Cap Volatility Surface Introduction

Cap Volatility

- Cap consists of a portfolio of caplets. In market, a cap is quoted by implied volatilities rather than prices.
- An interest rate cap volatility surface is a three-dimensional plot of the implied volatility as a function of strike and maturity.
- Caplets of a cap are of the same term. For example, all underlying of caplets are of 3-month or 6-month forward rates.
- Implied cap volatilities are market observable while caplet volatilities are not.
- To obtain caplet volatilities, a fitting procedure (or bootstrapping) is needed.

Cap Volatility

- We find a group of caps in market.
- There are two approaches to bootstrap caplet volatility.
- The first approach is to go through each cap one by one to make model price equal to market price.
- There are n x m minimization problems to be solved where n is the number of caplet tenors and m is the number of strikes.
- The second approach is to obtain all the caplet volatilities simultaneously by solving a single minimization problem with n x m variables.

Cap Volatility

- To construct a reliable volatility surface, it is necessarily to apply robust interpolation methods to a set of discrete volatility data. Arbitrage free conditions may be implicitly or explicitly embedded in the procedure. Typical approaches are
  - Local Volatility Model: a generalisation of the Black-Scholes model.
  - Stochastic Volatility Models: such as SABR, Heston, Levy
  - Parametric or Semi-Parametric Models: such as SVI, Omega

Cap Volatility

- Vertical arbitrage free and horizontal arbitrage free conditions for cap volatility surfaces are based on different strikes
- There is no calendar arbitrage in cap volatility surfaces as caps with different maturities have different cash flows and are associated with different indices. In other words, they can be treated independently.

Cap Volatility

- The SABR model is a stochastic volatility model for the evolution of the forward price of an asset, which attempts to capture the volatility smile/skew in derivative markets.
- There is a closed-form approximation of the implied volatility of the SABR model.
- In the cap volatility case, the underlying asset is the forward interest rate.

Cap Volatility

- The dynamics of the SABR model

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta}dW_1$$

$$d\hat{\alpha} = v\hat{\alpha}dW_2$$

$$dW_1 dW_2 = \rho dt$$

$$\hat{\alpha}(0) = \alpha$$

where

$\hat{F}$	the forward rate
$\hat{\alpha}$	the forward volatility
W_1, W_2	the standard Brownian motions
ρ	the instantaneous correlation between W_1 and W_2

Cap Volatility

- There are four parameters (a , b , r, n) in SABR model
- For each maturity of a cap/floor, conduct the following calibration procedure.
- The β parameter is estimated first and typically chosen a priri according to how the market prices are to be observed.
- Alternatively β can be estimated by a linear regression on a time series of ATM volatilities and of forward rates.
- After β is set, we can obtain α by using σ_{ATM} to solve the following equition

$$\sigma_{ATM} = \frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{\alpha^2(1-\beta)^2}{24f^{2(1-\beta)}} + \frac{\rho\beta v\alpha}{4f^{1-\beta}} + \frac{(2-3\rho^2)v^2}{24}\right]T\right\}$$

The Viete method is used to solve this equation

Cap Volatility

- Given the α and β solved above, we can find the optimized value of (ρ, v) by minimizing the distance between the SABR model output volatilities and market volatilities across all strikes for each term and tenor.

$$min \sum_{i=1}^{n} \left[\sigma_i^{SABR}(\alpha, \beta, \rho, v) - \sigma_i^{Market}\right]^2$$

  The Levenberg-Marquardt least-squares optimization routine is used for optimization.

- After α, β, ρ, v calibrated, one can generate SABR volatility (cap volatility) for any moneyness.
- Repeat the above process for each term and tenor.

Thank You

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