On the strengthening of Nyman-Beurling criteria and Riemann hypothesis

Kwok Kwan Wong

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Abstract

We show that the span of the Beurling sequence is dense in the sequence space with a weighted inner product. Based on the condition posed by Bagchi and Baez-Duarte, which is a strengthening condition of Nyman-Beurling criteria, is equivalent to showing the Riemann hypothesis does hold.

Mathematics Subject Classification: 11Mxx, 46Cxx

1 Introduction

The Riemann hypothesis was raised by Riemann in 1859 [17]. The hypothesis is about the zeros of the Riemann-Zeta function ζ, ζ has the trivial zero, which are negative integers, and the nontrivial zeros. Riemann posed a hypothesis that the real part of the nontrivial zeros are $\frac{1}{2}$, which we call the *Riemann hypothesis* From 1859 to now, many scholars attempted to prove or disprove the Riemann hypothesis[16, 23, 13, 7, 21, 9, 11, 22], since the Riemann hypothesis had a big impact on the field of mathematics, such as the distributions of prime numbers [24], the large prime gap conjecture [10], etc. To prove or disprove the Riemann

hypothesis, many mathematicians try to formulate the Riemann hypothesis in another way [18, 19, 20]. In particular, Nyman and Beurling show that the Riemann hypothesis is true if and only if the space of the Beurling function is dense in Hilbert space $L^2((0,1))$ [15, 4]. Baez-Duarte has restated and strengthened this condition to be the Riemann hypothesis is true if and only if the characteristic function $\chi_{(0,1]}$ belongs to the closure of the space of the *natural Beurling function* in the Hilbert space $L^2((0,\infty))$ [1]. Bagchi reformulates the condition to be if the constant sequence belongs to the closure of the span of the *Beurling sequence* in the $l^2(\mathbb{N})$ with a weighted inner product [2].
There are numerous working on this approach[14, 25, 12, 3, 5, 8]. Our contribution is proving that the constant sequence does belong to the closure of the span of the Beurling sequence, thus proving the Riemann hypothesis.

2 The main result

The Hilbert space we consider is $l^2(\mathbb{N}) := H$ over $\mathbb{C}$ with the norm induced by the inner product.

$$\langle a, b\rangle = \sum_{n=1}^{\infty} \frac{a_n^* b_n}{n(n+1)}.$$

Observe that bounded sequence belongs to H as well.

We adopt the notion in [2], we introduce the sequence $\gamma_l = (\{\frac{n}{l}\}) = (1/l, 2/l, ...)$ for $l \in \mathbb{N}$, where $\{x\}$ is the fractional part function. It is easy to see that $\gamma_l \in H$ for all l. Denote the $span(\gamma_l, l \in \mathbb{N}) = B$, we call the B the space of the *Beurling sequences*. Let $\gamma = (1, 1, ...)$ be the constant sequence, it is easy to see that it belongs to H. In [2], it state the following theorem.

Theorem 2.1. The Riemann hypothesis is equivalent to $\gamma \in \overline{B}$, and is equivalent to B is dense in H.

Proof. See the proof of Theorem 1 of [2]. □

Let e_i be the sequence with 1 in the i-th entry, zero otherwise. Let $T : H \to H$ by T by $T(e_i) = \frac{\gamma_{i+1}}{(i+1)^2}$. Clearly this is linear map.

Define $R_i : \mathbb{N} \to \mathbb{N}$, by sending p to $p\, mod\, i$.

First, we prove a inequality.

lemma 2.2. $0 < \langle \gamma_k, \gamma_k \rangle < 1$.

Proof. By direct calculation, $\langle \gamma_k, \gamma_k \rangle =$

$$\begin{aligned} &\tfrac{1}{k^2}(\textstyle\sum_{n=1}^{k-1} \tfrac{n}{n+1} + \sum_{n=k+1}^{\infty} (R_k(n))^2 \tfrac{1}{n(n+1)}) \\ &< \tfrac{1}{k^2}(k-1+(k-1)^2 \textstyle\sum_{n=k+1}^{\infty} \tfrac{1}{n(n+1)}) \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$,

$$\begin{aligned} &< \tfrac{1}{k^2}(k-1+(k-1)^2 \textstyle\sum_{n=k+1}^{\infty} \tfrac{1}{n(n+1)}) \\ &\leq \tfrac{1}{k^2}(k-1+(k-1)^2) \\ &\leq \tfrac{k-1}{k} < 1. \end{aligned}$$

□

Now we show T is a bounded operator.

Theorem 2.3. *T is a bounded linear operator.*

Proof. Let $x = \sum_{n=1}^{\infty} a_n e_n \in H$. $Tx = \sum_{n=1}^{\infty} \frac{a_n}{(n+1)^2} \gamma_{n+1}$.
$||Tx||^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n^* a_m}{(n+1)^2 (m+1)^2} \langle \gamma_{n+1}, \gamma_{m+1} \rangle$. Which is less than or equal to $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|a_n^* a_m|}{(n+1)^2 (m+1)^2} \langle \gamma_{n+1}, \gamma_{m+1} \rangle$.
By Cauchy-Schwarz inequality,

$$||Tx||^2 \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|a_n^* a_m|}{(n+1)^2 (m+1)^2} ||\gamma_{n+1}|| ||\gamma_{m+1}||.$$

By lemma 2.2,

$$||Tx||^2 < \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|a_n^* a_m|}{(n+1)^2(m+1)^2}$$
$$= (\sum_{n=1}^{\infty}\frac{|a_n|}{(n+1)^2})^2.$$

Now we separate $\sum_{n=1}^{\infty}\frac{|a_n|}{(n+1)^2}$ into two parts, one part with all $|a_n| < 1$, another part is $|a_n| \geq 1$. For the latter part, since $||x||^2 = \sum_{n=1}^{\infty}\frac{|a_n|^2}{n(n+1)} < \infty$, and the latter part is less than or equal to $||x||^2$, so it converges. For another part, since $|a_n| < 1$, the sum is bounded by $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \infty$, so it converges too. So $||Tx||^2$ is finite provided that $||x||$ is finite. In particular $\sum_{n=1}^{\infty}\frac{|a_n|}{(n+1)^2}$ is absolute converge.
Now we consider $\frac{||Tx||^2}{||x||^2}$, by the above estimation, it is less than $\frac{(\sum_{n=1}^{\infty}\frac{|a_n|}{(n+1)^2})^2}{\sum_{n=1}^{\infty}\frac{|a_n|^2}{n(n+1)}}$.
Now the denominator can be rewrite as $(\sum_{n=1}^{\infty}\frac{|a_n|}{\sqrt{n(n+1)}}\frac{\sqrt{n(n+1)}}{(n+1)^2})^2$. By Cauchy-Schwarz inequality, the denominator is less than or equal to $(\sum_{n=1}^{\infty}\frac{|a_n|^2}{n(n+1)})(\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+1)^4})$, then

$$\frac{(\sum_{n=1}^{\infty}\frac{|a_n|}{(n+1)^2})^2}{\sum_{n=1}^{\infty}\frac{|a_n|^2}{n(n+1)}} \leq \sum_{n=1}^{\infty}\frac{n(n+1)}{(n+1)^4}.$$

Since $\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+1)^4}$ converges, $||Tx||^2/||x||^2$ is bounded, so T is bounded. □

Now we show T is injective.

Theorem 2.4. *T is injective, that is, $ker(T) = \{0\}$.*

Proof. Let $x = \sum_{n=1}^{\infty} a_n e_n \in ker(T)$. $Tx = 0$ implies $\sum_{n=1}^{\infty}\frac{a_n}{(n+1)^2}\gamma_{n+1} = 0$. Each entry of $Tx = 0$ forms a equation. For the first entry, $\sum_{n=2}^{\infty}\frac{a'_{n-1}}{n} = 0$, where $a'_n = a_n/(n+1)^2$. For the second entry, $\sum_{n=3}^{\infty}\frac{2a'_{n-1}}{n} = 0$. By combining equation 1 and 2, $a'_1 = a_1 = 0$.
Assume $a'_1, ...a'_k = 0$ for k is some positive integer. The equation in $k+1$ entry is $\sum_{n=k+1}^{\infty}\frac{(k+1)a'_{n-1}}{n} + \sum_{n=2}^{k+1}\frac{a'_{n-1}R_n(k+1)}{n} = 0$. Now $\sum_{n=2}^{k+1}\frac{a'_{n-1}R_n(k+1)}{n} = 0$, so $\sum_{n=k+1}^{\infty}\frac{a'_{n-1}}{n} = 0$. The equation of $k+2$ is $\sum_{n=k+2}^{\infty}\frac{(k+2)a'_{n-1}}{n} + \sum_{n=2}^{k+2}\frac{a'_{n-1}R_n(k+2)}{n} = 0$, it is easy to see that $a'_{k+1} = a_{k+1} = 0$ by comparing two equations. By strong induction, $a_n = 0$, so $x = 0$. Thus the conclusion. □

Define Mobius function $\mu : \mathbb{N} \to \mathbb{R}$ by $\mu(k) = 1$ if k is a square-free number with even number of prime factors, $\mu(k) = -1$ if k is a square-free number with odd number of prime factors. $\mu(k) = 0$ otherwise. The following result is standard.

lemma 2.5. $\sum_{i=1}^{l+1} floor(\frac{l+1}{i})\mu(i) = 1$, where $floor(x)$ is the floor function.

Proof. See [6]. □

Consider $x \in \overline{B}$, x can be represented as $\sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{nk}\gamma_{k+1}$ with some restriction on a_{nk}. We use this fact to prove a important theorem.

Theorem 2.6. $\gamma = (1, 1, ...)$ does belong to $\overline{B}$.

Proof. We claim $\gamma = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{nk}\gamma_{k+1}$ with $a_{nk} = -\frac{\mu(k+1)}{2^n}$. Let $m_k = -\frac{\mu(k+1)}{k+1}$. We first show the right hand side is bounded.
To show that $|| \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{nk}\gamma_{k+1}|| < \infty$, which means the quantity

$$\langle \sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} a_{n'k'}\gamma_{k'+1}, \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{nk}\gamma_{k+1}\rangle,$$

is bounded. By the continuity of inner product,

$$= \sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{n'k'}a_{nk}\langle\gamma_{k'+1}, \gamma_{k+1}\rangle$$
$$= \sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{n'k'}a_{nk}\langle \sum_{i'=1}^{\infty} \frac{R_{k'+1}(i')}{k'+1}e_{i'}, \sum_{i=1}^{\infty} \frac{R_{k+1}(i)}{k+1}e_i\rangle$$
$$= \sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{m_{k'}m_k}{2^{n+n'}} \sum_{i'=1}^{\infty} \sum_{i=1}^{\infty} R_{k'+1}(i')R_{k+1}(i)\langle e_{i'}, e_i\rangle$$
$$= \sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{m_{k'}m_k}{2^{n+n'}} \sum_{i=1}^{\infty} R_{k'+1}(i)R_{k+1}(i)/(i(i+1))$$

Since m_k are bounded, let $M = \sup_{k,k'}(|m_{k'}m_k|)$, then $M \sum_{n'=1}^{\infty} \sum_{n=1}^{\infty} \frac{n'(n'+1)n(n+1)}{2^{n+n'+2}}$ is finite. Since

$$M \sum_{n'=1}^{\infty} \sum_{n=1}^{\infty} \frac{n'(n'+1)n(n+1)}{2^{n+n'+2}} \geq$$
$$\sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{|m_{k'}m_k|}{2^{n+n'}} \sum_{i=1}^{\infty} k'k/(i(i+1)) \geq$$
$$\sum_{n'=1}^{\infty} \sum_{k'=1}^{n'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{m_{k'}m_k}{2^{n+n'}} \sum_{i=1}^{\infty} R_{k'+1}(i)R_{k+1}(i)/(i(i+1))$$

Then $||\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{nk}\gamma_{k+1}||$ is finite

Now apply T to both sides, $T(\gamma) = \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\gamma_{n+1} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{nk}T\gamma_{k+1}$.

$\gamma_{k+1} = \frac{1}{k+1}\sum_{n=1}^{\infty} R_{k+1}(n)e_n$, so,

$$T\gamma = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{a_{nk}}{(k+1)}\sum_{i=1}^{\infty} R_{k+1}(i)\gamma_{i+1}.$$

Now we find out the coefficients of γ_{i+1}, which are $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{a_{nk}}{(k+1)}R_{k+1}(i)/(i+1)^2$. Now we match the coefficients term by term,, which are $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{a_{nk}}{(k+1)}R_{k+1}(i) = 1$. Let $a_{nk}/(k+1) = b_{nk}$ for convenience. Since the series absolute converge, $\sum_{n=1}^{\infty} b_{n1}R_2(i) + \sum_{n=2}^{\infty} b_{n2}R_3(i) + ... = 1$. Let $\sum_{n=l}^{\infty} b_{nl} = c_l$ for convenience. Then we have system of linear equations:

$$c_1 + c_2 + ... = 1$$

$$2c_2 + 2c_3 + ... = 1$$

$$c_1 + 3c_3 + ... = 1$$

$$.$$

$$.$$

$$.$$

Now using equation multiply by l to minus the equation l, then the system becomes:

$$c_1 + c_2 + ... = 1$$

$$2c_1 = 1$$

$$2c_1 + 3c_2 = 2$$

$$.$$

$$.$$

$$.$$

We denote the infinite matrix representing the system of equation without equation 1 be A. Then it is easy to see that it is a lower triangular matrix and the diagonal is not zero.The c_l is given by $c_l = \frac{l-\sum_{i=1}^{l-1} A_{li}c_i}{A_{ll}}$, $A_{ij} = (j+1)floor(\frac{i+1}{j+1})$.

We claim the $c_l = -m_l$. Indeed,we use strong induction. Since $2c_1 = 1$, $c_1 = \frac{1}{2} = -\frac{\mu(2)}{2}$. Assume the statement holds for all positive integer less than l. Now $c_l = \frac{l-\sum_{i=1}^{l-1} A_{li}c_i}{A_{ll}}$, we obtain $\frac{l+\sum_{i=1}^{i-1} floor(\frac{l+1}{i+1})\mu(i+1)}{l+1}$, by lemma 2.5, it becomes $\frac{l+1-\mu(1)(l+1)-\mu(l+1)}{l+1}$, so $c_l = -\frac{\mu(l+1)}{l+1}$. By strong induction, $c_n = -\frac{\mu(n+1)}{n+1} = m_n$ for all positive integer n.
Now we have shown that $T\gamma = T(\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{nk}\gamma_{k+1})$, by Theorem 2.4, we conclude $\gamma = \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{nk}\gamma_{k+1}$, since $\sum_{k=1}^{n} a_{nk}\gamma_{k+1}$ belongs to B for all positive integer n, γ is an infinite sum of vectors from B which has finite norm, so it converges, So $\gamma \in \overline{B}$.

□

Theorem 2.7. The Riemann hypothesis is true.

Proof. Since γ does belong to $\overline{B}$ by Theorem 2.6, by Theorem 2.1, the Riemann hypothesis is true. □

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References

[1] L. Baez-Duarte. A strengthening of the nyman-beurling criterion for the riemann hypothesis. *Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali*, 14:5–11, 2003.

[2] B. Bagchi. On nyman, beurling and baez-duarte's hilbert space reformulation of the riemann hypothesis. *Proc. Indian Acad. Sci. (Math. Sci.)*, 116:137–146, 2006.

[3] S. Bettin and J. B. Conrey. A reciprocity formula for a cotangent sum. *International Mathematics Research Notices*, 2013:5709–5726, 2013.

[4] A. Beurling. A closure problem related to the riemann zeta-function. *Proc. Natl. Acad. Sci. USA*, 41(5):312–314, 1955.

[5] J. W. Bober. Factorial ratios, hypergeometric series, and a family of step functions. *Journal of the London Mathematical Society*, 79:422–444, 2009.

[6] bookofproofs. Möbius and floor functions combined. `https://www.bookofproofs.org/branches/moebius-and-floor-functions-combined-proof/#fn174474460962da397a785c4`, 2019.

[7] P. Borwein, R. Ferguson, and M. J. Mossinghoff. Sign changes in sums of the liouville function. *Mathematics of Computation*, 77:1681–1694, 2008.

[8] J. F. Burnol. A lower bound in an approximation problem involving the zeros of the riemann zeta function. *Advances in Mathematics*, 170:56–70, 2002.

[9] A. Connes. Trace formula in noncommutative geometry and the zeros of the riemann zeta function. *Selecta Mathematica, New Series*, 5:29–106, 1999.

[10] H. Cramer. On the order of magnitude of the difference between consecutive prime numbers. *Acta Arith.*, 2:23–46, 1936.

[11] I. Fesenko. Analysis on arithmetic schemes. ii. *Journal of K-theory*, 5:437–557, 2010.

[12] L. Habsieger. On the nyman-beurling criterion for the riemann hypothesis. *Funct. Approx. Comment. Math.*, 37:187–201, 2007.

[13] C. B. Haselgrove. A disproof of a conjecture of pólya. *Mathematika*, 5:141–145, 1958.

[14] H. Maier and M. T. Rassias. On the size of an expression in the nyman–beurling-báez–duarte criterion for the riemann hypothesis. *Canadian Mathematical Bulletin*, 61:622–627, 2018.

[15] B. Nyman. On the one-dimensional translation group and semi-group in certain function spaces. PhD Thesis, University of Uppsala, 1950.

[16] A. M. Odlyzko. On the distribution of spacings between zeros of the zeta function. *Mathematics of Computation*, 48:273–308, 1987.

[17] B. Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grösse. *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 1859.

[18] M. Riesz. Sur l'hypothèse de riemann. *Acta Math.*, 40:185–190, 1916.

[19] R. Salem. Sur une proposition équivalente à l'hypothèse de riemann. *Les Comptes rendus de l'Académie des sciences*, 236:1127–1128, 1953.

[20] A. Speiser. Geometrisches zur riemannschen zetafunktion. *Math. Ann.*, 110:514–521, 1934.

[21] R. Spira. Zeros of sections of the zeta function. ii. *Mathematics of Computation*, 22:163–173, 1968.

[22] M. Suzuki. Positivity of certain functions associated with analysis on elliptic surfaces. *Journal of Number Theory*, 131:1770–1796, 2011.

[23] P. Turan. On some approximative dirichlet-polynomials in the theory of the zeta-function of riemann. *Danske Vid. Selsk. Mat.-Fys. Medd*, 24:36, 1948.

[24] N. F. H. von Koch. Sur la distribution des nombres premiers. *Acta Math.*, 24:159–182, 1901.

[25] J. Yang. A note on nyman–beurling's approach to the riemann hypothesis. *Integral Equations and Operator Theory*, 83:447–449, 2015.