Binomial Coefficients in Combinatorial Geometric Series and its Combinatorial Identities

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: The coefficient of each term in combinatorial geometric series refers to a binomial coefficient. This paper discusses the binomial coefficients in combinatorial geometric series and presents its binomial expansions and combinatorial identities.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind to create a combinatorial geometric series [1-7]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-7] is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r.

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \& \ V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1$ and $n, r \in N = \{0, 1, 2, 3, \cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ denotes the combinatorial geometric series and V_n^r the binomial coefficient.

The traditional binomial coefficient denotes $\binom{n}{r} = nCr = \frac{n!}{r!\,(n-r)!}$, where $n, r \in N$.

The factorial function or factorial of a nonnegative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For examples, 5! = 1×2× 3×4×5 = 120 and 0! = 1.

Theorem 2.1: $\frac{(n+r)!}{n!\,r!} = \prod_{i=1}^{r} \frac{n+i}{r!} = V_n^r$, where $n, r \geq 0$ & $n, r \in N$.

$Proof.$ $\binom{n+r}{n} = \frac{(n+r)!}{n!\,(n+r-n)!} = \frac{(n+r)!}{n!\,r!} = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = V_n^r$.

$$\text{That is, } V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = \prod_{i=1}^{r} \frac{n+i}{r!}.$$

From the above expressions, we conclude that

$$\frac{(n+r)!}{n!\,r!} = \prod_{i=1}^{r} \frac{n+i}{r!} = V_n^r, \text{where } n, r \geq 0 \ \& \ n, r \in N.$$

Note that $(n+r)! = V_n^r n!\,r! \Longrightarrow (2n)! = 2V_{n-1}^n (n!)^2$.

The combinatorial or binomial identities of V_r^n are derived from Theorem 2.1 as follows:
(i) $V_n^0 = V_0^n = 1$ for $n = 0, 1, 2, 3, \cdots$
(ii) $V_r^m = V_m^r$, $(m, r \geq 1 \ \& \ m, r \in N)$.

$$Proof \text{ for identity } (i): \quad V_n^0 = V_0^n = \frac{(0+n)!}{0!\,n!} = 1 \Longrightarrow \quad V_0^0 == \frac{(0+0)!}{0!\,0!} = 1.$$

$$Proof \text{ for identity } (ii): \quad V_n^m = V_m^n = \frac{(m+n)!}{m!\,n!}.$$

$$\textbf{Theorem 2.2}: \frac{(n+p+r)!}{n!\,p!\,r!} = V_n^{p+r} \times V_p^r, \text{where } n, p, r \in N.$$

$$Proof: V_n^{p+r} \times V_p^r = \times \frac{(n+p+r)!}{n!\,(p+r)!} \times \frac{(p+r)!}{p!\,r!} = \frac{(n+p+r)!}{n!\,p!\,r!}.$$

Corollary 2.1

$$\frac{(n_1+n_2+n_3+\cdots+n_k)!}{n_1!\,n_2!\,n_3!\cdots n_k!} = V_{n_1}^{n_2+n_3+n_4+\cdots+n_k} \times V_{n_2}^{n_3+n_4+\cdots+n_k} \times V_{n_3}^{n_4+n_5+\cdots+n_k} \times \cdots \times V_{n_{k-1}}^{n_k},$$

where $k \geq 2; n_i \geq 0; and\ k, n \in N$.

3. Conclusion

In this article, the combinatorial geometric series and its binomial coefficients and combinatorial identities were introduced and theorems on binomial and multinomial coefficients and factorials discussed with detailed proofs for research and development further.

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