A Very Brief Note on the Riemann Hypothesis

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Abstract. Robin's criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. In this note, using Robin's inequality on superabundant numbers, we prove that the Riemann Hypothesis is true. This proof is an extension of the article "Robin's criterion on divisibility" published by The Ramanujan Journal on May 3rd, 2022.

Keywords: Riemann Hypothesis · Robin's inequality · Sum-of-divisors function · Superabundant numbers · Prime numbers.

1 Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann Hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$. We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The Ramanujan's Theorem stated that if the Riemann Hypothesis is true, then the previous inequality holds for large enough n. Next, we have the Robin's Theorem:

Proposition 1. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [4, Theorem 1 pp. 188].*

Superabundant numbers were defined by Leonidas Alaoglu and Paul Erdős (1944). In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [2, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following property for the superabundant numbers:

Proposition 2. *If n is superabundant, then n is a Hardy-Ramanujan integer [1, Theorem 1 pp. 450].*

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \;\; for \;\; (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 3. *Every colossally abundant number is superabundant [1, pp. 455].*

Several analogues of the Riemann Hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann Hypothesis might be false.

Proposition 4. *If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that $\mathsf{Robin}(n)$ fails (i.e. $\mathsf{Robin}(n)$ does not hold) [4, Proposition pp. 204].*

Putting all together yields the proof of the Riemann Hypothesis.

2 Main Results

The following is a key Lemma.

Lemma 1. *If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers n such that $\mathsf{Robin}(n)$ fails.*

Proof. This is a direct consequence of Propositions 1, 3 and 4. □

For every prime number $q_k > 2$, we define the sequence:

$$Y_k = \frac{e^{\frac{0.2}{\log^2(q_k)}}}{(1 - \frac{1}{\log(q_k)})}.$$

As the prime number q_k increases, the sequence Y_k is strictly decreasing [5, Lemma 6.1 pp. 6]. We use the following Propositions:

Proposition 5. *[5, Theorem 6.6 pp. 8]. Let* $\prod_{i=1}^{k} q_i^{a_i}$ *be the representation of a superabundant number* $n > 5040$ *as the product of the first* k *consecutive primes* $q_1 < \ldots < q_k$ *with the natural numbers* $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ *as exponents. Suppose that* $\mathsf{Robin}(n)$ *fails. Then,*

$$\alpha_n < \frac{\log\log(N_k)^{Y_k}}{\log\log n},$$

where $N_k = \prod_{i=1}^{k} q_i$ *is the primorial number of order* k *and* $\alpha_n = \prod_{i=1}^{k}\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right)$.

Proposition 6. *[3, Lemma 3.3 pp. 8]. Let* $x \geq 11$. *For* $y > x$, *we have*

$$\frac{\log\log y}{\log\log x} < \sqrt{\frac{y}{x}}.$$

This is the main insight.

Lemma 2. *Let* $\prod_{i=1}^{k} q_i^{a_i}$ *be the representation of a superabundant number* $n > 5040$ *as the product of the first* k *consecutive primes* $q_1 < \ldots < q_k$ *with the natural numbers* $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ *as exponents. Suppose that* $\mathsf{Robin}(n)$ *fails. Then,*

$$\alpha_n < \sqrt{\frac{(N_k)^{Y_k}}{n}},$$

where $N_k = \prod_{i=1}^{k} q_i$ *is the primorial number of order* k *and* $\alpha_n = \prod_{i=1}^{k}\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right)$.

Proof. When $n > 5040$ is a superabundant number and $\mathsf{Robin}(n)$ fails, then we have

$$\alpha_n < \frac{\log\log(N_k)^{Y_k}}{\log\log n}$$

by Proposition 5. We assume that $(N_k)^{Y_k} > n > 5040 > 11$ since $\alpha_n > 1$. Consequently,

$$\frac{\log\log(N_k)^{Y_k}}{\log\log n} < \sqrt{\frac{(N_k)^{Y_k}}{n}}$$

by Proposition 6. As result, we obtain that

$$\alpha_n < \sqrt{\frac{(N_k)^{Y_k}}{n}}$$

and thus, the proof is done. □

This is the main theorem.

Theorem 1. *The Riemann Hypothesis is true.*

Proof. We know there are infinitely many superabundant numbers [1, Theorem 9 pp. 454]. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. For every prime q, $\nu_q(n)$ goes to infinity as long as n goes to infinity when n is superabundant [3, Theorem 4.4 pp. 12], [1, Theorem 7 pp. 454]. Let $n_k > 5040$ be a large enough superabundant number such that q_k is the largest prime factor of n_k. Suppose that $\mathsf{Robin}(n_k)$ fails. In the same way, let $n_{k'}$ be another superabundant number much greater than n_k such that $\mathsf{Robin}(n_{k'})$ fails too. By Lemma 2, we have

$$\alpha_{n_k} < \sqrt{\frac{(N_k)^{Y_k}}{n_k}}$$

and

$$\alpha_{n_{k'}} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}}.$$

Hence,

$$\alpha_{n_{k'}} \cdot \alpha_{n_k} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}} \cdot \alpha_{n_k}.$$

Consequently,

$$\alpha_{n_{k'}} \cdot \alpha_{n_k} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}} \cdot \sqrt{\frac{(N_k)^{Y_k}}{n_k}}.$$

So,

$$(\alpha_{n_{k'}} \cdot \alpha_{n_k})^2 < \frac{(N_{k'})^{Y_{k'}}}{n_{k'}} \cdot \frac{(N_k)^{Y_k}}{n_k}.$$

However, we know that

$$(\alpha_{n_{k'}} \cdot \alpha_{n_k})^2 > 1.$$

Moreover, we can see that

$$\frac{(N_{k'})^{Y_{k'}}}{n_{k'}} \cdot \frac{(N_k)^{Y_k}}{n_k} \leq 1$$

since the following inequality

$$Y_k \leq \frac{\log(n_{k'} \cdot n_k)}{\log((N_{k'})^{\frac{Y_{k'}}{Y_k}} \cdot N_k)}$$

is satisfied for $n_{k'}$ much greater than n_k, because of $\frac{Y_{k'}}{Y_k} < 1$ and $\lim_{k \to \infty} Y_k = 1$. In this way, we obtain the contradiction $1 < 1$ under the assumption that $\mathsf{Robin}(n_k)$ fails. To sum up, the study of this arbitrary large enough superabundant number $n_k > 5040$ reveals that $\mathsf{Robin}(n_k)$ holds on anyway. Accordingly, $\mathsf{Robin}(n)$ holds for all large enough superabundant numbers n. This contradicts the fact that there are infinite superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann Hypothesis is false according to Lemma 1. By reductio ad absurdum, we prove that the Riemann Hypothesis is true. □

3 Conclusions

Practical uses of the Riemann Hypothesis include many propositions that are known to be true under the Riemann Hypothesis and some that can be shown to be equivalent to the Riemann Hypothesis. Indeed, the Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf Hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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