

Analysis of the Relationship between Integers and Factorial Functions

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Abstract: This paper presents theorems in factorials and relationship between integers and factorial functions. The results of factorial theorems can be used as applications in computing and cybersecurity to develop algorithms and computer programs.

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1. Introduction

Integers play a vital role in factorial functions or factorials [1-10] and answers of factorials and binomial coefficients such as combination and permutation are integers. Theorems in factorials have several applications in computing, science, and engineering.

Definition: Factorial of any non-negative integer p , denoted by $p!$, is defined as a product of all nonnegative integers less than or equal to p .

2. A Theorem in Factorials

The theorem states that binomial coefficient [1, 10] and the factorial of nonnegative integer q divided by integer p are always an integer.

Theorem 2.1: For integers $p \geq q \geq 0$, $\frac{q!}{p!} \leq \frac{(p+q)!}{p!} \leq \frac{(p+q)!}{q!}$ & $\frac{(p+q)!}{p!q!}$ are integers.

Proof. $0! = 1$; $1! = 1$; $2! = 1 \times 2 = 2$; $3! = 1 \times 2 \times 3 = 6$; \dots ; $p! (p \geq 0)$ are integers.

If $q = p \geq 0$, then $q! = p! \geq 1$. If $q > p$, $q! = p! \times (p+1)(p+2)(p+3) \dots (q-2)(q-1)q$.

$\frac{q!}{p!} = (p+1)(p+2)(p+3) \dots (q-2)(q-1)q$ is an integer. As $\frac{q!}{p!}$ is an integer,

$\frac{q!}{p!} \leq \frac{(p+q)!}{q!} \leq \frac{(p+q)!}{p!}$ are always integer. Here, $p! \leq q! \leq p!q! \leq (p+q)!$ is true.

The binomial coefficient is $\binom{q}{p} = \frac{q!}{p!(q-p)!}$. If $q = p$, then $\binom{p}{p} = \frac{p!}{p!(p-p)!} = 1$.

If $q > p$, $\binom{q}{p} = \frac{q!}{p!(q-p)!} > 1$. For example, $\binom{3}{2} = \frac{3!}{2!1!} = 3$.

Binomial coefficient $\binom{p+q}{p} = \frac{(p+q)!}{p!q!} = l$ is an integer, ($l \geq 0$).

Note that $\frac{(p+q)!}{p!q!} = l \Rightarrow (p+q)! = l \times p! \times q!$.

Theorem 2.2 : For any k nonnegative integers n_1, n_2, n_3, \dots and n_k ,

$$(n_1 + n_2 + n_3 + \dots + n_k)! = (a_1 \times a_2 \times a_3 \times \dots \times a_{k-1}) \times n_1! \times n_2! \times n_3! \times \dots \times n_k!,$$

$$\text{that is, } \left(\sum_{i=1}^k n_i \right)! = A \prod_{i=1}^k n_i!,$$

where $A = a_1 \times a_2 \times a_3 \times \dots \times a_{k-1}$ and $A, a_1, a_2, a_3, \dots, a_{k-1}$ are nonnegative integers.

Proof. Let $x = n_2 + n_3 + n_4 + \dots + n_k$. Then, $(n_1 + x) = a_1 \times n_1! \times x!$ (refer to theorem 2.1), that is, $(n_1 + n_2 + n_3 + \dots + n_k)! = a_1 \times n_1! \times (n_2 + n_3 + \dots + n_k)!$.

Similarly, if we apply the same way to prove each of the sums, we get as follows:

$$(n_2 + n_3 + n_4 + \dots + n_k)! = a_2 \times n_2! \times (n_3 + n_4 + \dots + n_k)!; (n_3 + n_4 + n_5 + \dots + n_k)! = a_3 \times n_3! \times (n_4 + n_5 + \dots + n_k)!; (n_4 + n_5 + n_6 + \dots + n_k) = a_4 \times n_4! \times (n_5 + n_6 + \dots + n_k)!; \dots, \text{ and } (n_{k-1} + n_k) = a_{k-1} \times n_{k-1}! \times n_k.$$

If we substitute these results step by step in $a_1 \times n_1! \times (n_2 + n_3 + \dots + n_k)!$, that is, $a_1 \times n_1! \times (n_2 + n_3 + n_4 \dots + n_k)! = a_1 \times n_1! \times (a_2 \times n_2! \times (n_3 + n_4 \dots + n_k)!)$ $= a_1 \times a_2 \times n_1! \times n_2! \times (a_3 \times n_3! \times (n_4 + \dots + n_k)!)$, etc., we obtain the following result.

$$(n_1 + n_2 + n_3 + \dots + n_k)! = (a_1 \times a_2 \times a_3 \times \dots \times a_{k-1}) \times n_1! \times n_2! \times n_3! \times \dots \times n_k!,$$

$$\text{that is, } \left(\sum_{i=1}^k n_i \right)! = A \prod_{i=1}^k n_i!, \quad \text{where } A = a_1 \times a_2 \times a_3 \times \dots \times a_{k-1}.$$

Hence, theorem is proved.

For instance,

If $n_1 = n_2 = n_3 = \dots = n_k = n$. Then, $(n_1 + n_2 + n_3 + \dots + n_k)! = (k \times n)!$.

If $n_1 = n_2 = n_3 = \dots = n_k = 0$. Then, $(n_1 + n_2 + n_3 + \dots + n_k)! = (k \times 0)! = 0! = 1$.

If $n_1 = n_2 = n_3 = \dots = n_k = 1$. Then, $(n_1 + n_2 + n_3 + \dots + n_k)! = (k \times 1)! = k!$.

If $n_1 = n_2 = n_3 = \dots = n_k = 2$. Then, $(n_1 + n_2 + n_3 + \dots + n_k)! = (k \times 2)! = (2k)!$.

If $n_1 = n_2 = n_3 = \dots = n_k = k$. Then, $(n_1 + n_2 + n_3 + \dots + n_k)! = (k \times k)! = k^2!$.

This idea can help to the researchers working in computational science, management, science, and engineering.

3. Conclusion

In this article, an innovative combinatorial technique and theorem are introduced and the theorem states that the factorial of sum of any k nonnegative integers is equal to multiple of the product of factorials of the k nonnegative integers. This methodological advance can enable the researchers working in computational science, management, science and engineering to solve the most real life problems and meet today's challenges [11].

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