{2+\delta}\le C_{2+\delta}^{2+\delta}\Delta_n(4pm)^{(2+\delta)/2},\quad 1\le i\le t+1,$$

And therefore

$$\sum_{i=1}^{t+1}E\left|U_{n,i}\right|^{2+\delta}/B_n^{2+\delta}\le Const.\Delta_n(d/p+1)(pm)^{(2+\delta)/2}/B_n^{2+\delta}$$

By assumption (3), finally,

$$\Delta_n(d/p)(pm)^{(2+\delta)/2}/B_n^{2+\delta}\le \Delta_n L_n^{-(2+\delta)/2}\frac{d}{p}\left(\frac{pm}{dm^{\gamma}}\right)^{(2+\delta)/2}$$

$$\le \Delta_n L_n^{-(2+\delta)/2}\left(\frac{p}{d}\right)^{\delta/2}m^{(1-\gamma)(2+\delta)/2}$$

$$=O(1)AB \text{ (by assumption (5));}$$

where $A=p^{\delta/2+(1-\gamma)(2+\delta)/2}d^{-\delta/2}$ and $B=\left(\frac{m}{p}\right)^{(1-\gamma)(2+\delta)/2}$. The second condition on p in (7) implies that A tends to zero. The first condition on p in (7), together with the fact that $\gamma\le 1$, imply that B tends to zero as well.

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