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# Note on the Riemann Hypothesis

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**Abstract** In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and and Planat stated that the Riemann hypothesis is true if and only if the inequality $\prod_{q \leq q_n} \left(1+\frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. We call this inequality the Dedekind inequality. We can deduce from that paper, if the Riemann hypothesis is false, then the Dedekind inequality is not satisfied for infinitely many prime numbers $q_n$. Using this argument, we prove the Riemann hypothesis is true when $\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$ holds for a sufficiently large prime number $q_n$. We show this is equivalent to state that the Riemann hypothesis is true when $(1-\frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1+\frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x}$ is always satisfied for every sufficiently large positive number $x$. However, we know that inequality is trivially satisfied for every sufficiently large positive number $x$. In this way, we prove the Riemann hypothesis is true.

**Keywords** Riemann hypothesis · Prime numbers · Dedekind function · Chebyshev function · Riemann zeta function

**Mathematics Subject Classification (2010)** 11M26 · 11A41 · 11A25

## 1 Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathe-

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matics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers $p$ that are less than or equal to $x$ [6]. We denote the $n^{th}$ prime number as $q_n$. We know the following properties for the Chebyshev function:

**Theorem 1.1** *For all $n \geq 2$, we have [3]:*

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}).$$

**Theorem 1.2** *For every $x \geq 19035709163$ [1]:*

$$\theta(x) > (1 - \frac{0.15}{\log^3 x}) \times x.$$

Besides, we define the prime counting function $\pi(x)$ as

$$\pi(x) = \sum_{p \leq x} 1.$$

We also know this property for the prime counting function:

**Theorem 1.3** *For every $x \geq 19027490297$ [1]:*

$$\pi(x) > \eta_x$$

*where*

$$\eta_x = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2 \times x}{\log^3 x} + \frac{5.85 \times x}{\log^4 x}$$
$$+ \frac{23.85 \times x}{\log^5 x} + \frac{119.25 \times x}{\log^6 x} + \frac{715.5 \times x}{\log^7 x} + \frac{5008.5 \times x}{\log^8 x}.$$

In mathematics, $\Psi = n \times \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind $\Psi$ function, where $q \mid n$ means the prime $q$ divides $n$. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm and $\zeta(x)$ is the Riemann zeta function. The importance of this inequality is:

**Theorem 1.4** $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann hypothesis is true [7].*

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [5]. We know from the constant $H$, the following formula:

**Theorem 1.5** *We have that [5]:*

$$\sum_{q} \left( \log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H.$$

We know this value of the Riemann zeta function:

**Theorem 1.6** *It is known that [7]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

We have the following result:

**Theorem 1.7** *For every $x > -1$ [4]:*

$$x \geq \log(1+x).$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

## 2 Results

**Theorem 2.1** *If the Riemann hypothesis is false, then there are infinitely many prime numbers $q_n$ for which $\mathsf{Dedekind}(q_n)$ do not hold.*

*Proof* If the Riemann hypothesis is false, then we consider the function [7]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the Riemann hypothesis is false, if there exists some $x_0$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ [7]. We know the bound [7]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where $f$ is introduced in the Nicolas paper [6]:

$$f(x) = e^{\gamma} \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

From the same paper [6], we know when the Riemann hypothesis is false, then there is a $0 < b < 1$ such that $\limsup x^{-b} \times f(x) > 0$ and hence $\limsup \log f(x) \gg \log x$, where the symbol $\gg$ means "much greater than" [7]. In this way, if the Riemann hypothesis is false, then there are infinitely many natural numbers $x$ such that $\log f(x) \geq \log x$ by the definition of limit superior. The result follows because of $\frac{2}{x} = o(\log x)$ and therefore, there would be infinitely many $x_0$ such that $\log g(x_0) > 0$ [7].

The following is a key theorem.

**Theorem 2.2**
$$\sum_{q}\left(\frac{1}{q}-\log(1+\frac{1}{q})\right)=\log(\zeta(2))-H.$$

*Proof* If we add $H$ to
$$\sum_{q}\left(\frac{1}{q}-\log(1+\frac{1}{q})\right)$$
then we obtain that
$$\begin{aligned}
H+\sum_{q}\left(\frac{1}{q}-\log(1+\frac{1}{q})\right) &= \sum_{q}\left(\log(\frac{q}{q-1})-\frac{1}{q}\right)+\sum_{q}\left(\frac{1}{q}-\log(\frac{q+1}{q})\right)\\
&= \sum_{q}\left(\log(\frac{q}{q-1})-\log(\frac{q+1}{q})\right)\\
&= \sum_{q}\left(\log(\frac{q}{q-1})+\log(\frac{q}{q+1})\right)\\
&= \sum_{q}\left(\log(\frac{q^2}{(q-1)\times(q+1)})\right)\\
&= \sum_{q}\left(\log(\frac{q^2}{(q^2-1)})\right)\\
&= \log(\prod_{q}\frac{q^2}{q^2-1})\\
&= \log(\zeta(2))
\end{aligned}$$
according to the Theorems 1.5 and 1.6. Therefore, the proof is done.

This is a new criterion based on the Dedekind inequality.

**Theorem 2.3** $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the inequality*
$$\sum_{q}\frac{1}{q}-\sum_{q>q_n}\log(1+\frac{1}{q})>B+\log\log\theta(q_n)$$
*is satisfied for all prime numbers* $q_n > 3$.

*Proof* We start from the inequality:
$$\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)>\frac{e^{\gamma}}{\zeta(2)}\times\log\theta(q_n).$$
If we apply the logarithm to the both sides of the inequality, then
$$\log(\zeta(2))+\sum_{q\leq q_n}\log(1+\frac{1}{q})>\gamma+\log\log\theta(q_n).$$

This is the same as

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

which is

$$\sum_{q} \left(\frac{1}{q} - \log(1 + \frac{1}{q})\right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

according to the Theorem 2.2. Let's distribute the elements of the inequality to obtain that

$$\sum_{q} \frac{1}{q} - \sum_{q > q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

when $\mathsf{Dedekind}(q_n)$ holds. The same happens in the reverse implication.

This is the main insight.

**Theorem 2.4** *The Riemann hypothesis is true if the inequality*

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

*is satisfied for all sufficiently large prime numbers $q_n$.*

*Proof* The inequality

$$\sum_{q} \frac{1}{q} - \sum_{q > q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{q} \frac{1}{q} - \sum_{q \geq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

is also satisfied. In the inequality

$$\sum_{q} \frac{1}{q} - \sum_{q \geq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

only change the value of

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes $q_n$ and $q_{n+1}$. Hence, it is enough to show that

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers $q_n$ according to the Theorems 2.1 and 2.3. Certainly, if the inequality

$$\log(1+\frac{1}{q_n})+\log\log\theta(q_n)\geq\log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers $q_n$, then it cannot exist infinitely many prime numbers $q_n$ for which $\mathsf{Dedekind}(q_n)$ do not hold. By contraposition, we know that the Riemann hypothesis should be true. This is the same as

$$\log\left((1+\frac{1}{q_n})\times\log\theta(q_n)\right)\geq\log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}}\geq\log\log\theta(q_{n+1}).$$

Therefore, the Riemann hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}}\geq\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers $q_n$.

**Theorem 2.5** *The Riemann hypothesis is true when the inequality* $(1-\frac{0.15}{\log^3 x})^{\frac{1}{x}}\times x^{\frac{1}{x}}\geq 1+\frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x}$ *is satisfied for all sufficiently large positive numbers x.*

*Proof* Because of the Theorem 2.4, we know that the Riemann hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}}\geq\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers $q_n$. This is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}}\geq\theta(q_n)+\log(q_{n+1})$$

which is

$$\theta(q_n)^{\frac{1}{q_n}}\geq 1+\frac{\log(q_{n+1})}{\theta(q_n)}.$$

We use the Theorem 1.2 to show that

$$\theta(q_n)^{\frac{1}{q_n}}>(1-\frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}}\times q_n^{\frac{1}{q_n}}$$

for a sufficiently large prime number $q_n$. Under our assumption in this theorem, we have that

$$(1-\frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}}\times q_n^{\frac{1}{q_n}}\geq 1+\frac{\log(1-\frac{0.15}{\log^3 q_n})+\log q_n}{q_n}.$$

Using the Theorems 1.1 and 1.3, we only need to show that

$$\begin{aligned}\frac{\theta(q_n)}{\log q_{n+1}} &\geq n \times (1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}) \\ &> \eta_{q_n} \times (1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}) \\ &> \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}\end{aligned}$$

for a sufficiently large prime number $q_n$ where

$$\eta_{q_n} = \frac{q_n}{\log q_n} + \frac{q_n}{\log^2 q_n} + \frac{2 \times q_n}{\log^3 q_n} + \frac{5.85 \times q_n}{\log^4 q_n}$$
$$+ \frac{23.85 \times q_n}{\log^5 q_n} + \frac{119.25 \times q_n}{\log^6 q_n} + \frac{715.5 \times q_n}{\log^7 q_n} + \frac{5008.5 \times q_n}{\log^8 q_n}.$$

Certainly, as the prime number $q_n$ increases, the value of $(1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n})$ gets closer to 1 and the inequality $\eta_{q_n} \gg \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}$ starts to become trivially satisfied, where the symbol $\gg$ means "much greater than" [7]. However, this implies that

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > \frac{\log(q_{n+1})}{\theta(q_n)}$$

which is equal to

$$1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

and finally, the proof is complete.

**Theorem 2.6** *The Riemann hypothesis is true.*

*Proof* From the Theorem 1.7, we have that:

$$\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} \geq \log(1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x})$$

since

$$\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} > -1$$

for all sufficiently large positive numbers $x$. We know that

$$\begin{aligned}\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} &= \frac{\log\left((1 - \frac{0.15}{\log^3 x}) \times x\right)}{x} \\ &= \log\left((1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}\right)\end{aligned}$$

by the properties of the logarithm. This implies that

$$\log((1-\frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}) \geq \log(1+\frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x})$$

which is equivalent to

$$(1-\frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1+\frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x}$$

and so, this final result is a direct consequence of the Theorem 2.5.

## 3 Discussion

The practical uses of the Riemann hypothesis include many propositions which are known as true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [2]. Certainly, the Riemann hypothesis is closed related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [2]. Indeed, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics in general [2]. We consider that our paper has achieved this goal considered as the Holy Grail of Mathematics by several authors [2].

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