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Three Related Problems of Bergman Spaces of Tube Domains over Symmetric Cones

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Abstract: **The Szego projection of tube domains over irreducible symmetric cones is unbounded in $L^{(1+\epsilon)}$. Indeed, this is a consequence of the fact that the characteristic function of a disc is not a Fourier multiplier, a fundamental theorem proved by C. Fefferman in the 70's. The same problem, related to the Bergman projection, deserves a different approach. In this survey, based on joint work of the author with D. Bekolle, G. Garrigos, M. Peloso and F. Ricci, we give partial results on the range of $1+\epsilon$ for which it is bounded. We also show that there are two equivalent problems, of independent interest. One is a generalization of Hardy inequality for holomorphic functions. The other one is the characterization of the boundary values of functions in the Bergman spaces in terms of an adapted Littlewood–Paley theory. This last point of view leads naturally to extend the study to spaces with mixed norm as well.**

Keywords: **Whitney decomposition; Symmetric cone; Bergman projector; Littlewood – Paley; Hardy inequality.**

1. INTRODUCTION

For V be an irreducible symmetric cone in the Euclidean space V , and $T_\Omega = V + i\Omega$the corresponding tube domain in the complexified space$V^{\mathbb{C}}$. We shall note n the dimension of V and r the rank of Ω . Moreover , we shall denote by $(x|y)$ the scalar product inV, and by Δ the determinant function . For the description of such cones in terms of Jordan , one may use the book of Faraut and Koranyi [8] . One may also have in mind the typical example that one obtains when V is the space of real symmetric $r \times r$ matrices and Ω is the cone of positive definite matrices . In this example , the scalar product on V is induced by the Hilbert-Schmidt norm of the matrices , and the determinant function is given by the determinant of the matrices .

The rank is r , while the dimension is $\frac{r(r+1)}{2}$.

We shall also make use of the generalized wave operator on V, given by

$$\blacksquare = \Delta\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$$

This is a differential operator of degree r , defined by the equality

$$\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\left[\sum\nolimits_k e^{i(x/\xi_k)}\right] = \sum\nolimits_k \Delta(\xi_k)e^{i(x/\xi_k)}\ ,\ \ \xi_k \in V$$

It is the usual derivative (up to a constant) when Ω is the half-line $(0,\infty)$. Its name is due to another fundamental example, given by the forward light cone in R^n,

$$\left\{x \in R^n \,;\, x_1 > \sqrt{x_1^2 + \cdots + x_n^2}\right\}$$

which is of rank 2 .In this case , the determinant function is equal to

$$\Delta(x) = x_1^2 - x_2^2 - \cdots - x_n^2 \,,$$

and the operator is wave operator . One may look at [5,2] , which deal with this particular case .

For $\epsilon \geq 0$, let $A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)} = A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}(T_\Omega)$ denote the weighted Bergman spaces in the tube domain T_Ω that is the space of holomorphic functions

$F_k \in \mathcal{H}(T_\Omega)$ satisfying the integrability condition

$$\sum_k \|F_k\|_{A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}} = \sum_k \|F_k\|_{L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}} = \sum_k \left[\int_\Omega \left[\int_V |F_k(x+iy)|^{(1+\epsilon)} dx\right]\right]^{\frac{1}{(1+\epsilon)}} < \infty \quad (1.1)$$

We shall impose $\epsilon > 0$ to avoid trivial cases where $A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)} = \{0\}$.

The mixed Lebesgue spaces $L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$ are defined in an obvious way . We write $L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$ and $A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon)}$ to simplify . The space $A_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$ is closed subspace of $L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$.

The case $\epsilon = 1$ is of special interest . It is well-known that every $F_k \in A_{(\frac{n}{r}-1+\epsilon)}^2$ can be written as

$$\sum_k F_k(z) = \mathfrak{T}_k \sum f_k(z) = \int_\Omega \sum e^{i(z|\xi_k)} \hat{f}(\xi_k) dz \quad , z \in T_\Omega (1.2)$$

for some functions $\widehat{f_k} \in (L^2\left(\Omega\,;\, \Delta^{-(\frac{n}{r}-1+\epsilon)}\right) d\xi_k)$ (see [8]) . The operator $\mathfrak{T}$ will be called the Fourier – Laplace transform of f_k (using the usual terminology , it is the Laplace transform of its Fourier transform) . The functions f_k may be seen as the (Shilov) boundary value of the holomorphic functions F_k . The orthogonal projection from $L_{(\frac{n}{r}-1+\epsilon)}^2$ onto $A_{(\frac{n}{r}-1+\epsilon)}^2$, which is called the (weighed) Bergman projection , is denoted by $P_{(\frac{n}{r}-1+\epsilon)}$ and explicitly given by

$$P_{(\frac{n}{r}-1+\epsilon)}\left(\sum_K F_K(Z)\right) = \int_{T_\Omega} \sum_k B_{(\frac{n}{r}-1+\epsilon)}(z-\overline{w}) F_k(w) \Delta(Im\ w)^{(\epsilon-1)} dw$$

where $B_{(\frac{n}{r}-1+\epsilon)}(z-\overline{w}) = d(\frac{n}{r}-1+\epsilon) \Delta^{-\left(\frac{2n}{r}-2+2\epsilon\right)}\left((z-\overline{w})/i\right)$ is the reproducing kernel of $A_{(\frac{n}{r}-1+\epsilon)}^2$(see [8]) . For simplification , we have written

$dw = du\, d\left(\frac{n}{r}-1+\epsilon\right)$, for $w = u + i\left(\frac{n}{r}-1+\epsilon\right)$an element of T_Ω . We can now state the three problems under consideration in this survey .

Problem .1 Boundedness of the Berman projection . The question , here , is know the exact range of $(1+\epsilon),(1+\epsilon)$ for which the projection $P_{(\frac{n}{r}-1+\epsilon)}$ extends as a bounded operator on $L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$. For obvious reasons (self-adjointness and interpolation), the set of couples $\left(\frac{1}{(1+\epsilon)}, \frac{1}{(1+\epsilon)}\right)$for which it is bounded is a convex set in $(0,1) \times (0,1)$, which is symmetric around $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Let us recall that , for the upper half-plane , this convex set is the whole square $(0,1) \times (0,1)$. For higher rank , the situation is different . From the convexity and symmetric given above , we may restrict our interest to value of $(1+\epsilon)$ which are larger than 2 . We shall first see that there is a small critical index $(1+\epsilon)_{(\frac{n}{r}-1+\epsilon)} > 2$ such that $P_{(\frac{n}{r}-1+\epsilon)}$ defines a bounded operator on $L_{(\frac{n}{r}-1+\epsilon)}^{(1+\epsilon),(1+\epsilon)}$ for

$1 \leq \epsilon \leq (1+\epsilon)_{(\frac{n}{r}-1+\epsilon)}$ for all value of $(1+\epsilon)$. Moreover , in this range , one has still a bounded operator when the kernel $B_{(\frac{n}{r}-1+\epsilon)}$ is replaced by its absolute value , that is a when one considers the positive operator given by

$$P^{+}_{(\frac{n}{r}-1+\epsilon)} \sum_{k} F_k(z) = \int_{T_\Omega} \sum_k \left| B_{(\frac{n}{r}-1+\epsilon)}(z-\varpi) \right| F_k(\omega) \Delta (Im\ \omega)^{2(\frac{n}{r}-1+\epsilon)} d\omega \quad (1.3)$$

We shall see that the index $(1+\epsilon)_{(\frac{n}{r}-1+\epsilon)}$ is sharp for this continuity property .

In the other direction , there is a large critical index , depending on $(1+\epsilon)$,that we shall call $\widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)'}$, such that , for $(1+\epsilon) \geq \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)'}$, the projection $P_{(\frac{n}{r}-1+\epsilon)}$ fails to be bounded for obvious reasons .

Let us recall that situation is completely different for the Szego projection , which is unbounded in $L^{(1+\epsilon)}(V)$ for $\epsilon \neq 1$ (see [11,9]) .

Problem 2 .

Hardy inequality in Bergman spaces .The question , here , is know the range of $(1+\epsilon, 1+\epsilon)$ for which one has a Hardy type inequality for holomorphic functions on the domain T_Ω

$$\sum_k (F_{k, L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}})_k \leq C_{(1+\epsilon)(1+\epsilon)} \sum_k \|\Delta(Im\ w) \blacksquare F_k\|_{L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}} \quad (1.4)$$

Again , for the upper half-plane , one knows the exact , and in fact it is valid for all $(1+\epsilon)$ and $(1+2\epsilon)$ in the interval $0 \leq \epsilon < \infty$. It is an easy consequence of the usual Hardy inequality , which givens an integral inequality between a function and its derivative . Let us remark that , since we deal with holomorphic functions , the differential operator $\blacksquare$ may be defined as a polynomial in $\partial/\partial x$, as we did , or in $\partial/\partial y$.

The converse inequality , where left and right hand side of (1.4) are exchanged , is always valid as a consequence of the mean value property .

Problem 3. Characterization of boundary values Bergman spaces . For the upper half-plane , Bergman spaces are spaces are characterized by the fact that their boundary value belong to some Basov space . So, the functions of the Bergman spaces may be obtained as Fourier-Laplace transforms of these boundary values ,a property which generalizes situation of A^2 .

One would like to have an equivalent characterization in higher rank . We will shall that it is indeed the case for some values of $(1+\epsilon),(1+\epsilon)$. We will need a precise description of the geometry of the cone to be able to describe these objects, which come from an adapted Littlewood-Paley decomposition . So , we will not be able to state properly Problem 3 .

It turns out that the three problem are some sense equivalent . The same critical indices occur in the three problems . In particular , all three possess a negative answer for $2-\epsilon > \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$, for obvious reasons . So, the equivalent between the three problems is only interesting for $1 > \epsilon > \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. We may see Problems 2 and 3 as equivalent formulations of Problems 1 which help to take care of the oscillations of the kernel .

We will give precise statements in the other sections ,and given a complete answer for $\epsilon \leq 1$, with the exact range of values $\epsilon < 1$ for which the projection $P_{(\frac{n}{r}-1+\epsilon)}$ is bounded on $L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$. For there is a gap in the results . We will see in the last section how the question opened by this gap may be related to Littlewood-Paley theory for functions on F_k with spectrum in Ω .We will then state a last problem 4.

The present survey is based on joint work of the author with David Bekolle , Gustavo Garrigos , Marco Peloso and Fulvio Ricci [1,3,5] . While the first papers dealt only with the forward light cone , te two last ones deal with the general case . Once the geometric aspects of the proofs have been developed , using the formalism of Jordan algebras as in [8] , there is on difficulty to write in the general case , which we do here . Sections 3 and 4 contain some new statements . We tried to give some easy proofs , when they help for the general understanding of the subject . We refer to the different papers for the difficult ones .

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Let us mention that part of the result of [1] , which are related to the small critical index have also been generalized by Bekolle and Temgoua Kagou [6] , using the formalism of Gindikin for the description of the cones . Let us also mention that one source of inspiration has been the work of Coifman and Rochberg on atomic decomposition of Bergman spaces [7] .

Finally , I would like to than Gustavo Garrigos , whose comments were very helpful . All this survey has been enriched by discussions with him .

2. GEOMETRY AND ANALYSIS ON THE CONE

In order to describe precisely the results , and specially to define Besov spaces , we start with the description of the geometry of the cone . We refer to [8] for the context , and to [3] and [4] for the geometric lemmas .

Considering as a Jordan algebra , we denote its unit element by **e**(think of the identity . matrix for the fundamental example of real symmetric matrices) . Let G be the identity component of the group of invertible linear transformations which leave the cone Ω invariant . It is well know that G acts transitively on Ω , which may be identified with the Riemannian symmetric space G/K , where Kis the compact subgroup of elements of G which leave e invariant . The G-invariant Riemannian metric can be defined by

$$\langle \xi_k, \eta_k \rangle_{(1+\epsilon)} := (t^{-1}\xi_k | t^{-1}\eta_k)$$

if $y = te$ and ξ_k , η_k are tangent vectors at $y \in \Omega$. We shall denote by d the corresponding distance ,and by ξ_k , η_k the invariant ball centered at ξ_k of radius δ . The invariance implies that , for $g_k \in G$, $B_\delta(g_k \xi_k) = g_k B_\delta(\xi_k)$.

The determinant function is also preserved by g_k , in such a way that

$$\sum \Delta(g_k y) = \Delta \sum g_k e \Delta(y) = \sum Det g_k^{\frac{r}{n}} \Delta(y) \qquad (2.1)$$

It follows from this formula that an invariant measure in Ω is given by $\Delta(y)^{-\frac{n}{r}} dy$.

The invariance properties allow also to prove that the determinant function is almost constant on the balls of a given radius , as well as scalar products .

Lemma 2.1 : There is a constant $\epsilon > 0$ such that ,for $y \in \overline{\Omega}$, if $\xi_k , \xi_k' \in \Omega$ with $d(\xi_k , \xi_k') \le 2$, then

$$\sum \frac{1}{\xi_k} \le \sum \frac{\Delta(\xi_k)}{\Delta(\xi_k')} \le (1+\epsilon) ; \qquad (2.2)$$

$$\sum \frac{1}{\xi_k} \le \sum \frac{\Delta(\xi_k|y)}{\Delta(\xi_k'|y)} \le (1+\epsilon) ; \qquad (2.3)$$

$$\frac{1}{(1+\epsilon)} \le \sum \frac{|\xi_k|}{|\xi_k'|} \le 1+\epsilon . \qquad (2.4)$$

Form the previous lemma , it follows that , for all $1+\epsilon \in \Omega$ and $0 < \delta \le 2$,

$$meas\big(B_\delta(y)\big) = meas\big(B_\delta(e)\big) \sim Vol\big(B_\delta(e)\big) \sim \delta^n .$$

where $Vol(B)$stands for the Euclidean volume of B , while meas (B) stands for its measure for the invariant .

Next ,. we need the analog for a general cone of the decomposition of the real half –line $(0 , +\infty)$ into an union of dyadic intervals $[2^j , 2^{j+1})$, which may be seen as invariable 5 balls of constant size . This is given by the next lemma .

Lemma 2.2 : There exists a sequence of $\{(\xi_k)_j\}_{j,k}$in Ω , and an associated family of disjoint sets $\{\xi_j\}_j$covering Ω , such that

$$B_{1/2}(\xi_k) \subset E_j \subset B_1\left((\xi)_{k_j}\right)$$

A sequence of points $\{\xi_k\}_j$with the above properties is called a lattice of the cone ,

and the associated partition $\{(\xi_k)_j\}_{j,k}$ a Whitney decomposition of the cone Ω .

From considerations on the volume of balls we get easily that , for a fixed radius $R \geq 1$, the balls $B_R(\xi_k)_j$. have the finite intersection property . That is , these is an integer $N = N(\Omega, R)$ so that each point in Ω belongs to at most N of these balls . Basov spaces . Its existence is given in the next proposition .

Proposition 2.3 : There exists a sequence of smooth function ψ_j such that

1. $\widehat{\psi_j} \in C_c^\infty\left(B\left((\xi_k)_j, 1\right)\right)$;

2. $0 \leq \widehat{\psi_j} \leq 1$ and $\sum_j \sum_k \widehat{\psi}_j(\xi)_k = 1 \quad , \forall \xi_k \in \Omega$;

3. The functions ψ_j are uniformly bounded in $L^1(\mathbb{R}^n)$.

This implies , in particular , the existence of some constant $c > 0$ such that

$$\left\|\sum_k \psi_j * f_k\right\|_{(1+\epsilon)} \leq c \sum \|f_k\|_{(1+\epsilon)}, \forall f_k \in L^{(1+\epsilon)}(\mathbb{R}^n),$$
$$\forall j, 1 \leq \epsilon < \infty \qquad (2.5)$$

Roughly speaking , the $\psi_j{}'s$ are obtained from a fixed function by the action of an element of Gwhich sends e to $(\xi_k)_j$. This allows to compute easily their $L^{(1+\epsilon)}$ norms .

Associated with the operator ■ and the Whitney decomposition , we can now introduce the family of Basov-type spaces $B_V^{(1+\epsilon),(1+\epsilon)}$, naturally adapted to the geometry of the cone . They are defined as equivalent classes of tempered distributions on V , by means of the seminorm :

$$\sum \|f_k\|_{B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}} = \left[\sum_j \sum_k \Delta^{-(\frac{n}{r}-1+\epsilon)}(\xi_k)_j \left[\!\left[f_k * \psi_j\right]\!\right]_{(1+\epsilon)}^{(1+\epsilon)}\right]^{\frac{1}{(1+\epsilon)}}, \; f_k \in \mathcal{S}'(V) \quad (2.6)$$

The Whitney decomposition of the cone has other applications . It allows to discretize integrals which involve almost constant quantities on each piece . Let us give an example of such a situation . The proof is a direct consequence of the lemma .

Proposition 2.4: Let $0 < \delta \leq 1$ be fixed , and $\{(\xi_k)_j\}_{j,k}$ be a lattice with associated Whitney decomposition $\{E_j\}_{j^*}$. Then , for every $s \in \mathbb{R}, y \in \bar{\Omega}$ and for every non-negative function f_k on

the cone , we have

$$\frac{1}{C}\sum_j \sum_k e^{-(1+\epsilon)(y|\xi_k)} \Delta'(\xi_k)_j \int_{E_j} f_k(\xi_k) \frac{d\xi_k}{\Delta(\xi_k)^{\frac{n}{r}}}$$
$$\leq \int_\Omega \sum f_k(\xi_k) e^{-(y|\xi_k)} \Delta'(\xi_k) \frac{d\xi_k}{(\xi_k)^{\frac{n}{r}}}$$
$$\leq \mathrm{C} \sum_j \sum_k e^{-\frac{1}{(1+\epsilon)}(y|\xi_k)_j^{-s}} \Delta'(\xi_k)_j \int_{E_j} f_k(\xi_k) \frac{d\xi_k}{\Delta(\xi_k)^{\frac{n}{r}}}$$

where $(1+\epsilon)$ is the constant in (2.3) and C depends only .

One may think at first view , that such estimates will be difficult to use because of the constant $(1+\epsilon)$. But a further integration in the y variable transforms into powers , as given in the next lemma .

Lemma 2.5 : For $y \in \Omega$ and $s \in \mathbb{C}$ with $Re\, s > \frac{1}{n} - 1$, then

$$\int_\Omega \sum_k e^{-(\xi_k|y)} \Delta'(\xi_k) \frac{d\xi_k}{\Delta(\xi_k)^{\frac{n}{r}}} = \Gamma_\Omega(s)\Delta^{-s}(y).$$

Moreover , the integral does not converge for other values of s **.**

$\Gamma_\Omega(s)$ is the Gamma function in Ω , which may be computed in terms of the usual Gamma function . We will need the following lemma , which is as easy consequence of the previous one . Here $L^{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}(\Omega)$ denotes the space of functions on (Ω) whose $(1+\epsilon)-th$ power is integrable for the measure $\Delta(y)^{(\epsilon-1)}dy$.

Lemma 2.6 : The function $\Delta(y+e)^{-s}$ is in $L^{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}(\Omega)$ if and only if $L^{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}s > \frac{1}{(1+\epsilon)}\left(V+\frac{n}{r}-1\right)$.

In fact , we also need in the proofs the generalized powers of Δ . We give here their definitions for completeness , but refer to [8 ,4] for their use in estimates . Let $\{c_1,\dots,c_r\}$ bee a fixed Jordan frame in V (think of diagonal matrices for which the diagonal entries are all zero except for one equal to 1) . Let $\Delta_1(x),\dots,\Delta_r(x)$ the principal minors of $x\in V$, with respect to the fixed Jordan frame $\{c_1,\dots,c_r\}$. The generalized function in Ω is defined as

$$\Delta_s = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\dots\Delta_r^{s_r}(x), s = (s_1,s_2,\dots,s_r)\in\mathbb{C}^r, x\in\Omega$$

When all s_j are equal to , we see that $\Delta_s = \Delta^r$.

3. THE SMALL CRITICAL INDEX

Th small critical index is given by

$$(1+\epsilon)_{(\frac{n}{r}-1+\epsilon)} = 1+\frac{n-r+r\epsilon}{n-r} \quad (3.1)$$

Let us mention , even if we will not use it , that for the forward light cone and the usual Bergman space $r=2$, $\left(\frac{n}{r}-1+\epsilon\right)=\frac{n}{2}$, it is the critical index for Bochner-Riesz means in $\mathbb{R}^{n-1}$.

We see first that the small critical index is relation to Problem 2 . More precisely , it occurs when generalizing the Hardy inequality on the real half-line , given by

$$\int_0^\infty \sum_k \left(\int_{(\frac{n}{r}-1+\epsilon)}^\infty f_k(y)dy\right)^{(1+\epsilon)} \left(\frac{n}{r}-1+\epsilon\right)^{(\frac{n}{r}-2+\epsilon)} d\left(\frac{n}{r}-1+\epsilon\right) \le C\int_0^\infty \sum_k f_k(y)^{(\frac{n}{r}-2+\epsilon)}d\,y$$

for positive f_k . To replace the integration , the first idea that one has in mind is to use an explicit solution of the equation $\blacksquare^m g_k$ inside the cone , with large enough so that its elementary solution , given by $c\Delta^{m-\frac{n}{r}}\chi_\Omega$, is locally integrable (which is the case $m>\frac{n}{r}-1$) . Then

$$T_m\left(\sum_k f_k\left(\frac{n}{r}-1+\epsilon\right)\right) = \int_\Omega \sum_k f_k\left((\frac{n}{r}-1+\epsilon)+y\right)\Delta(y)^{m-\frac{n}{r}}d\,y$$

satisfies the equation $\blacksquare^m g_k$. We will prove the following proposition , which can be called the Hardy inequality of order on Ω .

Proposition 2.4 : There exists a constant C such that , for all positive functions f_k ,

$$\int_\Omega \sum_k \left[T_m f_k\left(\frac{n}{r}-1+\epsilon\right)\right]^{(1+\epsilon)} \Delta\left(\frac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\frac{n}{r}-1+\epsilon\right) \le \int_\Omega \sum_k [\Delta(y)^m f_k(y)]^{(1+\epsilon)}\Delta(y)^{(\epsilon-1)}dy$$

if and only if $1+\epsilon < (1+\epsilon)_{(\frac{n}{r}-1+\epsilon)}$.

Proof : It is equivalent to prove that the operator C , with kernel given by

$$\Delta\left(y-\left(\frac{n}{r}-1+\epsilon\right)\right)^{m-\frac{n}{r}} \chi_\Omega\left(y-\left(\frac{n}{r}-1+\epsilon\right)\right)\Delta(y)^{(-m+1-\epsilon)},$$

is bounded in $L^{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}(\Omega)$. A necessary condition is that f_k belong to $L^{(1+\epsilon)'}_{(\frac{n}{r}-1+\epsilon)}(\Omega)$, with T^* the formal adjoint of T , when f_k is the characteristic function of the ball $B_1(e)$. One can show that , for $y\in N\,e+\Omega$, with N a fixed integer which is large enough , one has the bound below

$$T^*\left(\sum_k f_k(y)\right) > c\Delta(y)^{-(\frac{n}{r}-1+\epsilon)}$$

necessary condition follows from Lemma 2.6 .

The sufficiency follows from a routine argument , using Schur 's lemma and generalized powers of Δ : we find that there exists some generalized power s such that

$$T\left(\Delta_s^{1/(1+\epsilon)}\right) \leq C\, \Delta_s^{1/(1+\epsilon)} T^* \left(\Delta_s^{1/(1+\epsilon)'}\right) \leq C \Delta_s^{1/(1+\epsilon)'}$$

This is based on integrability conditions . We shall not go into details , and refer to the bibliography for this kind of computations .

This proposition leads to a Hardy inequality of order m for holomorphic functions on the tube domain T_Ω ,

$$\left\|\sum_k F_k \right\|_{L^{(1+\epsilon),(1+\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}} \leq C_{(1+\epsilon),(1+\epsilon)} \sum_k \|\Delta(Im\ \omega)^m \blacksquare^m F_k\|_{L^{(1+\epsilon),(1+\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}} \quad (3.3)$$

for all $0 \leq \epsilon < \infty$, with $1+\epsilon < (1+\epsilon)_{\left(\frac{n}{r}-1+\epsilon\right)}$

We will see that this range of values can always be extended . So this first method , inspired by the method on the upper half-plane , does not give optimal results .

Let us now show the role of the small index in Problem 1 .We will prove the following theorem .

Theorem 3.2 : The operator given by (1.3), is bounded on $P^+_{\left(\frac{n}{r}-1+\epsilon\right)}$ if and only if $(1+\epsilon)'_{\left(\frac{n}{r}-1+\epsilon\right)} < (1+\epsilon) < (1+\epsilon)_{\left(\frac{n}{r}-1+\epsilon\right)}$.

Proof : Let us first prove the sufficient condition . Clearly , $P^+_{\left(\frac{n}{r}-1+\epsilon\right)}$ acts as a convolution operator in the x, u variable (if we note $z = x + iy$,

$w = u + i\left(\frac{n}{r}-1+\epsilon\right)$) . Moreover , the norm of this convolution operator acting in $L^{(1+\epsilon)}(V)$ is bounded by the L^1 norm of the kernel $\left|\Delta\left(\cdot + i\left(y + \left(\frac{n}{r}-1+\epsilon\right)\right)\right)\right|^{-\left(2\frac{n}{r}-1+\epsilon\right)}$ .This norm is easily computed , using Lemma

(2.5) and Plancherel formula . Indeed , we have that

Lemma 3.3 : For $\alpha \in \mathbb{R}$, the integral

$$\int_V |\Delta(x+iy)|^{-\alpha} dx\ , \quad y \in \Omega$$

is finite if and only if $\alpha > \frac{2n}{r} - 1$. In this case , it is equal to $c(\alpha)\Delta(y)^{-\alpha+\frac{n}{r}}$.

Using also Minkowski inequality , we see that

$$\sum_k \left(\int_V \left|P^+_{\left(\frac{n}{r}-1+\epsilon\right)} f_k(x+iy)\right|^{(1+\epsilon)} dx\right)^{\frac{1}{(1+\epsilon)}} \leq c \int_\Omega \sum_k \Delta\left(y + \tfrac{n}{r} - 1 + \epsilon\ \right)^{-\left(\frac{n}{r}-1+\epsilon\right)} F_k\left(\tfrac{n}{r} - 1 + \epsilon\ \right) Q\left(\tfrac{n}{r} - 1 + \epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r} - 1 + \epsilon\right)$$

with

$$\sum_k F_k\left(\frac{n}{r} - 1 + \epsilon\right) = \left(\sum_k \int_V \left|f_k\left(u + i\left(\frac{n}{r} - 1 + \epsilon\right)\right)\right|^{(1+\epsilon)} du\right)^{\frac{1}{(1+\epsilon)}} .$$

By assumption , F_k belongs to $L_V^{(1+\epsilon)}(\Omega)$, and has norm equal to the norm of f_k in $L^{(1+\epsilon),(1+\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}(\Omega)$. To conclude , we use the next proposition .

Proposition 3.4 : The operator with kernel $\Delta\left(y + \left(\frac{n}{r} - 1 + \epsilon\right)\right)^{-\left(\frac{n}{r}-1+\epsilon\right)}$ is bounded on $L_V^{(1+\epsilon)}(\Omega)$ if and only if $(1+\epsilon)'_{\left(\frac{n}{r}-1+\epsilon\right)} < (1+\epsilon) < (1+\epsilon)_{\left(\frac{n}{r}-1+\epsilon\right)}$.

We do not give the this proposition . It follows the same lines as the proof of Proposition (3.1) .

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Let us now prove the necessary condition of the theorem . We test the operator $P^+_{(\frac{n}{r}-1+\epsilon)}$ on functions $f_k(x+iy) = \chi_{|x|<2}(x)g_k(y)$, with g_k a positive function supported in the intersection of the cone with the Euclidean ball of radius $\frac{1}{2(1+\epsilon)}$ centered at 0 . The constant $(1+\epsilon)$, here , is the constant of Lemma (2.1) **.** Let us take for granted that there exists a constant c such that , for $y \in \Omega$ with $|y| < \frac{1}{2(1+\epsilon)}$, one has

$$\int_{|x|<1} |\Delta(x+iy)|^{-a} dx \geq c\Delta(y)^{-a+\frac{n}{r}}. \qquad (3.4)$$

We postpone the proof of this inequality , and go on with the proof of the theorem . For x such that $|x| < 1/2$, and $y \in \Omega$ such that $|y| < \frac{1}{(2+2\epsilon)}$, one has the inequality

$$P^+_{(\frac{n}{r}-1+\epsilon)} \sum_k f_k(x+iy) \geq c\int_\Omega \sum_k \Delta\left(y+\left(\tfrac{n}{r}-1+\epsilon\right)\right)^{-(\frac{n}{r}-1+\epsilon)} g_k\left(\tfrac{n}{r}-1+\epsilon\right) Q\left(\tfrac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r}-1+\epsilon\right).$$

By assumption , there exists a constant C independent of g_k , such that

$$\int_{y\in\Omega,|y|<\frac{1}{(2+2\epsilon)}} \sum_k \left(\int_\Omega \Delta\left(y+\tfrac{n}{r}-1+\epsilon\right)^{-(\frac{n}{r}-1+\epsilon)} g_k\left(\tfrac{n}{r}-1+\epsilon\right)\Delta\left(\tfrac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r}-1+\epsilon\right)\right)^{(1+\epsilon)} \Delta(y)^{(\frac{n}{r}-1+\epsilon)} dy$$

$$\leq C\int_\Omega \sum_k g_k\left(\tfrac{n}{r}-1+\epsilon\right)^{(1+\epsilon)} \Delta\left(\tfrac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r}-1+\epsilon\right)$$

By homogeneity of the kernel , we can replace the constant $\frac{1}{(2+2\epsilon)}$ by any positive constant N : for every positive function g_k on Ω , we have the inequality

$$\int_{y\in\Omega,|y|<N} \sum \left(\int_\Omega \Delta\left(y+\tfrac{n}{r}-1+\epsilon\right)^{-(\frac{n}{r}-1+\epsilon)} g_k\left(\tfrac{n}{r}-1+\epsilon\right)\Delta\left(\tfrac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r}-1+\epsilon\right)\right)^{(1+\epsilon)} \Delta(y)^{(\epsilon-1)} dy$$

$$\leq C\int_{y\in\Omega,|\frac{n}{r}-1+\epsilon|<N} \sum_k g_k\left(\tfrac{n}{r}-1+\epsilon\right)^{(1+\epsilon)} \Delta\left(\tfrac{n}{r}-1+\epsilon\right)^{(\epsilon-1)} d\left(\tfrac{n}{r}-1+\epsilon\right).$$

Using the density of compactly supported functions , we get the same inequality without any bound on integrals . The necessary condition of the theorem is then a consequence of the necessary condition in Proposition 3.4 .

It remains to prove (3.4) . It is sufficient to prove the inequality

$$\int_{B_1(y)} |\Delta(x+iy)|^{-a} dx \geq c\Delta(y)^{-a+\frac{n}{r}}.$$

Indeed , we deduce from Lemma (2.1) that the invariant ball $B_1(y)$is contained in the Euclidean ball $\{|x|<1\}$. Now , we can use the fact that Δ is almost constant on the invariant ball , which allows to write that the left hand side is equivalent to

$$\Delta(y)^{\frac{n}{r}} \int_{B_1(y)} |\Delta(x+iy)|^{-a} \frac{dx}{\Delta(x)^{\frac{n}{r}}}$$ تراجع هذه

Using the action of G_K and the formula of change of variable for Δ , we see that this last quantity is equal to $\Delta(y)^{-a+\frac{n}{r}}$, multiplied by the same integral when computed for $y = e$. This last factor is clearly a positive constant .

For both Problems 1 and 2 , we see that the study below the small critical index can be deduced from the boundedness of positive operators on the cone Ω . One needs different methods to take into account the oscillations of the Bergman kernel .

The large Critical Index4.

The large critical index is equal to

$$\widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)} = \frac{\frac{n}{r}-1}{\left(\frac{n}{r(1+\epsilon)'}-1\right)_+}(1+\epsilon)_k = \frac{\left(2\frac{n}{r}-2+\epsilon\right)}{\left(\frac{n}{r(1+\epsilon)'}-1\right)}$$

with the convention that $\widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)} = \infty$ if $(1+\epsilon)' \geq \frac{n}{r}$, that is $(1+\epsilon) \leq 1 + \left(\frac{n}{r}-1\right)^{-1}$.

Let us first consider its relation with Problem 1. It follows from Lemma (2.6) that the Bergman kernel $B_{(\frac{n}{r}-1+\epsilon)}(\cdot+ie) = d(\frac{n}{r}-1+\epsilon)\Delta^{-(\frac{n}{r}-1+\epsilon)+c}(c-i)$ is in $L^{(1+\epsilon)',(1+\epsilon)'}_{(\frac{n}{r}-1+\epsilon)}$ if and only when $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. But , if the Bergman projection of the characteristic function of an Euclidean ball , which is centered at ie and contained inside T_Ω , belongs to $L^{(1+\epsilon)'}_{(\frac{n}{r}-1+\epsilon)'}, (1+\epsilon)'$. By the mean value equality , this is the function $B_{(\frac{n}{r}-1+\epsilon)}(\cdot+ie)$, up to a constant . So the condition $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$ is necessary for the boundedness of the Bergman projection $P_{(\frac{n}{r}-1+\epsilon)}$.

We do not know whether this condition is sufficient for the boundedness of the Bergman projection . Nevertheless , it is sufficient to have a reproducing formula for $A^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ functions in terms the Bergman kernel : for $f_k \in A^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ and $z \in T_\Omega$, we may write

$$\sum_k f_k(z) = \int_k \sum_k B_{(\frac{n}{r}-1+\epsilon)}(z-\bar{w}) f_k(w) \Delta(Im\, w)^{(\epsilon-1)}\, dw .$$

Indeed , this identity is valid for $f_k \in A^2_{(\frac{n}{r}-1+\epsilon)}$. Such functions are dense in $A^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$, and we can pass to the limit since $B_{(\frac{n}{r}-1+\epsilon)}(z-\cdot)$ is in $L^{(1+\epsilon)',(1+\epsilon)'}_{(\frac{n}{r}-1+\epsilon)}$.

Let us now consider Problem 2 . We will only consider the values of $(1+\epsilon)$ for which $\epsilon > 1+\left(\frac{n}{r}-1\right)^{-1}$, and refer to [4] for other values . We prove first that there is no Hardy inequality for $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. Indeed , there exists a function which is in $A^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$, and which is annihilated by the ■ operator. It is sufficient to consider the function $\Delta(z+ie)^{-\frac{n}{r}+1}$, and to use again Lemma (2.6) . When $(1+\epsilon) = \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$ (we only consider here the case when $\widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)} < \infty$) , the proof is more technical : one considers the function

$$F_k(z) = \Delta((z+ie)/i)^{-\frac{n}{r}+1}\left(1+log\Delta\left((z+ie/i)\right)\right)^{-\frac{1}{(1+\epsilon)}} ,\ z \in T_\Omega$$

It is possible to compute explicitly ■ F_k , and to see that , in its expression , the Logarithm appears with a square (see [8]) . It is also possible to compute the $L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ norms of both functions , and see that his an infinite norm , while ■ F_k has a finite one . Similar computations are done in [4] for another counter example which is used later . We conclude that , as for problem 1 . , one can find easily that problem 2 has a negative answer above the large critical index .

Let us mention a related problem , the injectivity of the ■ $F_k$operators . We have seen that , for $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$ and $\epsilon > 1+\left(\frac{n}{r}-1\right)^{-1}$, there exists a function $F_k \in A^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ such that ■ $F_k = 0$. Can one prove that it is not the case for the other values such that $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. For $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$ using the representative formula (4.1) , we can write that :

$\sum_k$ ■$^m(\sum_k F_k(z)) = c\int_{T_\Omega} \sum_k B_{(\frac{n}{r}-1+\epsilon+m)}(z-\bar{w}) F_k(w) \Delta(Im\, w)^{(\epsilon-1)} dw$.

We used the fact that ■^{m}■$^{-\alpha} = c\Delta^{-\alpha-m}$, which implies that ■$^m B_{(\frac{n}{r}-1+\epsilon)}(\cdot - w) = c\, B_{(\frac{n}{r}-1+\epsilon+m)}(\cdot - w)$. To prove that there does not exist such a function with ■$^m F_k = 0$. It is sufficient to prove the density of the functions $B_{(\frac{n}{r}-1+\epsilon+m)}(\bar{z}-\cdot)$ in $A^{(1+\epsilon)',(1+\epsilon)'}_{(\frac{n}{r}-1+\epsilon)}$. Indeed , if it is the case and if ■$^m F_k(z) = 0$ for all z , then the scalar product of F_k with $B_{(\frac{n}{r}-1+\epsilon)}(\bar{z}-\cdot)$ is also 0 , which implies , by the representative formula ,that F_k is identically 0 . For m large , the density follows the fact that the projection $P_{(\frac{n}{r}-1+\epsilon+m)}$ is bounded in $L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$, see [5] . We have proved that the ■ operator is injective for $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. It remains to consider the other case s , for which we have no conjecture .

Let us finally consider Problem 3 , and the Basov spaces that we have introduced in Section 2 . For F_k a consider set in V , let us denote by $S'_{F_k} = S'_{F_k}(V)$ the space of tempered distributions with Fourier transform supported in F_k. It is clear that the natural definition for $L^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ is the following .

Definition (4.1) : Given $\left(\frac{n}{r}-1+\epsilon\right) \in \mathbb{R}, 0 \leq \epsilon < \infty$, we define $B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ as the space of equivalence classes of tempered distributions .

$$B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)} = \left\{ f_k \in S'_{\overline{\Omega}} \, \| \|F_K\|_{B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}} < \infty \right\} \Big/ S'_{\partial\Omega} .$$

One would like to identify an element of $B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}$ with a representative of the equivalence class , and also to define its Fourier-Laplace transform . Again , we claim that this possibility is related with the condition $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$. This is based on the next proposition .

Proposition (4.2) : Let $(f_k)_j$ a sequence of functions on V such that $B_1(\xi_k)_j$ has spectrum in

and

$$\sum \Delta(\xi_k)_j^{-(\frac{n}{r}-1+\epsilon)} \|(F_k)_j\|^{(1+\epsilon)}_{(1+\epsilon)} < \infty.$$

Then , if $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$ the series $\sum_j (\sum_k f_k)_j$ converges in S'. Moreover , this property holds for all such sequences only if $1+\epsilon < \widetilde{(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon),(1+\epsilon)}$.

Proof : Let us prove the necessary condition . Assume that φ is a smooth function whose Fourier transform has compact support , and is 1 in a neighborhood of 0 .Since the series $\sum_j \sum_k |\langle (f_k)_j, \varphi\rangle|$ converges for any order which is chosen on the $(\xi_k)_j{}'s$, it means that $\sum_j \sum_k |\langle (f_k)_j, \varphi\rangle|$ converges , with the sun taken for $(\xi_k)_j$ in a neighborhood of 0 . If $(f_k)_j$ has spectrum in $B_{1/2}(\xi_k)_j$, then this means that $\sum_j \sum_k |\langle (f_k)_j, \psi_j\rangle|$ converges . Using the action of G we may assume that $\psi_j$is equal to $\Delta\left((\xi_k)_j\right)^{-\frac{n}{r}} \psi_j \, o \, g_k$, and take also $(f_k)_j = a_j f_k \, o \, g_k$ where $(\xi_k)_j$ is a fixed function whose spectrum is contained in $B_{1/2}(e)$ and such that $\langle f_k, \psi_j \rangle$ is not zero . We take for g_k an element of G such that $(\xi_k)_j = g_k e$. Then , we have that $\sum_j |a_j| < \infty$ whenever $\sum \Delta(\xi_k)_j^{-(\frac{n}{r}-1+\epsilon)-\frac{n(2-\epsilon)}{r(2+\epsilon)}} |a_j|^{(1+\epsilon)} < \infty$. This implies the inequality

$$\sum_j \sum_k \Delta(\xi_k)_j^{\left(\frac{n}{r}-1+\epsilon\right)(2-\epsilon)'/(2-\epsilon)+\frac{n(2-\epsilon)'}{r(2+\epsilon)}} < \infty.$$

where the sum is taken for $(\xi_k)_j$ in a neighborhood of 0 . Using Proposition 2.4 , the fact that this sum is finite is equivalent to the fact that

$$\int_{y\in V, |y|<\epsilon} \Delta(y)^{\left(\frac{n}{r}-1+\epsilon\right)\frac{(2-\epsilon)'}{(2-\epsilon)}+\frac{n(2-\epsilon)'}{r(2+\epsilon)}-\frac{n}{r}} dy < \infty .$$

This last inequality is valid if and only if $(2-\epsilon) < \widetilde{(2-\epsilon)}_{(\frac{n}{r}-1+\epsilon),(2+\epsilon)}$.

We refer to [4] for the proof of the fact that , whenever $(2-\epsilon) < \widetilde{(2-\epsilon)}_{(\frac{n}{r}-1+\epsilon),(2+\epsilon)}$ and ψ_j belongs to S', then the semi-norm of ψ_j in $B^{(2+\epsilon)',(2-\epsilon)'}_{-\left(\frac{n}{r}-1+\epsilon\right)(2-\epsilon)'/(2-\epsilon)}$ is finite . So the series $\sum_j |\langle (f_k)_j, \psi_j\rangle|$ is absolutely convergent .

Assume that $(2-\epsilon) < \widetilde{(2-\epsilon)}_{(\frac{n}{r}-1+\epsilon),(2+\epsilon)}$. If we use the previous proposition for $(f_k)_j = f_k * \psi_j$with $f_k \in S'_{\overline{\Omega}}$ such that $\sum_k \|f_k\|_{B^{(1+\epsilon),(1+\epsilon)}_{(\frac{n}{r}-1+\epsilon)}} < \infty$, we see that $\sum_j \sum_k f_k * \psi_j$ converges in S' to an element $f_k^{\approx}$ which depends only on the equivalence class of f_k . The mapping $f_k \longmapsto f_k^{\approx}$defines a mapping from to .

Moreover , it is an injective mapping . To prove this , it is sufficient to prove that $f_k * \psi_j = 0$ for all whenever $f_k^{\approx} = 0$. But , then

$$\sum_k f_k * \psi_j(x) = \sum_j \sum_k \langle f_k, \overline{\psi_j(x-\cdot)}\rangle = \langle \sum_j \sum_k f_k * \psi_j , \overline{\psi_j(x-\cdot)}\rangle$$

$$= \sum_k \langle f_k^{\ast} , \overline{\psi_j(x-\cdot)}\rangle = 0$$

In the previous identities , the infinite one because of the finite intersection property , and this allows to pass to the limit . This means that , below the large critical index , $B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ indentifies with a space of tempered distributions . One many also define the space of holomorphic functions $B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ as the Fourier Laplace transform of $B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$. Indeed , for $f_k \in B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ with $(2-\epsilon) < \widetilde{(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right),(2+\epsilon)}$, we can define $\sum_k \mathcal{L}_k(f_k) = \sum_k \mathcal{L}_k(f_k^{\ast}) = \sum_j \sum_k \mathcal{L}_k\big(f_k * \psi_j\big)$. As before , to prove that this last sum is well defined , itsufficient to prove that , for $y \in B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$, the function whose Fourier transform is $exp(-y|\cdot)$ has a finite semi-norm in $B^{(2+\epsilon)',(2-\epsilon)'}_{-\left(\frac{n}{r}-1+\epsilon\right)(2-\epsilon)'/(2-\epsilon)}$. The computation is nearly the same as the previous one , and it is the case when $(2-\epsilon) < \widetilde{(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right),(2+\epsilon)}$. When $(2-\epsilon) < \widetilde{(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right),(2+\epsilon)}$, it makes sense to ask whether $B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ is equal to $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$. This is Problem 3.

4. IN BETWEEN; RESULTS AND OPEN QUESTIONS

We now state the results , and refer to [4] for the proofs . Let us first consider Problem 2 and 3 for **Theorem (5.1)**: For $\epsilon \le 0$ and for all $0 \le \epsilon < \infty$, the spaces $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ and $B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ coincide. Moreover , for m a positive integer , the $\blacksquare^m$ operator is an isomorphism between $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ and $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)+(2-\epsilon)m}$.

We will only sketch the proof for Problem 2 , to show again the importance of representation formulas . Remember that $\blacksquare^m B_{\left(\frac{n}{r}-1+\epsilon\right)}(\cdot - w) = cB_{\left(\frac{n}{r}-1+\epsilon\right)+m}(\cdot - w)$. For F_k a function in $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)+(2-\epsilon)m}$, which many be written as

$$\sum_k F_k(z) = \int_{T_\Omega} \sum_k B_{\left(\frac{n}{r}-1+\epsilon\right)}(z-\overline{w}) F_k(w) \Delta(Im\, w)^{(\epsilon-1+m)} dw$$

a natural solution of Equation $\blacksquare^m G_k = F_k$ is given by

$$\sum_k G_k(z) = \int_{T_\Omega} \sum_k B_{\left(\frac{n}{r}-1+\epsilon\right)}(z-\overline{w}) F_k(w) \Delta(Im\, w)^{(\epsilon-1+m)} dw$$

It remains to see that this makes sense , and gives the only solution (remember that we have proved the uniqueness) .In the next theorem , we state the equivalence between Problem 1,2 and 3 .

Theorem (5.2) : Let $0 \le \epsilon < \infty$ and $2 < (2-\epsilon) < \widetilde{(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right),(2+\epsilon)}$. Then there exists an integer m_0 such that the three properties are equivalent :

1. The projection $P_{\left(\frac{n}{r}-1+\epsilon\right)}$ extends into a continuous operator in $L^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$.

2. A holomorphic function F_k belongs to $A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$ if and only if it may be written as $\mathcal{L}(f)$, with $f \in B^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$.

3. For some m larger thatm_0 , then there exists$C$ such that the Hardy inequality of order ,

$$\left\|\sum_k F_k\right\|_{L^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}} \le C_{(2+\epsilon),(2-\epsilon)} \sum_k \|(Im\, w)^m \blacksquare^m F_k\|_{L^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}}$$

holds for all $F_k \in A^{(2+\epsilon),(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right)}$.

Moreover , if one of the properties is satisfied , then inequalities of Hardy are valid at all orders.

We do not know whether m_0 can be taken equal to 1 , or whether there exists a range for which the Hardy inequality of order 1 holds , while Hardy inequalities of higher order do not . It follows from this theorem and theorem 1.3 that Problem 1 , 2 and 3 have a positive answer in the range $[2, 2-\epsilon)$. It remains to explore the range $\left[(2-\epsilon)_{\left(\frac{n}{r}-1+\epsilon\right)}, \widetilde{(2-\epsilon)}_{\left(\frac{n}{r}-1+\epsilon\right),(2+\epsilon)}\right)$. The next theorem will give a partial answer . Let us first consider another related problem , which has its own interest .

Problem 4 : Basov spaces for $\frac{n}{r}+\epsilon = 1$, and purely Fourier analysis approach . Up to now , we only considered values of $\left(\frac{n}{r}-1+\epsilon\right)$ for which $\epsilon > 0$. Such values are related to the weighted Bergman spaces . The case $\frac{n}{r}+\epsilon = 1$ is related to Hardy spaces : by this , we mean the Laplace transforms of functions which are in $L^2(V)$ are functions of the Hardy space H^2, and conversely .

Once one has a Whitney decomposition of the cone Ω , one may ask whether there is an associated Littlewood-Paley inequality for functions in $L^{(2+\epsilon)}(V)$, that is , whether there exists a constant C such that , for $(f_k)_j \in L^{(2+\epsilon)}(V)$,

$$\left\|\left(\sum_j \sum_k \left|(f_k)_j * \psi_j\right|^2\right)^{\frac{1}{2}}\right\|_{(2+\epsilon)} \leq C \sum_k \|(f_k)_j\|_{(2+\epsilon)}$$

By duality , it implies that , for $(f_k)_j$ with spectra in $B_j(\xi_k)_j$, one has the inequality

$$\left\|\sum_j \sum_k (f_k)_j\right\|_{(2+\epsilon)} \leq C \left\|\left(\sum_j \sum_k \left|(f_k)_j\right|^2\right)^{\frac{1}{2}}\right\|_{(2+\epsilon)'}$$

For $\epsilon \neq 2$, both Littlewood-Paley inequalities for $(2+\epsilon)$ and $(2+\epsilon)'$ cannot be valid in the same time , since the characteristic function of the cone Ω is not a Fourier multiplier .

We shall in fact consider a different property , which is weaker when $\epsilon \geq 0$ and $s \geq 2$: the existence of some constant C such that

$$\left(\sum_j \sum_k \left\|(f_k)_j * \psi_j\right\|_{(2+\epsilon)}^s\right)^{\frac{1}{2}} \leq C \left\|\sum_k (f_k)_j\right\|_{(2+\epsilon)}$$

Such an inequality can only be valid for $s \geq 2$. Indeed , take $(f_k)_j$ with disjoint spectra in $B_{\frac{1}{2}}(\xi_k)_j$ and $\sum f_k = \sum \varepsilon_j \sum_k (f_k)_j$ where the $\varepsilon_j' s$ are independent ± 1 given by Rademacher functions . Then , using Khintchine inequalities and assuming that (5.2) holds , we find that

$$\left(\sum_j \sum_k \left\|(f_k)_j * \psi_j\right\|_{(2+\epsilon)}^s\right)^{\frac{1}{2}} \leq C \left\|\left(\sum_j \sum_k \left|(f_k)_j\right|^2\right)^{\frac{1}{2}}\right\|_{(2+\epsilon)}$$

We test last inequality on N functions $(f_k)_j$ with same modulus (taking translations of the same function on Fourier side) to find a contradiction if $s < 2$.

In the other direction , (5.2) is certainly valid for $s = max\left((2+\epsilon), (2+\epsilon)'\right)$, by interpolations between the cases $\epsilon = 1, \epsilon < \infty$, for which it is a consequence of the fact that the norms of ψ_j in L^1 are uniformly bounded , and $\epsilon = 0$, for which it follows from the finite intersection property .

By duality , it is equivalent to the fact that , for a finite sequence of functions $(f_k)_j$ whose transforms are supported in $B_1(\xi_k)_j$, one has the inequality

We call $\left(C_{(2+\epsilon)'}(s')\right)$ this last property . This means in particular that , when $\left(C_{(2+\epsilon)}(s)\right)$ holds , one has the following inclusion related to the Basov space for $\frac{n}{r}+\epsilon = 1$

$$\left\|\sum_j \sum_k (f_k)_j\right\|_{(2+\epsilon)} \leq C \left(\sum_j \left\|\sum_k (f_k)_j\right\|_{(2+\epsilon)'}^{s'}\right)^{\frac{1}{s'}}$$

Problem 4 consists in finding the critical index for $\left(C_{(2+\epsilon)}(s)\right)$, between $min\left((2+\epsilon), (2+\epsilon)'\right)$ and 2 . Let us remark that $\left(C_{(2+\epsilon)}(s)\right)$ implies , in particular , that infinite sums $\sum_j \sum_k (f_k)_j$ for which $\sum_j \sum_k \left\|(f_k)_j\right\|_{(2+\epsilon)}^s$ is finite , converge in

S' . This indicates , by Proposition 4.2 that $s < \widetilde{(2-\epsilon)}_{0,(2+\epsilon)} = \frac{\frac{n}{r}-1}{\left(\frac{n}{r(2+\epsilon)'}-1\right)}$ .This constraint is only interesting when this number is smaller than 2 , that is we answer to Problem 4 when $\epsilon \leq 0$: then $(2-\epsilon)$ is the best possible index . For $\epsilon > 0$, it seems to be a difficult problem , which is related to the other ones as it can be see in the next theorem .

Theorem(5.3) : If the condition $\left(C_{(2+\epsilon)}(s)\right)$holds , then Problems 1,2 and 3 have a positive answer for $(2-\epsilon)$ in the range $\left[2.s(2-\epsilon)_{(\frac{n}{r}-1+\epsilon)}\right)$. It is the case , in particular , when $s = min\left((2+\epsilon),(2+\epsilon)'\right)$ Moreover , this last result is optimal when $\epsilon < 1$.

Again , we refer to [4] for the proof . The counterexamples are given , as before , by functions of the determinant function involve powers and logarithms . We also prove there that a necessary condition for a positive answer to Problems 1, 2 and 3 is the existence of a constant C such that ,for all finite sequences of functions $(f_k)_j$whose Fourier s transforms are supported are supported in $B_1(\xi_k)_j$, one has the inequality

$$\left\|\sum_j \sum_k (f_k)_j\right\|_{(2+\epsilon)}^{(2-\epsilon)} \leq C \sum_j \sum_k \Delta(\xi_k)_j^{-(\frac{n}{r}-1+\epsilon)} \left\|(f_k)_j\right\|_{(2+\epsilon)}^{(2-\epsilon)}$$

where the sum is restricted to those $(\xi_k)_j{}'s$ which are of Euclidean norm less than 1 . An easy consequence of this , using Khintchine inequalities as before , is the necessary condition $(2-\epsilon) < 2(2-\epsilon)_{(\frac{n}{r}-1+\epsilon)}$for all $(2+\epsilon)$ *:* the larger range is obtained for $\epsilon = 0$.

These result leave a gap , for Problems 1to 3 , as well as for Problem 4 , for $\epsilon > 0$ .It is possible that solving the problems in the gap is of consider difficulty . Moreover, the sufficient conditions given by Problem 4 and the necessary conditions (5,3) seem very close, and give a purely Fourier analysis formulation of the different problems . Indeed , work in progress allows to fill part of the gap when using it for the forward light cone in dimension 3 .

Among other open problems , let us mention the boundedness of the projection $P_{(\frac{n}{r}-1+\epsilon)}$ for the limit case $\epsilon = 0$ (see [10]) .One does not know whether there is an interval of$(2+\epsilon)$ for which it is bounded in $L^{(2+\epsilon)}$.

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