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Artículo de Investigación

NEW CONSTRUCTION OF ALGEBRAS AS QUOTIENTS

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ABSTRACT

In this article we have presented a new approach to define algebras using for a natural number $k \geq 2$, the set of natural numbers in base k, none of their digits equal to zero. The study was developed in the context of vector $\mathbb{R}$ -spaces and the vector space definitions of the formal multiples of any element x of the field $\mathbb{R}$, of the direct sum of vector spaces and binary operations on vector spaces were used. The results obtained were the construction of a vector space denoted by $\mathbb{V}$, on the basis of the particular set of natural numbers in base k mentioned, which allowed novel ways of defining the well-known and very important algebras of complex numbers and that of quaternions on $\mathbb{R}$ as quotients of ideals of $\mathbb{V}$, for suitably chosen ideals I. With this new approach and with the help of the vector spaces $\mathbb{V}$, known algebras can be presented in a different way than those found up to now, by using certain ideals of those spaces in their quotient form. The spaces $\mathbb{V}$ can be over any field K and other algebras such as Clifford algebras can be constructed using this procedure.

Keywords: Algebras, Quotients in algebras, Complex numbers and quaternions as quotients of algebras.

NUEVA CONSTRUCCIÓN DE ÁLGEBRAS COMO COCIENTES

RESUMEN

En este artículo se ha presentado un nuevo enfoque para definir álgebras usando para un número natural $k \geq 2$, el conjunto de números naturales en base k, ninguno de sus dígitos iguales a cero. El estudio se desarrolló en el contexto de los $\mathbb{R}$-espacios vectoriales y se usaron las definiciones de espacio vectorial de los múltiplos formales de un elemento cualquiera x del cuerpo $\mathbb{R}$, de la suma directa de espacios vectoriales y operaciones binarias sobre espacios vectoriales. Los resultados obtenidos fueron la construcción de un espacio vectorial denotado por $\mathbb{V}$, sobre la base del particular conjunto de números naturales en base k mencionado, que permitió novedosas formas de definir las conocidas y muy importantes álgebras de los números complejos y la de los cuaterniones sobre $\mathbb{R}$ como cocientes de ideales de $\mathbb{V}$, para ideales I convenientemente elegidos. Con este nuevo enfoque y con la ayuda de los espacios vectoriales $\mathbb{V}$ se pueden presentar álgebras conocidas de manera distinta a las encontradas hasta ahora, al usar en su forma de cociente ciertos ideales de esos espacios $\mathbb{V}$. Los espacios $\mathbb{V}$ pueden ser sobre cualquier cuerpo K y otras álgebras como las álgebras de Clifford se pueden construir usando este procedimiento.

Palabras clave: Algebras, cocientes en álgebras, Números complejos y quaterniones como cocientes en álgebras.

NOVA CONSTRUÇÃO DE ÁLGEBRAS COMO QUOCIENTES

RESUMO

Neste artigo apresentamos uma nova abordagem para definir as álgebras usando para um número natural $k \geq 2$, o conjunto de números naturais na base k, nenhum de seus dígitos igual a zero. O estudo foi desenvolvido no contexto de espaços vetoriais R e foram utilizadas as definições de espaço vetorial dos múltiplos formais de um elemento qualquer x do campo R, da soma direta de espaços vetoriais e operações binárias em espaços vetoriais. Os resultados obtidos foram a construção de um espaço vetorial denotado por V, com base no conjunto particular de números naturais na base k mencionados, o que permitiu novas formas de definir as conhecidas e muito importantes álgebras de números complexos e dos quatérnions em R como quocientes de ideais de V, para ideais I adequadamente selecionados. Com esta nova abordagem e com a ajuda dos espaços vetoriais V, as álgebras conhecidas podem ser apresentadas de uma forma diferente das encontradas até agora, usando certos ideais desses espaços na sua forma de quociente. Os espaços V podem estar em qualquer campo K e outras álgebras, tais como álgebras de Clifford, podem ser construídas usando este processo.

Palavras-chave: Álgebras, quocientes nas álgebras, Números complexos e quatérnions como quocientes nas álgebras

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1. INTRODUCTION

Given any symbol x, we define the vector space over $\mathbb{R}$ denoted by <x>, consisting of all formal real multiples of x as the set $\{ rx \,/\, r \in \mathbb{R} \}$, together with the sum $+: < x > \times < x > \to < x >$, defined for all rx, $sx \in < x >$ by $rx + sx = (r + s)x$ and the product $\cdot: \mathbb{R} \times <x> \to <x>$, defined $\forall$ $a \in \mathbb{R}$ and $\forall$ $rx \in < x >$ by $a(rx) = (ar)x$.

We will also deal with the (external) “direct sum” of vector spaces:

Let $\mathcal{C}$ be a collection of vector spaces, the direct sum of all vector spaces V of $\mathcal{C}$ is the set denoted by $\oplus_{V\in\mathcal{C}}$ V, whose elements are the tuples $(u_V)_{V\in\mathcal{C}}$, where $u_V \in V$, for each $V\in\mathcal{C}$ and $u_V = 0_V$ for almost all $V\in\mathcal{C}$ (that means $u_V = 0_V$ for all but a finite number of vector spaces V of $\mathcal{C}$).

This set turns into a vector space over $\mathbb{R}$ with the operations:

Sum: $+:[\oplus_{V\in\mathcal{C}}V]\times[\oplus_{V\in\mathcal{C}}V] \to [\oplus_{V\in\mathcal{C}}V]$, $\forall$ $(x_V)_{V\in\mathcal{C}}$, $(y_V)_{V\in\mathcal{C}} \in [\oplus_{V\in\mathcal{C}}V]$, $(x_V)_{V\in\mathcal{C}} + (y_V)_{V\in\mathcal{C}} = (x_V + y_V)_{V\in\mathcal{C}}$ and the product $\cdot:\mathbb{R}\times[[\oplus_{V\in\mathcal{C}}V]\to[\oplus_{V\in\mathcal{C}}V]$, defined for all $(x_V)_{V\in\mathcal{C}} \in [\oplus_{V\in\mathcal{C}}V]$ and all $r\in\mathbb{R}$ by $r\cdot(x_V)_{V\in\mathcal{C}} = (rx_V)_{V\in\mathcal{C}}$.

Notation: If $U \in \mathcal{C}$ and $u\in U$, with $\bar{u}$ we will understand the tuple $(x_V)_{V\in\mathcal{C}} \in [\oplus_{V\in\mathcal{C}}V]$, where $x_V = 0$ if $V \neq U$ and $x_U = u$.

Given basis $B_V = \{ u_{1_V}, \dots, u_{n_V} \}$ of each vector space $V \in \mathcal{C}$, we consider the subset $\overline{B_v} = \{\overline{u_{1_V}}, \dots, \overline{u_{n_V}}\}$ of $[\oplus_{V\in\mathcal{C}}V]$. The union $U_{V\in\mathcal{C}}\, \overline{B_v}$ is a basis of $[\oplus_{V\in\mathcal{C}}V]$.

Next we consider for all natural number $k \geq 2$, the set $\mathbb{N}_k^*$ of all natural numbers written in basis k, none of its digits null. The digits used to express the numbers of $\mathbb{N}_k^*$ are the elements of $\mathcal{D} = \{1_{\mathbb{N}*}, \dots, 9_{\mathbb{N}*}, d_{10}, \dots, d_{k-1}\}$

($\mathcal{D} = \mathcal{D}_k$ when needed).

The next step is to consider the direct sum $< 1_{\mathbb{R}} >$ $[\oplus_{x\in\mathbb{N}_k^*}<x>]$ or $\mathbb{R}V[\oplus_{x\in\mathbb{N}_k^*}<x>]$, which is a vector space denoted by $\mathbb{V}$ ($\mathbb{V}_k$ when needed).

Vectors of $\mathbb{V}$ are the tuples $\left(r, (r_x x)_{x\in\mathbb{N}_k^*}\right)$, where r, $r_x \in \mathbb{R}$ and $r_x = 0$ for almost all $x \in \mathbb{N}_k^*$ with $r_x = 0\ \forall\ x \in \mathbb{N}_k^*$, will be denoted by $\overline{1_{\mathbb{R}}}$ and, for

The tuple $\left(1, (r_x x)_{x\in\mathbb{N}_k^*}\right)$, with $r_x = 0\ \forall\ x \in \mathbb{N}_k^*$, will be denoted by $\overline{1_{\mathbb{R}}}$ and for all real number r, with $\bar{r}$ the tuple $r - \overline{1_{\mathbb{R}}} = \left(r, (r_x x)_{x\in\mathbb{N}_k^*}\right)$ with $r_x = 0$, $\forall\ x \in \mathbb{N}_k^*$ We recall for all $y \in \mathbb{N}_k^*$ we denote the tuple $\bar{y} = \left(0, (\mathrm{r_x x})_{\mathrm{x}\in\mathbb{N}_k^*}\right)$, where $r_k = 0, \forall\, x \in \mathbb{N}_k^*\ \{y\}$ and $r_y = 1$. Once these has been said, any tuple $\left(r, (\mathrm{r_x x})_{\mathrm{x}\in\mathbb{N}_k^*}\right)$, where $r_x = 0$ for all digit x different from $x_1, \dots, x_n$, can be written as $\bar{r} = \sum_{i=1}^{n} r_{x_i}\overline{x_i}$.

In order to provide $\mathbb{V}$ with an algebra over $\mathbb{R}$ structure, we define the operation $\cdot$: $\mathbb{V} \times \mathbb{V}$ as follows:

i) $\forall \alpha \in \mathbb{V}$, $\overline{1_{\mathbb{R}}}\cdot\alpha = \alpha\cdot\overline{1_{\mathbb{R}}} = \alpha$

ii) $\forall \alpha,\ \ \beta \in \mathbb{N}_k^*,\ \ \alpha = x_0 x_1 \dots x_n,\ \ \beta = y_0 y_1 \dots y_m,$ where $x_0 \dots x_n,\ \ y_0 \dots y_m \ \in \mathcal{D} = \{1, \dots, 9, d_{10}, \dots, d_{k-1}\}$, $\alpha\cdot\beta$ is the ordered concatenation of the α-digits with the β-digits, meaning $\alpha\cdot\beta = [x_0 x_1 \dots x_n y_0 y_1 \dots y_m]$.

iii) Lastly, $\forall u,\ v \in \mathbb{V}$, where $u = \sum_{i=1}^{i=n} r_i \overline{\alpha_i}$ and $v = \sum_{j=1}^{j=m} s_j \overline{\beta_j}$ with $\alpha_i,\ \beta_j \in \mathbb{N}_k^*\ \ \forall\ 1 \le i \le n$ and $\forall\ 1 \le j \le m$ then $u\cdot v = \sum_{i=1}^{i=n}\sum_{j=1}^{j=m}(r_i s_j)\left(\overline{\alpha_i \beta_j}\right)$

It can be easily proven that this operation is associative and bilinear, which makes $\mathbb{V}$ an algebra over $\mathbb{R}$ with $\overline{1_{\mathbb{R}}}$ the multiplicative identity, only commutative when k = 2, because for all k > 2, $1\cdot2 = 12 \ne 21 = 2\cdot1$.

In order to keep the usual exponential notation for the digits $= \{1, \dots, 9, d_{10}, \dots, d_{k-1}\}$, we write $x^n = xx \dots x$ (n-times x) and $x^0 = \overline{1_{\mathbb{R}}}$.

For themes related with general algebra, like vector spaces and direct sums see (Atiyah & Macdonald, 1969; Hartley & Hawkes, 1983).

2. ALGEBRAS AS QUOTIENTS

Our aim here is to build algebras over $\mathbb{R}$ as quotients of conveniently chosen two sided ideals of $\mathbb{V}$.

For instance, lets choose the ideal I of $\mathbb{V}_2$ generated by $11 + \overline{1_{\mathbb{R}}}$ and the quotient $\mathcal{A}_2 = \mathcal{A}_{(2,I)} = \frac{\mathbb{V}}{I}$.

The set of digits $\mathbb{V}_2$ of is the singleton $\mathcal{D}_2 = \{1\}$ and a basis of $\mathbb{V}_2$ is $\{\overline{1_{\mathbb{R}}}\} \cup \mathbb{N}_2^* = \{\overline{1_{\mathbb{R}}}\} \cup \{1, 11, 111, \dots\} = \{\overline{1_{\mathbb{R}}}\} \cup \{1^n/n \in \mathbb{N}\}$, therefore the set $\mathcal{G} = \{\overline{1_{\mathbb{R}}} + I\} \cup \{1^n + I/n \in \mathbb{N}\}$ generates $\mathcal{A}$.

We claim that one basis of the quotient over I is the set $\mathcal{B} = \left\{\overline{1_{\mathbb{R}}} + I, 1 + I\right\}$, which we will proceed to prove right now.

i) <u>$\mathcal{B}$ generates $\mathcal{A}_2$</u>

It can be easily check that $\forall\ k \epsilon \mathbb{N} \cup \{0\}$

$$1^{k} + \mathrm{I} = \begin{cases} \pm(\overline{1_{\mathrm{R}}} + \mathrm{I}) & \text{si k es par} \\ \pm(1 + \mathrm{I}) & \text{si k es impar} \end{cases}$$

Which implies $\mathcal{B}$ actually generates $\mathcal{A}_2$.

ii) $\mathcal{B}$ is linearly independent:

Consider the null linear combination of vectors of $\mathcal{B}$: $a(\overline{1_{\mathbb{R}}} + I) + b(1 + \mathrm{I}) = 0 + I$, where $a, b \epsilon \mathbb{R}$. With the product of $\mathcal{A}_2$ inherited from $\mathbb{V}_2$ multiply by $a(\overline{1_{\mathbb{R}}} + I) + b(1 + \mathrm{I}) = 0 + I$ at both sides of the equality to obtain $(a^2 + b^2)(\overline{1_{\mathbb{R}}} + I) = 0 + I$, from which $a^2 + b^2 = 0$ and $a = b = 0$

It turns out, that $1_{\mathcal{A}} = \overline{1_{\mathbb{R}}} + I$, and denoting with $i = 1 + I$ we have the following table for the product on the basis $\mathcal{B}$ of $\mathcal{A}_2$.

$\cdot$	$1_{\mathcal{A}}$	I
$1_{\mathcal{A}}$	$1_{\mathcal{A}}$	I
i	i	$-1_{\mathcal{A}}$

Which means $\mathcal{A}_2 \approx \mathbb{C}$ and, algebraically speaking, both fields: $\mathcal{A}_2$ and $\mathbb{C}$, are the same object. (Yaglom, 1968).

Another interesting construction is the one of the quaternions, usually denoted by $\mathbb{H}$ (Gürlebeck 1997; Hamilton, 1866).

On that purpose consider the two-sided ideal I of $\mathbb{V}_4$ generated by all elements of the form:

i) $x^2 + \overline{1_{\mathbb{R}}}$, for all $x \epsilon \mathcal{D}$

ii) $xy + yx$, for all x, $y \epsilon \mathcal{D}$, such that $x \neq y$

iii) $12 - 3$, $23 - 1$ and $31 - 2$

In this case the set $\mathcal{B} = \{\overline{1_{\mathbb{R}}} + I, 1 + I, 2 + I, 3 + I\}$ is a basis of $\mathcal{A}_4$ fact that will be proven right now.

i) $\mathcal{B}$ generates $\mathcal{A}_4$

A basis of is $\mathbb{V}_4$ is $\{\overline{1_{\mathbb{R}}}\} \cup \mathbb{N}_4^*$ therefore the set $\left\{ \mathrm{x}_1^{\delta_1} \mathrm{x}_2^{\delta_2} \ldots \mathrm{x}_k^{\delta_k} + \mathrm{I} \,/\, \mathrm{k} \in \mathbb{N} \wedge \forall\, \mathrm{i} = 1, \ldots, \mathrm{n}, x_i \in \mathcal{D} \wedge \delta_{\mathrm{i}} = 0 \text{ ó } \delta_{\mathrm{i}} = 1 \right\}$ generates $\mathcal{A}_4$.

Check that for all $x \epsilon \mathcal{D}$ and $k \epsilon \mathbb{N}$

$$x^{k} + I = \begin{cases} \pm(\overline{1_{R}} + \mathrm{I}) & \text{if k is even} \\ \pm(\mathrm{x} + \mathrm{I}) & \text{if k is odd} \end{cases}$$, which implies that $\mathcal{B}$ generates $\mathcal{A}_4$

ii) $\mathcal{B}$ is linearly independent:

Consider the null linear combination $a(\overline{1_{\mathbb{R}}} + I) + b(1 + \mathrm{I}) + c(2 + I) + d(3 + I) = 0$, where a, b, c, $d \epsilon \mathbb{R}$ and right-multiply at both sides of the equality by $a(\overline{1_{\mathbb{R}}} + I) - b(1 + \mathrm{I}) - c(2 + I) - d(3 + I)$, to obtain $(a^2 + b^2 + c^2 + d^2)(\overline{1_{\mathbb{R}}} + I) = 0$, which implies $a^2 + b^2 + c^2 + d^2 = 0$ and $a = b = c = d = 0$.

To conclude $\mathcal{A}_4$ just rename the elements of $\mathcal{B}$ as $1_{\mathcal{A}} = \overline{1_{\mathbb{R}}} + I, I = 1 + I, j = 2 + I, k = 3 + I$

and consider the following table of the product restricted to $\mathcal{B}$:

·	$1_{\mathcal{A}}$	i	J	k
$1_{\mathcal{A}}$	$1_{\mathcal{A}}$	i	J	K
i	i	-$1_{\mathcal{A}}$	K	-j
j	j	-k	-$1_{\mathcal{A}}$	I
k	k	j	-i	-$1_{\mathcal{A}}$

3. CONCLUDING REMARKS

With this new approach and with the help of the vector spaces $\mathbb{V}$, known algebras can be presented in a different way than those found up to now, by using certain ideals of those spaces in their quotient form. The spaces $\mathbb{V}$ can be over any field K and other algebras can be constructed using this procedure. In particular, as quotients of $\mathbb{V}_k$ the Clifford Algebras (Brackx, Delanghe & Sommen, 1982; Játem & Vanegas, 2018), may also be built, which will appear in a second article now in preparation.

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