Entropies and logarithms

Amelia Carolina Sparavigna
Department of Applied Science and Technology, Politecnico di Torino, Torino, Italy

Abstract: Here we assume a discrete random variable, possessing a one-to-one correspondence with the set of natural numbers. Its Shannon entropy is considered and some distributions, obtained by means of the Maximum Entropy Principle, will be discussed too. Then two entropies, the q-entropy and κ-entropy proposed by C. Tsallis and G. Kaniadakis respectively, will be considered. These entropies have as their limit the Shannon entropy when entropic parameters q and κ approach specific values. We will show some relationships existing between the Shannon entropy and the above mentioned generalized entropies and give some links regarding other functions, in particular the related logarithms. Exponentials will be discussed too. We will also address ourselves to the generalization of the sum, in the framework of the κ-calculus proposed by Kaniadakis.

Keywords: Entropy, Shannon entropy, Hartley entropy, Rényi entropy, Maximum Entropy Principle, Uniform distribution, Geometric distribution, Generalized Entropies, q-entropy, κ-entropy, q-logarithm, κ-logarithm, q-exponential, κ-exponential, κ-calculus.

1. Introduction

Depending on the process which we are studying, the related random variable X may be continuous or discrete. In the first case, the random variable is given in a continuous range of values while, in the second case, the variable possesses a one-to-one correspondence with the set of natural numbers. If X is discrete, $p(x_i)$ is the probability distribution at each point x_i. In the framework of a discrete random variable, in 1948 [1], Claude Shannon defined the entropy H as the following expected value [2]:

$$H(x)=\sum_i p(x_i)I(x_i) = -\sum_i p(x_i)\log_b p(x_i)$$

In this expression, I is the information content of X, the probability of i-event is p_i and b is the base of the used logarithm. Common values of the base are 2, Euler's number e and 10.

Besides Shannon entropy, other entropies are used in information theory; here we will consider the generalized entropies proposed by C. Tsallis and by G. Kaniadakis, also known as q-entropy and κ-entropy [2-4]. Both entropies have simple functional forms. Actually, as told in [5], the q-entropy was introduced in Information Theory and Statistics from 1967, [6],

[7],[8], and after it was re-discovered in 1988 by Tsallis (its complete name is Havrda-Charvát-Tsallis entropy). Kaniadakis proposed a new formulation of entropy in 2001.

Here we will remember the relationship existing between q-entropy and κ-entropy [9]. Then, we will also propose a discussion of the relationships between the generalized form of some functions (such as exponential and logarithm).

2. The entropies

In the following formulas we can see defined the entropies (Shannon, Tsallis q-entropy and Kaniadakis κ-entropy), with a corresponding choice of measurement units equal to 1:

(1) entropy $$S=-\sum_i p_i \ln p_i$$

(2) q-entropy $$T=T_q=\frac{1}{q-1}\left(1-\sum_i p_i^q\right)$$

Here, let us add the Réyni entropy [10]. It has, like the q-entropy, its entropic parameter usually indicated by the letter q:

$$R_q=\frac{1}{1-q}\ln\left(\sum_i p_i^q\right)$$

In [3], the Réyni entropy is fundamental for the given discussion. Then, we have the Kaniadakis entropy:

(3) κ-entropy $$K_\kappa=-\sum_i \frac{p_i^{1+\kappa}-p_i^{1-\kappa}}{2\kappa}$$

In (2),(3) we have entropic parameters q and κ, so that $\lim_{q\to 1} T=S$; $\lim_{\kappa\to 0} K=S$.

3. The Hartley function

In the Shannon's article we can find Ralph Hartley mentioned.

"If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded ad a measure of the information produced when one message is chosen from the set, all choices being equally likely. As was pointed out by Hartley the most natural choice is the logarithmic function. Although this definition must be generalized considerably when we consider the influence of the statistics of the message and when we have a continuous range of messages, we will in all cases use an essentially logarithmic measure." And Shannon explain three reasons for using logarithm as a measure.

The Hartley measure was introduced in 1928. "If a sample from a finite set A uniformly at random is picked, the information revealed after the outcome is known is given by the Hartley function:

$$H_0(A) := \log_b |A|$$

where |A| denotes the cardinality of A". https://en.wikipedia.org/wiki/Hartley_function

Hartley used a base-ten logarithm. In fact, we can call it the Hartley entropy. "The Hartley function coincides with the Shannon entropy (as well as with the Rényi entropies of all orders) in the case of a uniform probability distribution. It is a special case of the Rényi entropy since:

$$H_0(X) = \frac{1}{1-0} \log \sum_{i=1}^{|X|} p_i^0 = \log |X|$$

But it can also be viewed as a primitive construction, since, as emphasized by Kolmogorov and Rényi, the Hartley function can be defined without introducing any notions of probability (see Uncertainty and information by George J. Klir, p. 423)".

4. Uniform distribution

Let us assume a uniform discrete distribution: $p_i = \frac{1}{n}$. The Shannon entropy is given by:

$$S = -\sum_i p_i \ln p_i = -\sum_i \frac{1}{n} \ln \frac{1}{n} = \frac{n}{n} \ln n = \ln n$$

In the case of Tsallis entropy, we have:

$$T = T_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right) = \frac{1}{q-1}\left(1 - \sum_i \frac{1}{n^q}\right) = \frac{1}{q-1}\left(1 - n^{1-q}\right) = \frac{n^{1-q}-1}{1-q}$$

As we will see in the following discussion, in the Tsallis calculus it has been defined the q-logarithm in the form:

$$\ln_q^T(f) = \frac{f^{1-q}-1}{1-q}$$

Therefore, the Tsallis entropy for the uniform distribution becomes:

$$T = T_q = \frac{n^{1-q} - 1}{1-q} = \ln_q^T(n)$$

Let us consider the Réyni entropy:

$$R_q = \frac{1}{1-q} \ln\left(\sum_i \frac{1}{n^q}\right) = \frac{1}{1-q} \ln n^{1-q} = \ln n$$

In the case of the Kaniadakis entropy:

$$K_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa} = -\frac{1}{2\kappa}\left(n^{-\kappa} - n^{\kappa}\right) = \frac{1}{2\kappa}\left(n^{\kappa} - n^{-\kappa}\right)$$

In the κ-calculus, the κ-logarithm is $\ln_\kappa f = \frac{f^\kappa - f^{-\kappa}}{2\kappa}$, then:

$$K_\kappa = \frac{1}{2\kappa}\left(n^{\kappa} - n^{-\kappa}\right) = \ln_\kappa n$$

Using again the Shannon’s words, as told by Hartley “the most natural choice is the logarithmic function”, and this function can be generalized considerably.

5. Maximum entropy principle

“The maximum entropy principle (Jaynes, 1957) provides a bridge between information theory and probability theory. It states that given certain a priori knowledge, the distribution that best represents the state of knowledge is the one with maximal entropy. As such, this principle explains why certain probability distributions take the forms they do” [11],[12].

Abstract of Jaynes’ article [12] tells that “Information theory provides a constructive criterion for setting up probability distributions on the basis of partial knowledge, and leads to a type of statistical inference which is called the maximum-entropy estimate. It is the least biased estimate possible on the given information; i.e., it is maximally noncommittal with regard to missing information. If one considers statistical mechanics as a form of statistical inference rather than as a physical theory, it is found that the usual computational rules, starting with the determination of the partition function, are an immediate consequence of the maximum-entropy principle”.

In the Appendix A of [12], “Entropy of a probability distribution”, Haynes is giving the Shannon entropy H using three conditions. Let us consider a variable with discrete values x_i, with probabilities p_i. Conditions are:

1) H is a continuous function of p_i,

2) if all p_i are equal, the quantity $A(n)=H(1/n,...,1/n)$ is a monotonic increasing function of n,

3) a composition exists such as $A(m)+A(n)=A(mn)$.

The solution is:

$$S=-\sum_i p_i \ln p_i \quad .$$

Let us consider the uniform distribution that we have previously used for entropy.

“It can be shown that a uniform distribution maximizes the entropy of a probability distribution *P*(*X*) subject to no more prior knowledge than that the probability masses need to sum to 1” [11].

Let us use the Lagrange multiplier:

$$L(X)=S(X)+\lambda\left(\sum_i p_i-1\right)=-\sum_i p_i \ln p_i+\lambda(\sum_i p_i-1)$$

$$\frac{\partial L}{\partial p_i}=-\ln p_i-1+\lambda=0 \quad ; \quad \frac{\partial L}{\partial \lambda}=\sum_i p_i-1=0$$

$$p_i=e^{\lambda-1} \quad ; \quad n e^{\lambda-1}=1 \quad \text{therefore} \quad p_i=\frac{1}{n}$$

n is the number of possible values of i. The uniform distribution is a probability distribution that “satisfies the a priori constraint and maximizes the uncertainty under that constraint” [11].

6. Shannon entropy and the geometric discrete distribution

Let us assume a geometric distribution:

$$p_i=(1-p)^{i-1}p \quad \text{, where } \quad i\in\mathbb{Z}^+ \quad \text{from 1 to infinity.}$$

Parameter p is $0<p\leqslant 1$.

The Shannon entropy (base 2) is:

$$-\sum_{i=1}^{\infty}(1-p)^{i-1}p\cdot\log_2((1-p)^{i-1}p)$$

$$= \quad -p\cdot\log_2 p\sum_{i=1}^{\infty}(1-p)^{i-1}-p\cdot\log_2(1-p)\sum_{i=1}^{\infty}(i-1)(1-p)^{i-1}$$

$$= \quad -\log_2 p-\frac{(1-p)\log_2(1-p)}{p}$$

The first sum is a geometric series: $\sum_{i=1}^{\infty}(1-p)^{i-1}=\frac{1}{p}$

https://www.wolframalpha.com/input?i=sum_%28k%3D1%29%5Einfinity+%281-p%29%5E%28k-1%29

The second sum is:

$$\sum_{k=0}^{\infty}k(1-p)^k=\sum_{k=1}^{\infty}k(1-p)^k=(1-p)\sum_{k=1}^{\infty}k(1-p)^{k-1}=(1-p)\frac{d}{d(1-p)}\sum_{k=0}^{\infty}(1-p)^k$$

https://www.wolframalpha.com/input?i=sum_%28k%3D0%29%5Einfinity++%281-p%29%5E%28k%29

$$= \quad (1-p)\frac{d}{d(1-p)}\left(\frac{1}{p}\right)=-(1-p)\frac{d}{dp}\left(\frac{1}{p}\right)=(1-p)\frac{1}{p^2}$$

For a given mean value μ, the entropy maximizing probability distribution on the non-negative integers is the geometric distribution. We have to maximize $S=\sum_{i}^{\infty}p_i\log p_i$ with the constraints $\sum_i i\,p_i=\mu$ and $\sum_i^{\infty}p_i=1$.

The problem has been discussed by Pierre Brémaud, in his “Discrete Probability Models and Methods. Probability on Graphs and Trees, Markov Chains and Random Fields, Entropy and Coding” [13], by means of the following example.

Example 11.1.6: Geometric Distribution Maximizes Entropy. Prove that the geometric distribution maximizes entropy among all positive integer-values random variables with given finite mean μ . The corresponding maximization problem is solved by the Lagrangian method [13].

Let us use the function:

$$-\sum_i p_i \log p_i - \lambda\left(\sum_i i\, p_i - \mu\right) \quad \text{, then} \quad \frac{\partial}{\partial p_i}\left(\sum_i p_i \log p_i\right) + \lambda \frac{\partial}{\partial p_i}\left(\sum_i i\, p_i\right) = 0$$

$$\log p_i + 1 + \lambda i = 0 \quad , \quad \text{therefore} \quad p_i = e^{-1-\lambda i} = \frac{1}{e} e^{-\lambda i} \qquad p_i = (1-p)^{i-1} p$$

In [13], it is told that the constraint $\sum_i p_i = 1$ finally yields $p_i = p(1-p)^{i-1}$. Moreover, $p = \mu^{-1}$, where μ is the mean of the geometric distribution.

This is what we find in [13]. However, let us discuss with more detail the calculation. Since we use i in exponentials, please remember it is a positive integer, not the imaginary unit.

Let us start from:

$$L = -\sum_i p_i \ln p_i - \lambda_o\left(\sum_1 p_i - 1\right) - \lambda_1\left(\sum_i i\, p_i - \mu\right)$$

$$\frac{\partial L}{\partial p_i} = 0 \quad \text{gives :} \quad 1 + \ln p_i + \lambda_o + \lambda_1 i = 0 \quad \text{, so that:} \quad \ln p_i = -1 - \lambda_o - \lambda_1 i$$

Therefore, the distribution is:

$$p_i = e^{-1-\lambda_o} e^{-\lambda_1 i} \quad .$$

$$\sum_i e^{-1-\lambda_o} e^{-\lambda_1 i} = 1 \rightarrow e^{-1-\lambda_o} \sum_i e^{-\lambda_1 i} = 1 \rightarrow e^{-1-\lambda_o} \cdot \frac{1}{e^{\lambda_1} - 1} = 1$$

The infinite sum is given by

https://www.wolframalpha.com/input?i=sum_%28k%3D1%29%5Einfinity++e%5E%28-lambda*k%29

$$e^{-1-\lambda_o}=e^{\lambda_1}-1$$

Now, let us consider the other constraint:

$$\sum_i i\, e^{-1-\lambda_o}\, e^{-\lambda_1 i}=\mu \;\rightarrow\; e^{-1-\lambda_o}\sum_i i\, e^{-\lambda_1 i}=\mu \;\rightarrow\; e^{-1-\lambda_o}\cdot\frac{e^{\lambda_1}}{\left(e^{\lambda_1}-1\right)^2}=\mu$$

$$(e^{\lambda_1}-1)\cdot\frac{e^{\lambda_1}}{\left(e^{\lambda_1}-1\right)^2}=\frac{e^{\lambda_1}}{e^{\lambda_1}-1} = \mu$$

The infinite sum is given by
https://www.wolframalpha.com/input?i=sum_%28k%3D1%29%5Einfinity+k*+e%5E%28-lambda*k%29

Let us write: $e^{\lambda_1}=y$, then:

$$\frac{y}{y-1}=\mu \;\rightarrow\; \frac{y-1}{y}=\frac{1}{\mu} \;\rightarrow\; 1-\frac{1}{y}=\frac{1}{\mu} \;\rightarrow\; \frac{1}{y}=1-\frac{1}{\mu} \;\rightarrow\; y=\frac{\mu}{\mu-1} \quad .$$

Then: $\lambda_1=\ln\frac{\mu}{\mu-1}$ and $e^{-1-\lambda_o}=e^{\lambda_1}-1=\frac{\mu}{\mu-1}-1=\frac{1}{\mu-1}$.

The distribution is:

$$p_i=e^{-1-\lambda_o}\, e^{-\lambda_1 i}=\frac{1}{\mu-1}\cdot e^{-i\ln\left(\frac{\mu}{\mu-1}\right)}=\frac{1}{\mu-1}\cdot\left(\frac{\mu-1}{\mu}\right)^i=\mu^{-i}(\mu-1)^{i-1}$$

$$= \mu^{-i+1}\cdot\mu^{-1}(\mu-1)^{i-1}=\mu^{-1}\cdot\left(\frac{1}{\mu}\right)^{i-1}(\mu-1)^{i-1}=\frac{1}{\mu}\left(1-\frac{1}{\mu}\right)^{i-1}=p(1-p)^{i-1}$$

Assuming $\mu=1/p$, we have the geometric distribution.

Other distributions are given here:
https://en.wikipedia.org/wiki/Maximum_entropy_probability_distribution#Other_examples

7. Maximizing *q*-entropy

Let us consider the natural constraint $\sum_i p_i = 1$ for the Tsallis entropy. Let us use the Lagrange multiplier λ_o , and consider n the number of occurrences.

$$L = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right) + \lambda_o\left(\sum_i p_i - 1\right)$$

$$\frac{\partial L}{\partial p_i} = -\frac{q}{q-1} p_i^{q-1} + \lambda_o = 0 \rightarrow p_i = \left(\frac{q-1}{q}\lambda_o\right)^{\frac{1}{q-1}}$$

$$\sum_i p_i = n\left(\frac{q-1}{q}\lambda_o\right)^{\frac{1}{q-1}} = 1 \rightarrow n^{q-1}\left(\frac{q-1}{q}\lambda_o\right) = 1 \rightarrow \lambda_o = \frac{q}{q-1}\frac{1}{n^{q-1}}$$

Therefore:

$$p_i = \left(\frac{q-1}{q}\frac{q}{q-1}\frac{1}{n^{q-1}}\right)^{\frac{1}{q-1}} = \frac{1}{n}$$

The distribution is the uniform distribution, as in the case of the Shannon entropy.

As told in [3], is is extremized, for all values of q, in the case of equiprobability.

From [14]: "The concept of entropy is closely linked with the concept of uncertainty, information, chaos, disorder, surprise or complexity. Indeed, there are often different interpretations of entropy in different fields. For instance, in statistics, entropy is regarded as a measure of randomness, objectivity or unbiasedness, dependence, or *departure from the uniform distribution*. In ecology, it is a measure of diversity of species or lack of concentration. In water engineering, it is a measure of information of uncertainty. In industrial engineering, it is a measure of complexity. In manufacturing, it is a measure of interdependence. In management, it is a measure of similarity. In social sciences, it is measure of equality".

We have seen that the uniform distribution is maximizing the Shannon and Tsallis entropies. It means that it is maximizing the randomness. If we add constraints to the distribution, we reduce randomness and increase the departure from uniformity (equiprobability).

8. Maximizing *q*-entropy (canonical ensemble)

In Ref. [3], C. Tsallis wanted to extremize T_q under the conditions:

$$\sum_i p_i = 1 \quad ; \quad \sum_i p_i \varepsilon_i = U_q$$

where ε_i and U_q are known real numbers. In [3], they are referred to as generalized spectrum and generalized internal energy.

The function used by Tsallis is the following:

$$L_q = T_q + \alpha \sum_i p_i - \alpha \beta (q-1) \sum_i p_i \varepsilon_i$$

The function had been written in this way for convenience.

Imposing $\partial L_q / \partial q_i = 0$, in [3] it is told that we can find:

$$p_i = \frac{[1-\beta(q-1)\varepsilon_i]^{1/(q-1)}}{Z_q} \quad \text{with} \quad Z_q \equiv \sum_i [1-\beta(q-1)\varepsilon_i]^{1/(q-1)}$$

9. Maximizing *κ*-entropy

Let us consider again the natural constraint $\sum_i p_i = 1$ for the Kaniadakis entropy. Let us use the Lagrange multiplier λ_o , and consider n the number of occurrences.

$$L = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa} + \lambda_o \left(\sum_i p_i - 1 \right)$$

$$\frac{\partial L}{\partial p_i} = -\frac{1}{2\kappa}(1+\kappa) p_i^{\kappa} + \frac{1}{2\kappa}(1-\kappa) p_i^{-\kappa} + \lambda_o = 0$$

$$p_i^{2\kappa} + \frac{2\kappa \lambda_o}{1+\kappa} p_i^{k} + \frac{1-\kappa}{1+\kappa} = 0 \rightarrow p_i^{\kappa} = f(\lambda_o, \kappa) \rightarrow p_i = \left(f(\lambda_o, \kappa) \right)^{1/\kappa} = \frac{1}{n}$$

Therefore we have the uniform distribution in this case too.

10. Joint entropy

Let us consider the joint entropy $H(A,B)$ of two independent subsystems *A,B*. We have the additivity for the Shannon entropy, but a generalized additivity for Tsallis and Kaniadakis entropies.

$$(4) \quad S(A,B)=S(A)+S(B)$$

It means:

$$S(A,B)=-\sum_{i,A}\sum_{j,B} p_i p_j \ln(p_i p_j) = -\sum_i \sum_j p_i p_j (\ln p_i + \ln p_j)$$

$$-\sum_i p_i \ln p_i \sum_j p_j = -\sum_j p_j \ln p_j \sum_j p_i = -\sum_{i,A} p_i \ln p_i \; -\sum_{j,B} p_i \ln p_j$$

where we used $\sum_i p_i = 1$, $\sum_j p_j = 1$.

For *q*-entropy:

$$(5) \quad T(A,B)=T(A)+T(B)+(1-q)T(A)T(B)$$

$$\frac{1}{q-1}\left(1-\sum_i p_i^q\right)+\frac{1}{q-1}\left(1-\sum_j p_j^q\right) -\frac{1}{q-1}\left(1-\sum_i p_i^q\right)\left(1-\sum_j p_j^q\right)$$

$$=\frac{1}{q-1}\left(1-\sum_i \sum_j p_i^q p_j^q\right) = T(A,B)$$

Let us note that the two independent subsystems *A,B* must have the same entropic parameter *q*.

In the case of κ-entropy:

$$(6) \quad K(A,B)=K(A)\Im(B)+K(B)\Im(A) \quad \text{with} \quad \Im=\sum_i \frac{p_i^{1+\kappa}+p_i^{1-\kappa}}{2}$$

In fact:

$$-\sum_i \frac{p_i^{1+\kappa}-p_i^{1-\kappa}}{2\kappa} \sum_j \frac{p_j^{1+\kappa}+p_j^{1-\kappa}}{2} -\sum_j \frac{p_j^{1+\kappa}-p_j^{1-\kappa}}{2\kappa} \sum_i \frac{p_i^{1+\kappa}+p_i^{1-\kappa}}{2}$$

$$=-\sum_i \sum_j \frac{p_i^{1+\kappa} p_j^{1+\kappa} - p_i^{1-\kappa} p_j^{1-\kappa}}{2\kappa} = K(A,B)$$

Note that for the generalized additivity of κ -entropy, we need another function containing probabilities (see [15] and references therein). As in the case of q-entropy, the parameter κ must be the same.

For the uniform distribution:

$$\Im = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2} = \frac{1}{2}\left(n^{-\kappa} + n^{\kappa}\right)$$

11. The *κ*-entropy expressed by means of an Euler infinite product expansion

Before discussing the link between entropies, let us stress an expansion discussed in [16], which is also helping us in understating the role of function $\Im$ as a generalization of unit.

As previously seen, the Kaniadakis entropy has the discrete form:

$$K_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}$$

Let us consider a term in the sum and use in it the Euler number and logarithm:

$$\frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa} = \frac{p_i}{2\kappa}\left(p_i^{\kappa} - p_i^{-\kappa}\right) = \frac{p_i}{2\kappa}\left(e^{\kappa \ln p_i} - e^{-\kappa \ln p_i}\right) = \frac{p_i}{2\kappa}\left(e^{u_i} - e^{-u_i}\right)$$

where we defined $u_i = \kappa \ln p_i$.

In [17] we can find mentioned a useful formula [18], which is an Euler infinite product expansion:

$$e^u - e^{-u} = 2u\left(1 + \frac{u^2}{\pi^2}\right)\left(1 + \frac{u^2}{2^2\pi^2}\right)\left(1 + \frac{u^2}{3^2\pi^2}\right)\ldots = 2u \prod_{j=1}^{\infty}\left(1 + \frac{u^2}{j^2\pi^2}\right)$$

Then the *κ*-entropy can be written as:

$$K_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa} = -\frac{1}{2\kappa}\sum_i 2\, p_i u_i \prod_{j=1}^{\infty}\left(1 + \frac{u_i^2}{j^2\pi^2}\right)$$

$$= \; -\frac{1}{2\kappa}\sum_i 2\, p_i \kappa \ln p_i \prod_{j=1}^{\infty}\left(1 + \frac{(\kappa \ln p_i)^2}{j^2\pi^2}\right) = -\sum_i (p_i \ln p_i) \prod_{j=1}^{\infty}\left(1 + \frac{(\kappa \ln p_i)^2}{j^2\pi^2}\right)$$

So we can write:

$$K_\kappa = -\sum_i (p_i \ln p_i) \prod_{j=1}^{\infty}\left(1 + \frac{(\kappa \ln p_i)^2}{j^2\pi^2}\right)$$

And here we can see clearly that κ-entropy becomes Shannon entropy for $\kappa \to 0$:

$$\lim_{\kappa \to 0} K_\kappa = S_{Shannon} = -\sum_i p_i \ln p_i$$

12. The *κ*-logarithm

Kaniadakis proposed a new form of logarithm, in the framework of his κ-calculus (see Ref. [19] for all details about the calculus). The κ-logarithm is used to define the entropy (3):

$$K_\kappa = -\sum_i p_i \ln_\kappa p_i$$

where

$$\ln_\kappa p_i = \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa}$$

As a consequence, using the Euler function previously seen, we can tell that [16]:

$$\ln_\kappa p_i = (\ln p_i) \prod_{j=1}^{\infty}\left(1 + \frac{(\kappa \ln p_i)^2}{j^2\pi^2}\right)$$

Again, when $\kappa \to 0$ the κ-logarithm becomes the logarithm.

13. Generalizing the unit

Let us add the following too: in the generalized additivity of *κ*-entropy, (6), it appears another function:

$$\Im_\kappa = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

In [17] we find another Euler formula [20]:

$$e^u + e^{-u} = 2\left(1+\frac{4u^2}{\pi^2}\right)\left(1+\frac{4u^2}{3^2\pi^2}\right)\left(1+\frac{4u^2}{5^2\pi^2}\right)\dots = 2\prod_{j=0}^{\infty}\left(1+\frac{4u^2}{(2j+1)^2\pi^2}\right)$$

And then [16]:

$$\Im_\kappa = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2} = \sum_i \frac{p_i}{2}\left[2\prod_{j=0}^{\infty}\left(1+\frac{4(\kappa \ln p_i)^2}{(2j+1)^2\pi^2}\right)\right]$$

In the case that $\kappa \to 0$, we have $\Im_\kappa \to 1$.

We will consider again the *κ*-logarithm in the following discussion, but before let us show the relationship between Tsallis and Kaniadakis entropy.

Here we have used two infinite products for Euler's book. In [21], other expressions by means of infinite products are given.

14. Basic relationship between *κ*-entropy and *q*-entropy

Let us consider to apply the *q*-entropy and *κ*-entropy to the same discrete distribution.

For the κ-entropy, we have that [9]:

$$(7)\quad K_\kappa = \frac{T_{1+\kappa} + T_{1-\kappa}}{2}$$

In (7) we used the Tsallis entropies:

$$T(q=1+\kappa) = -\frac{1}{\kappa}\sum_i p_i^{1+\kappa} + \frac{1}{\kappa} \quad \text{and} \quad T(q=1-\kappa) = \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa}$$

Then:

$$K = \frac{1}{2}\left\{-\frac{1}{\kappa}\sum_i p_i^{1+\kappa} + \frac{1}{\kappa} + \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa}\right\} = -\frac{1}{2\kappa}\left\{\sum_i p_i^{1+\kappa} - \sum_i p_i^{1-\kappa}\right\}$$

Eq.(7) is a simpler form of an expression given in [22],[23]. However, besides this link, because of the generalized additivity possessed by the Kaniadakis entropy, we need also another relationship, concerning function $\Im$. It is:

$$(8) \quad \Im_\kappa = \frac{\kappa}{2}\left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

In fact:

$$\Im = \frac{\kappa}{2}\left\{\frac{1}{\kappa}\sum_i p_i^{1+\kappa} - \frac{1}{\kappa} + \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa} + \frac{2}{\kappa}\right\} = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

In (7) and (8), we have the Kaniadakis functions expressed by the Tsallis entropy.

Let us consider again the uniform discrete distribution. We have already seems that:

$$T = T_q = \frac{n^{1-q} - 1}{1-q} = \ln_q^T(n) \qquad \text{and} \qquad K_\kappa = \frac{1}{2\kappa}\left(n^\kappa - n^{-\kappa}\right) = \ln_\kappa n$$

Eq. (7), which is $K_\kappa = \frac{T_{1+\kappa} + T_{1-\kappa}}{2}$, becomes $K_\kappa = \frac{1}{2\kappa}\left(n^\kappa - n^{-\kappa}\right) = \frac{1}{2}\left(-\frac{n^{-\kappa} - 1}{\kappa} + \frac{n^\kappa - 1}{\kappa}\right)$.

Of course, we can also write q-entropy expressed by means of κ-entropy.

$$2K + \frac{2}{\kappa}\Im \ = \ T_{1+\kappa} + T_{1-\kappa} + \left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

And then:

$$(9) \quad K_\kappa + \frac{1}{\kappa}\Im_\kappa = T_{1-\kappa} + \frac{1}{\kappa}$$

Let us have: $\kappa = 1 - q$.

$$(10) \quad T_q = K_{1-q} + \frac{\Im_{1-q} - 1}{(1-q)}$$

We can have also:

$$2K_\kappa - \frac{2}{\kappa}\Im_\kappa = T_{1+\kappa} + T_{1-\kappa} - \left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

So that:

$$(11) \quad K_\kappa - \frac{1}{\kappa}\Im_\kappa = T_{1+\kappa} - \frac{1}{\kappa}$$

Let us have: $\kappa = q-1$. We have again Eq. (10).

15. Generalized product of $\Im$

Let us consider two subsystems A and B. We can find a relationship between the joint Tsallis and Kaniadakis entropies. Using (10), we obtain [9]:

$$(12) \quad T_q(A,B) = K_{1-q}(A,B) + \frac{\Im_{1-q}(A,B)-1}{1-q}$$

From (12), when $q \to 1$, we have:

$$\mathrm{T}_{q\to 1}(A,B) = K_{\kappa\to 0}(A,B) = S(A)+S(B)$$

When the subsystems are independent, for Tsallis entropy we have Eq.5, and then:

$$(13) \quad T_q(A,B) =$$

$$K_{1-q}(A,B) + \frac{\Im_{1-q}(A,B)-1}{1-q} = K_{1-q}(A) + \frac{\Im_{1-q}(A)-1}{1-q} + K_{1-q}(B) + \frac{\Im_{1-q}(B)-1}{1-q} + (1-q)\left(K_{1-q}(A) + \frac{\Im_{1-q}(A)-1}{1-q}\right)\left(K_{1-q}(B) + \frac{\Im_{1-q}(B)-1}{1-q}\right)$$

To continue, we can assume:

$$(14) \quad \Im_{1-q}(A,B) = \Im_{1-q}(A)\,\Im_{1-q}(B) + (1-q)^2 K_{1-q}(A) K_{1-q}(B)$$

This relationship was proposed in [24], but it is clear that it can obtained from (13). In fact:

$$K_{1-q}(A,B)+\frac{\Im_{1-q}(A,B)}{1-q}=$$

$$(1-q)K_{1-q}(A)K_{1-q}(B)+K_{1-q}(A)\Im_{1-q}(B)+\Im_{1-q}(A)K_{1-q}(B)+(1-q)\frac{\Im_{1-q}(A)}{1-q}\frac{\Im_{1-q}(B)}{1-q}$$

Remembering that $K_{1-q}(A,B)=K_{1-q}(A)\Im_{1-q}(B)+\Im_{1-q}(A)K_{1-q}(B)$, Eq.(13) becomes (14):

$$\frac{\Im_{1-q}(A,B)}{1-q}= (1-q)K_{1-q}(A)K_{1-q}(B)+(1-q)\frac{\Im_{1-q}(A)}{1-q}\frac{\Im_{1-q}(B)}{1-q}$$

And here we can find the generalization of the product:

$$\Im_\kappa(A,B)= \kappa^2 K_\kappa(A)K_\kappa(B)+\Im_\kappa(A)\Im_\kappa(B)$$

16. Conditional entropies

In [9], it had been proposed a detailed discussion of conditional Kaniadakis entropy too. If the entropic parameter has a low value, a formula previously given in [25] can be considered an approximation of the expression obtained in [9].

The conditional Tsallis entropy is given by [26]:

$$(15) \quad \mathrm{T}_q(A|B)=\frac{T_q(A,B)-T_q(B)}{1+(1-q)T_q(B)}$$

In [9], we have given for Kaniadakis entropy:

$$(16) \quad K_\kappa(A|B)=\frac{K_\kappa(A,B)-K_\kappa(B)\Im_\kappa(A|B)}{\Im_\kappa(B)}$$

$$(17) \quad \Im_\kappa(A|B)=\frac{\Im_\kappa(A,B)-\kappa^2 K_\kappa(A|B)K_\kappa(B)}{\Im_\kappa(B)}$$

In the case that $\kappa \rightarrow 0$, we find from (16) the expression of conditional Shannon entropy, which is given by $S(A|B)=S(A,B)-S(B)$.

17. Entropic measures

The Tsallis and Kaniadakis approaches are two generalizations of statistical mechanics, which had been involved in the discussion of several phenomena [27],[19]. Among the several possible examples, one is coming from astrophysics. It is related to polytropes, the polytropic solutions of the Lane–Emden equation. This is an equation which gives the pressure as a function of density [28],[29],[30]. Since Boltzmann distribution yields unphysical results, a generalized entropy, the Tsallis entropy, was used in [31] instead of Boltzmann entropy. The use of Kaniadakis entropy had been proposed in [32]. In [33], we compared the two entropic measures related to the polytropic solutions given in [31],[32].

As previously seen, the Tsallis entropy is:

$$\frac{1}{q-1}\left(1-\sum_i p_i^q\right)=\frac{1}{q-1}\left(\sum_i p_i-\sum_i p_i^q\right)=\frac{1}{q-1}\sum_i p_i\left(1-p_i^{q-1}\right)$$

In Ref. [31], it is used letter f for probability. From now on, we will use this notation. In [31], the measure from q-entropy is given as:

$$(18) \qquad \mathrm{C}_q(f)=\frac{f\left(1-f^{q-1}\right)}{q-1}$$

We can write Eq.(18) in the following manner:

$$(19) \qquad \mathrm{C}_q(f)=\frac{f-f^q}{q-1}=\frac{1}{2(q-1)}\left(f^{2-q}-f^q\right)-\frac{1}{2(q-1)}\left(f^{2-q}+f^q\right)+\frac{f}{q-1}$$

Of course, (18) and (19) are the same equation. Let us write the κ-entropy again:

$$K_{q-1}=-\sum_f \frac{f^{1+q-1}-f^{1-q+1}}{2(q-1)} \qquad \text{so that:} \quad k_{q-1}=-\frac{f^q-f^{2-q}}{2(q-1)}$$

$$\Im_{q-1}=\sum_f \frac{f^q+f^{2-q}}{2} \qquad \text{so that:} \qquad g_{q-1}=\frac{f^q+f^{2-q}}{2}$$

Therefore, the Kaniadakis measures are linked to Tsallis measure by:

$$t_q(f)=C_q(f)=\frac{(f-f^q)}{q-1}=k_{1-q}-\frac{g_{1-q}}{q-1}+\frac{f}{q-1}$$

Let us remember that:

$$T_q=\sum_f t_q,\ K_{1-q}=\sum_f k_{1-q}\ ,\ \Im_{1-q}=\sum_f g_{1-q},\ 1=\sum_f f$$

18. *q*-logarithm and *κ*-logarithm

Following the previously given approach, let us consider again the logarithms. In the frameworks of Tsallis and Kaniadakis approached, the q-logarithm and the κ-logarithm are defined as:

$$\ln_q^T(f)=\frac{f^{1-q}-1}{1-q}\qquad ;\qquad \ln_\kappa^K(f)=\frac{f^\kappa-f^{-\kappa}}{2\kappa}$$

First, let us note that:

$$\ln_q^T(f)=\frac{f^{1-q}-1}{1-q}=\frac{e^{(1-q)\ln f}-1}{1-q}$$

$$\ln_\kappa^K(f)=\frac{f^\kappa-f^{-\kappa}}{2\kappa}=\frac{e^{\kappa\ln f}-e^{-\kappa\ln f}}{2\kappa}=\frac{1}{\kappa}\sinh(\kappa\ln f)$$

Let us write the *κ*-logarithm with $\kappa=1-q$: $\ln_{1-q}^K(f)=\frac{f^{1-q}-f^{q-1}}{2(1-q)}$.

Then, as we did before:

$$\ln_q^T(f)=\frac{f^{1-q}-1}{1-q}=\frac{1}{2(1-q)}(f^{q-1}+f^{1-q})+\frac{1}{2(1-q)}(-f^{q-1}+f^{1-q})-\frac{1}{1-q}$$

$$=\ \ln_{1-q}^K(f)+\frac{f^{1-q}+f^{q-1}-2}{2(1-q)}$$

In [33], it has been used the measure g_{q-1} for a further expression of the link between logarithm. Here, let us propose the following approach:

$$\ln_q^T(f)=\ln_{1-q}^K(f)+\frac{f^{1-q}+f^{q-1}-2}{2(1-q)} \; = \; \ln_{1-q}^K(f)+\frac{e^{(1-q)\ln f}+e^{-(1-q)\ln f}-2}{2(1-q)}$$

$$(20) \quad \ln_q(f)^T \; = \; \ln_{1-q}^K(f)+\frac{1}{1-q}[\cosh((1-q)\ln f)-1]$$

Let us remember that $\ln_{1-q}^K(f)=\frac{1}{1-q}\sinh((1-q)\ln f)$, and then:

$$\ln_q^T(f)=\frac{1}{1-q}\sinh((1-q)\ln f)+\frac{1}{1-q}[\cosh((1-q)\ln f)-1]$$

We could define a function :

$$\xi_{1-q}^K(f)=\frac{1}{1-q}[\cosh((1-q)\ln f)-1]$$

that is:

$$\xi_\kappa^K(f)=\frac{1}{\kappa}[\cosh(\kappa\ln f)-1]$$

Eq. (20), which is giving the relationship between *q*- and *κ*- logarithms, turns out into:

$$\ln_q^T(f)=\ln_{1-q}^K(f)+\xi_{1-q}^K(f)$$

For what concerns the logarithms, let us note the following properties:

$$\ln_q^T(f_a f_b)=\ln_q^T(f_a)+\ln_q^T(f_b)+(1-q)\ln_q^T(f_a)\ln_q^T(f_b)$$

$$\ln_\kappa^K(f_a f_b)=\ln_\kappa^K(f_a)\sqrt{1+\kappa^2\ln_\kappa^K(f_b)}+\ln_\kappa^K(f_b)\sqrt{1+\kappa^2\ln_\kappa^K(f_b)}$$

In fact:

$$\ln_\kappa^K(f_a f_b)=\frac{1}{\kappa}\sinh(\kappa\ln(f_a f_b)) \; = \; \frac{1}{\kappa}\sinh(\kappa\ln f_a+\kappa\ln f_b)$$

$$= \frac{1}{\kappa}\sinh(\kappa \ln f_a)\cosh(\kappa \ln f_b) \;+\; \frac{1}{\kappa}\cosh(\kappa \ln f_a)\sinh(\kappa \ln f_b)$$

$$= \ln_\kappa^K(f_a)\cosh(\kappa \ln f_b) \;+\; \cosh(\kappa \ln f_a)\ln_\kappa^K(f_b)$$

$$= \ln_\kappa^K(f_a)\sqrt{1+\kappa^2 \ln_\kappa^K(f_b)} \;+\; \ln_\kappa^K(f_b)\sqrt{1+\kappa^2 \ln_\kappa^K(f_a)}$$

Being $(\cosh(\kappa \ln f))^2-(\sinh(\kappa \ln f))^2=1$.

19. *q*-exponential and *κ*-exponential

In the framework of Tsallis approach, the *q*-exponential is defined as:

$$\mathrm{e}_q^T(f)=[1+(1-q)f]^{1/(1-q)}$$

We have that:

$$\mathrm{e}_q^T(\ln_q^T(f))=\left[1+(1-q)\frac{f^{1-q}-1}{1-q}\right]^{1/(1-q)}=f \ .$$

Let us stress that the *q*-exponential in the framework of the Tsallis entropy is not the *q*-exponential given by the *q*-calculus, also known as “quantum calculus” [34].

In the framework of Kaniadakis theory, the *κ*-exponential is given by:

$$\mathrm{e}_\kappa^K(f)=\left[\sqrt{1+\kappa^2 f^2}+\kappa f\right]^{1/\kappa}$$

Let us note that:

$$\mathrm{e}_\kappa^K(f)=\left[e^{\operatorname{arsinh}(\kappa f)}\right]^{1/\kappa}$$

Therefore, we have $\mathrm{e}_\kappa^K(\ln_\kappa^K(f))=f$.

Now, let us consider $(e_q^T(f))^{(1-q)}=[1+(1-q)f]$, that is: $(e_q^T(f))^{(1-q)}-1=(1-q)f$.

We can write:

$$(e_{1-q}^K(f))^{(1-q)}=\left[\sqrt{1+(1-q)^2 f^2}+(1-q)f\right]$$

and use in it the q-exponential. We can find:

$$e_{1-q}^{K}(f)=\left[\sqrt{1+((e_q^T(f))^{(1-q)}-1)^2}+(e_q^T(f))^{(1-q)}-1\right]^{1/(1-q)}$$

Then:

$$(28) \quad e_{1-q}^{K}(f)=\left[e^{\operatorname{arsinh}((e_q^T(f))^{(1-q)}-1)}\right]^{1/(1-q)}$$

In [33], other expressions had been given.

20. Remark

Let us stress once more that the q-exponential is not the q-exponential of the quantum calculus. For the discussion of the two q-exponentials defined in the quantum calculus, see please Ref. [35] and [36], besides [34].

21. Generalized calculus

As we have previously seen, the κ-logarithm and the κ-exponential can be written as:

$$\ln_\kappa^K(f)=\frac{1}{\kappa}\sinh(\kappa \ln f) \;\; ; \quad \mathrm{e}_\kappa^K(f)=\left[e^{\operatorname{arsinh}(\kappa f)}\right]^{1/\kappa}$$

It means that these functions are coming from a deformation of a calculus based on the Euler exponential function. In his κ-calculus, [19], Giorgio Kaniadakis has generalized the sum, and therefore the product too, by means of this deformation based on hyperbolic functions, so that:

$$x \oplus y=\frac{1}{\kappa}\sinh(\operatorname{arsinh}(\kappa x)+\operatorname{arsinh}(\kappa y))$$

Actually, we can see that if we have a function $G(x)$, which is invertible $G^{-1}(G(X))=1$, we can use it as a deformation generator [37], to generate a consequent algebra [37],[38]. Therefore, we can use the generator G to define the formal group law $\Phi(x,y)$, such as in [39]:

$$(29) \quad \Phi(x,y)=G(G^{-1}(x)+G^{-1}(y))$$

or:

$$x \oplus y = G(G^{-1}(x) + G^{-1}(y))$$

The simplest example of a formal group law is the following:

$$\Phi(x, y) = x + y = G(G^{-1}(x) + G^{-1}(y)) = \exp(\ln(x) + \ln(y))$$

Let us stress that the binary operation between two elements of a given set $x \oplus y$ has been defined by Giorgio Kaniadakis as the "generalized sum", using the analogy of the generalization of entropy.

In [40], I discussed some generalized sums based on transcendental functions. We can apply the same approach to the sequences of integer numbers, such as the Mersenne, Fermat, Cullen and Woodall Numbers [41], or to the calculus of the q-integers [42], (other applications are given in [43] and [44]). Actually, let us stress that the framework of generalized calculus is the group theory; in this framework, the groupoids related to the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers possess different binary operators. As an application of the given approach, several different integer sequences can be obtained by means of the same binary operators, and therefore can be used to represent the related groupoids.

Eq. (29) is the same expression of the Lazard universal formal group law [45],[46]. In [47], the formal group law is considered the Rényi entropy, which is the first example of a new family of infinitely many multi-parametric entropies, defined as Z-entropies.

References

[1] Shannon, C.E. (1948). A Mathematical Theory of Communication. Bell System Technical Journal 2 (3):379–423. DOI: 10.1002/j.1538-7305.1948.tb01338.x

[2] Borda, M. (2011). Fundamentals in Information Theory and Coding. Springer. ISBN 978-3-642-20346-6.

[3] Tsallis, C. (1988). Possible Generalization of Boltzmann-Gibbs Statistics, Journal of Statistical Physics, 52: 479–487. DOI:10.1007/BF01016429

[4] Kaniadakis, G. (2002). Statistical Mechanics in the Context of Special Relativity, Phys. Rev. E, 2002, 66, 056125. DOI: 10.1103/physreve.66.056125

[5] Kalogeropoulos, N. (2020). Toward a relative q-entropy, Physica A: Statistical Mechanics and its Applications, 545, 123270, DOI: 10.1016/j.physa.2019.123270.

[6] Havrda, J., & Charvát, F. (1967). Quantificaton Method of Classification Processes. Concept of structural α-entropy, Kybernetika 3(1) 30-35.

[7] Vajda, I. (1968). Axioms for α-entropy of a generalized probability scheme, Kybernetika 4, 105-112 (in Czech).

[8] Daróczy, Z. (1970). Generalized information functions. Information and control, 16(1), 36-51.

[9] Sparavigna, A. C. (2015). Relations between Tsallis and Kaniadakis entropic measures and rigorous discussion of conditional Kaniadakis entropy. International Journal of Sciences, 4(10), 47-50. DOI: 10.18483/ijSci.866

[10] Rényi, A. (1960).Some fundamental questions of information theory, MTA III. Oszt. Közl. 1960, 10, 251-282. Reprinted in Selected Papers of Alfréd Rényi, Turán, P. (Ed.); Akadémiai Kiadó, Budapest, 1976.

[11] Xu, Yang. Technical notes on information theory.

http://www.cs.toronto.edu/~yangxu/information-theory-v3.pdf

[12] Jaynes, E. T. (1957). Information theory and statistical mechanics. Physical review, 106(4), 620.

[13] Brémaud, P. (2017). In Discrete Probability Models and Methods. Probability on Graphs and Trees, Markov Chains and Random Fields, Entropy and Coding. 2017. ISBN: 9783319434766, 3319434764, Springer International Publishing.

[14] Singh, V. P., Sivakumar, B., & Cui, H. (2017). Tsallis entropy theory for modeling in water engineering: A review. Entropy, 19(12), 641.

[15] Sparavigna, A. C. (2015). On the Generalized Additivity of Kaniadakis Entropy, Int. J. Sci. 4(2):44-48. DOI: 10.18483/ijSci.627

[16] Sparavigna, A. C. (2020). Entropy and logarithm of Kaniadakis calculus expressed by means of an Euler infinite product expansion. Zenodo. DOI: 10.5281/zenodo.4088214

[17] Piola. G. (1834). Sul Calcolo degli Integrali Definiti. Pag.120, in Opuscoli Matematici e Fisici di Diversi Autori, Tomo Secondo, Milano, presso Paolo Emilio Giusto.

[18] Euler, L. (1748). Introductio in analysin infinitorum. T. I, num.156.

[19] Kaniadakis, G. (2013). Theoretical foundations and mathematical formalism of the power-law tailed statistical distributions. Entropy, 15(10), 3983-4010.

[20] Euler, L. (1748). Introductio in analysin infinitorum. T. I, num.157.

[21] Sparavigna, A. C. (2020). Infinite product expansions from Euler's Introductio in Analysin Infinitorum. Zenodo. https://doi.org/10.5281/zenodo.4139216

[22] Kaniadakis, G. (2001). Non-linear kinetics underlying generalized statistics, Physica A 296(3-4):405-425. DOI: 10.1016/s0378-4371(01)00184-4

[23] Santos, P., Silva R., Alcaniz J.S., & Anselmo, D.H.A.L. (2011). Generalized quantum entropies. Physics Letters A 375:3119–3123. DOI: 10.1016/j.physleta.2011.07.001

[24] Scarfone, A. M., & Wada, T. (2005). Thermodynamic equilibrium and its stability for microcanonical systems described by the Sharma-Taneja-Mittal entropy. Physical Review E, 72(2), 026123. DOI: 10.1103/PhysRevE.72.026123

[25] Sparavigna, A. C. (2015). Mutual Information and Nonadditive Entropies: A Method for Kaniadakis Entropy, International Journal of Sciences 10(2015):5-8. DOI: 10.18483/ijSci.846

[26] Abe, S., & Rajagopal, A.K. (2000). Nonadditive Conditional Entropy and its Significance for Local Realism, arXiv:quant-ph/0001085, 24 Jan 2000.

[27] Sumiyoshi, A. and Yuko O., Editors (2001). Nonextensive Statistical Mechanics and Its Applications, Springer Science & Business Media, ISBN: 978-3-540-41208-3 (Print) 978-3-540-40919-9 (Online)

[28] Horedt, G.P. (2004). Polytropes. Applications in Astrophysics and Related Fields, Kluwer. ISBN 1-4020-2350-2

[29] Chandrasekhar, S. (1958). An Introduction to the Study of Stellar Structure, Dover.

[30] Knapp, J. (2015). Polytropes, retrieved 9 November 2015 from http://www.astro.princeton.edu/~gk/A403/polytrop.pdf

[31] Plastino, A.R., & Plastino, A. (1993). Stellar Polytropes and Tsallis Entropy. Physics Letters A 174(5):384-386. DOI: 10.1016/0375-9601(93)90195-6

[32] Bento, E.P., Silva, J.R.P., & Silva, R. (2013). Non-Gaussian Statistics, Maxwellian Derivation and Stellar Polytropes. Physica A: Statistical Mechanics and its Applications 392.4 (2013): 666-672. DOI: 10.1016/j.physa.2012.10.022

[33] Sparavigna, A. C. (2015). Tsallis and Kaniadakis Entropic Measures in Polytropic, Logarithmic and Exponential Functions. International Journal of Sciences, 4(11), 1-4. DOI: 10.18483/ijSci.873

[34] Kac, V., & Cheung, P. (2011). Quantum Calculus. Springer. p. 31. ISBN 146130072X.

[35] Sparavigna, A. C. (2016). Graphs of q-exponentials and q-trigonometric functions. 2016. HAL, HAL Id: hal-01377262, https://hal.archives-ouvertes.fr/hal-01377262

[36] Sparavigna, A. C. (2021). Nozioni di q-calcolo nell'ambito del quantum calculus. Zenodo. https://doi.org/10.5281/zenodo.4982846

[37] Kaniadakis, G. (2005). Statistical mechanics in the context of special relativity II. Phys. Rev. E, 72, 036108.

[38] Scarfone, A. M. (2013). Entropic forms and related algebras. Entropy, 15(2), 624-649.

[39] Sicuro, G., & Tempesta, P. (2016). Groups, information theory, and Einstein's likelihood principle. Phys. Rev. E 93, 040101(R)

[40] Sparavigna, A. C. (2018). Generalized Sums Based on Transcendental Functions. Zenodo. https://doi.org/10.5281/zenodo.1250020

[41] Sparavigna, A. C. (2019). On the generalized sums of Mersenne, Fermat, Cullen and Woodall Numbers. Zenodo. https://doi.org/10.5281/zenodo.2634312

[42] Sparavigna, A. C. (2018). The q-integers and the Mersenne numbers. Zenodo. https://doi.org/10.5281/zenodo.1251833

[43] Sparavigna, A. C. (2019). Some Groupoids and their Representations by Means of Integer Sequences, International Journal of Sciences 10(2019), 1-5. DOI: 10.18483/ijSci.2188

[44] Sparavigna, A. C. (2019). Composition Operations of Generalized Entropies Applied to the Study of Numbers, International Journal of Sciences 04(2019), 87-92. DOI: 10.18483/ijSci.2044

[45] Carrasco, J., & Tempesta, P. (2019). Formal rings. ArXiv. https://arxiv.org/abs/1902.03665 or https://doi.org/10.48550/arXiv.1902.03665

[46] Hazewinkel, M. (1978). Formal Groups and Applications, Academic Press, New York

[47] Tempesta, P. (2016). Formal groups and Z-entropies. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 472(2195), 20160143.