

## Classical Action and the Bohm Aharonov Effect

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The Bohm Aharonov phase shift for a free quantum particle with momentum  $p$  moving through a magnetic potential  $A(x)$  with no magnetic force is well known as  $\exp(i \int dx \frac{e}{cA(x)})$ . In the literature this has been obtained in two very different ways. The first way (standard) (1) is to use the classical equation for a charged particle in an electromagnetic field with electrostatic potential i.e.  $(p - e/cA)^2/2m = E$  with  $p = -i\hbar/dx$  representing an overall momentum, not just that of the particle. In such a case, there is coupling at the momentum level, but  $E$  is the energy of a free particle with momentum  $p_1 = p - e/cA$  (classically) i.e.  $p_1^2/2m$ . In other words, momentum considerations are treated differently from energy ones. The time dependent wavefunction is  $\exp(-iEt)$  where  $E$  shows only a noninteracting free particle.

The second approach (2) is the opposite. It ignores the Schrodinger equation and the momentum picture, but retains the idea that for a nonaccelerating free particle  $p/m = \text{velocity} = dx/dt$ . It then considers an interaction which changes  $E$  namely  $\Delta E = p e/cA(x)/m$ . This leads to  $\int pA(x)/m dt = \int pA/m dx/v$  which yields the same Bohm Aharonov phase shift.

In a previous note (3) we considered a free particle classical action as  $A = -Et + px$ . We argued that  $Et$  and  $px$  should be kept separate leading to  $\exp(-iEt)$  and  $\exp(ipx)$  (not combined) if one creates an eigenvalue equation. We argued that scaling exists in both time (by  $E$ ) and space (by  $p$ ). At the same time, the particle moves classically with a speed  $v = p/m$ , but has an associated  $\exp(ipx)$  momentum wave (and independently an energy wave  $\exp(-iEt)$ ). Even if one uses  $v = dx/dt$  as in (2) these waves are different i.e.  $Et = \int E dt = \int .5pv dx/v = .5 px$  while  $px$  is the phase from the momentum consideration. Thus one might argue that it is unusual that a change in  $E$  i.e.  $\Delta E = p/m A(x)$  from (2) applied to  $\exp(-iEt)$  yields the same Bohm Aharonov phase as the Schrodinger equation with  $p$ ,  $A$  coupling, but  $E$  left untouched.

In this note, we consider the classical action  $A = -Et + px$  and try to explain this feature directly from this expression.

### Classical Free Particle Action and Independent Energy and Momentum Waves

The classical action for a free particle is relativistically  $A = -m_0 t \sqrt{1-v^2}$  ( $c=1$ ) and nonrelativistically  $A = t .5mv^2$ . Setting  $v=x/t$  and varying  $x$  and  $t$  independently yields:

$$\partial A/\partial x = p \quad ((1a)) \quad \text{and} \quad \partial A/\partial t = -E \quad ((1b)) \quad \text{yielding} \quad A = -Et + px \quad ((1c))$$

$((1a))$  and  $((1b))$  show that there are separate, independent differential equations for  $E$  and  $p$ . Creating eigenvalue equations to physically manifest the scaling of  $x$  by  $p$  and  $t$  by  $E$  yields:

$$-i\hbar/dx \exp(ipx) = p \exp(ipx) \quad ((2a)) \quad \text{and} \quad i\hbar/dt \exp(-iEt) = E \exp(-iEt) \quad ((2b)) \quad \text{where} \quad E = p^2/2m$$

Thus we suggest a free particle moves with a speed  $v=x/t$  through space, but is associated with independent momentum  $\exp(ipx)$  and energy  $\exp(-iEt)$  waves. The momentum wave keeps

track of space through the phase  $\exp(ipx)$  and we argue that interactions are based on  $\exp(ipx)$  and not on speed. For example, one may have  $p=m_1v_1=m_2v_2$  yielding the same interference pattern. The same is true for the Bohm-Aharonov phase  $\int (a,x) dx_1 A(x_1)$ . Basing interactions on  $\exp(ipx)$  means that one does not follow the particle in space  $x,t$  while describing the interactions (i.e one does not ask which slit the particle passes through in a 2-slit experiment). Nevertheless, the particle is moving with  $v=p/m$  as it interacts, this just does not govern the outcome of the interaction.

### Bohm-Aharonov Phase from a Free Particle Classical Action

If one considers the free particle action  $A = -Et + px$  one may add a term:

Integral  $dx F(x)$  ((3a)) Given that  $v = \text{constant} = dx/dt$  this is equivalent to: Integral  $dt F(x(t)) v$  dt. ((3b))

Thus the extra term could just as well be associated with  $Et$  as with  $px$  leaving the other term unchanged. If one uses ((3a)) and keeps  $x$  and  $t$  independent then:

$$A = -Et + px + \int dx_1 (a,x) F(x) \quad ((4))$$

Creating an eigenvalue equation to physically incorporate the effects on space yields:

$$-i\hbar \frac{d}{dx} \exp(i(px) + i \int dx_1 (a,x) F(x_1)) = (px + F(x)) \exp(i(px) + i \int dx_1 (a,x) F(x_1)) \quad ((5))$$

This suggests a phase shift in addition to the free particle phase  $px$ . Thus momentum of a free particle does not change, but there is a physical presence  $F(x)$  somehow akin to momentum which changes the tracking of phase. This phase is important for interference between two identical free particles traveling different paths leading to different phases.  $F(x)$  for the Bohm Aharonov case is  $e/cA(x)$  the magnetic vector potential. One may note that  $E = p^2/2m$ , the free particle energy, is completely unchanged in ((5)). Thus the  $x$  part of ((4)) is treated independently of the  $t$  part. This approach is compatible with the taking the classical equation:

$$(p(\text{total}) - e/cA)^2 / 2m = E \quad ((6)) \text{ and creating a Schrodinger equation.}$$

$p(\text{total})$  is set to  $-i\hbar \frac{d}{dx}$ , but  $E$  is the energy of noninteracting free particle i.e.  $E = p(\text{noninteract})^2 / 2m$ .

Thus the two are treated differently. This is the approach of (1) which seems to be the standard one in describing the Bohm Aharonov effect.

Alternatively, one may ((3b)) i.e. consider the free particle momentum unchanged and analyse an interaction affecting energy. This interaction does not change the momentum, but there is

also a phase now because eigenfunction equations ((2a)) and ((2b)) physically introduce phases picked up due to energy and momentum in time and space.

In (2) it is ultimately suggested that the interaction is:

$\Delta E = p e/cA(x)/m$  ((7a)) so one has an extra phase  $\exp(-i \int_0^t dt_1 p e/cA(x)/m)$ .  
((7b))

In ((3a)) and ((3b)) we show that this phase must ultimately be the same as that from the momentum approach i.e.  $\int dx_1 e/cA(x_1)$ . In this case  $E$  becomes  $E + \Delta E$ , but momentum  $p$  of the particle is unchanged. In both approaches, there is an interaction which does not ultimately change the kinematics of the free particle, but leads to different phases.

To evaluate ((7b)) one may make use of the idea that even though there are phases in the quantum particle, it moves freely with velocity  $v = p/m = dx/dt$ . Thus ((7b)) becomes:

$\int dt_1 (0,t) p/m e/cA(x) = \int dx_1 (a,x) p/m e/cA(x_1) dx/v$  or  $\int dx_1 e/cA(x_1)$   
((8))

Thus the two approaches lead to the same phase as argued in ((3a)) and ((3b)).

## Conclusion

In conclusion, we argue that the Bohm Aharonov effect and generalizations to it may be obtained directly from the free particle action  $A = -Et + px$  which holds both relativistically and nonrelativistically. This form separates  $Et$  and  $px$  and treats them as scaling time and space. One may create two independent eigenvalue equations showing how this is physically manifested. For example for momentum:  $-i\hbar/dx \exp(ipx) = p \exp(ipx)$ . Thus a free particle moves through space with velocity  $v = x/t = dx/dt$ , but carries with it a momentum wave ( $\exp(ipx)$ ) or alternatively an energy wave ( $\exp(-iEt)$ ). The wave  $\exp(ipx)$  is responsible for interactions which are not based on speed and tracking the particle in  $x$  and  $t$ . Rather  $\exp(ipx)$  behaves as a kind of probability and one adds different  $\exp(ipx)$ s for different scenarios.

Given  $A = -Et + px$  it is possible to add an additional term  $\int dx_1 F(x_1) = \int dt_1 F(t_1)/v$  using  $v = dx/dt$ . In one scenario the term is associated with  $px$  and free particle energy  $E = p^2/2m$  remains unchanged. The alternative is to have  $p$  remain unchanged (i.e. that of a free particle) and have  $E$  change i.e. receive  $\int dt_1 F(t_1)/v dt_1$ . Both ways should lead to the same phase change when one creates an eigenvalue equation showing the physical manifestation.

In the momentum case, the new momentum type wave  $\exp(ipx + i \int (a,x) dx_1 F(x_1))$  satisfies the classical equation  $(p_{total} - e/cA)(p_{total} - e/cA)/2m = E$  converted into a quantum mechanical equation.  $E$  is the free particle energy, but  $p_{total} = -i\hbar/dx$  is the total momentum indicating that an unusual interaction is occurring which changes phase, but not the magnitude of  $p$ . In other words an interaction is associated with  $p$ , but not with  $E$ .

In the energy case  $E + \Delta E$ , (2) uses  $\Delta E = p/m A(x)$  which does not make use of the Schrodinger equation, but directly leads to the same Bohm Aharonov phase. Then  $E$  is involved

in an interaction, but not  $p$ . In the above  $F(x_1) = e/cA(x_1)$  for the Bohm Aharonov phase, but one may see that  $F(x_1)$  is more general and may apply to other problems.

## References

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