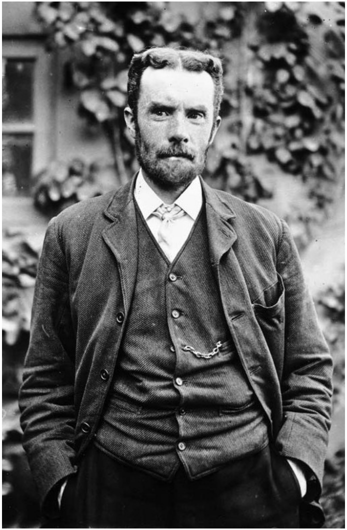
Transmission Lines and Light Bulbs: A Case Study

Predrag Pejović

Tuesday 15th February, 2022

dedicated to the memory of Oliver Heaviside 1850-1925



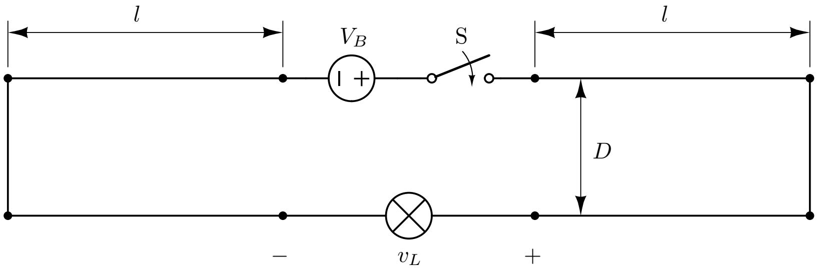
a word about Oliver Heaviside . . .

Wikipedia article: "Oliver Heaviside FRS (18 May 1850 – 3 February 1925) was an English mathematician and physicist who brought complex numbers to circuit analysis, invented **a new technique for solving differential equations** (equivalent to the Laplace transform), independently developed **vector calculus**, and rewrote **Maxwell's equations** in the form commonly used today. He significantly shaped the way Maxwell's equations are understood and **applied** in the decades following Maxwell's death. His formulation of the **telegrapher's equations** became commercially important during his own lifetime, after their significance went unremarked for a long while, as few others were versed at the time in his novel methodology. Although **at odds with the scientific establishment for most of his life**, Heaviside **changed the face of telecommunications, mathematics, and science.**"

how did we get here?

- a popular science video by Veritasium
- a bit of oversimplification ... and fine print ...
- I posted an initial complaint, a commenting video ...
- but, what actually goes on?
- it turned out to be analytically (and numerically) solvable!
- ... easier than by plain numerical simulation ...
- ... and providing more insight!
- I had a lot of fun in solving the problem ...
- this much that I bothered to make this presentation ...
- to share that joy with you!
- I really admire Oliver Heaviside and Jean le Rond d'Alembert ... and some others ...
- who actually solved the problem ...

the problem: setup



$D = 1\,\mathrm{m}$

$l = c \times \frac{1}{2}\,\mathrm{s} \approx 1.5 \times 10^8\,\mathrm{m} = 150\,\mathrm{Mm} = 150000\,\mathrm{km}$

the problem: question

In my perception: How long after closing the switch the light bulb will turn on? Offered answers:

a) $0.5\,\mathrm{s}$
b) $1\,\mathrm{s}$
c) $2\,\mathrm{s}$
d) $\frac{1\,\mathrm{m}}{c}$
e) none of the above

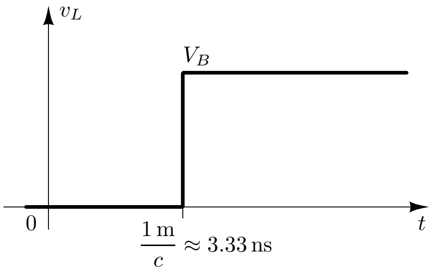
Suggested answer: d).

Some truth **is** there. But, is this representative part of the process?

Fine print followed: any voltage makes the light bulb shine!

I'd rather opt for e), and I'll explain that here.

a misbelief

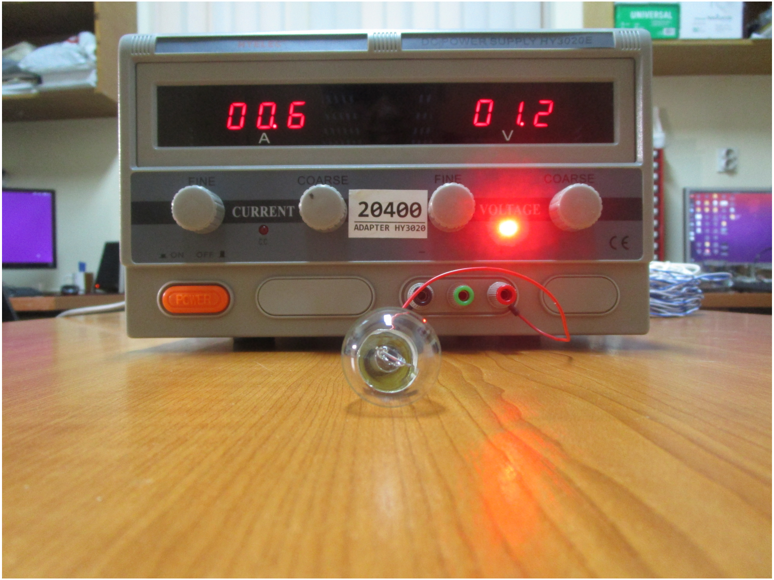


- if you believe that after $\frac{1\,\mathrm{m}}{c} \approx 3.33\,\mathrm{ns}$ the light bulb voltage reaches its final voltage, or even the light bulb starts to shine, this is not what would happen
- however, if this is not, what actually would happen?
- we'll cover that here!
- something quite different and more complex would happen!

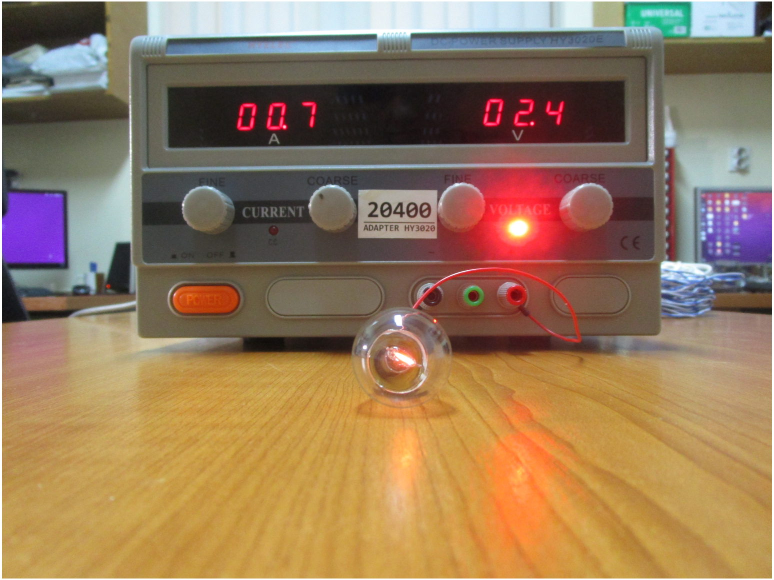
our setup ... for numerical examples

- we have $D = 1\,\mathrm{m}$ and $l = 150\,\mathrm{Mm}$ ...
- but we do not have d, diameter of the wire ...
- which is **really important** ...
- I have chosen $d = 1.382\,\mathrm{mm}$, corresponding to $A_{Cu} = 1.5\ (\mathrm{mm})^2$, sufficient for $10\,\mathrm{A}$ RMS
- I have also chosen $V_B = 12\,\mathrm{V}$ ...
- and $P_L = 21\,\mathrm{W}$, resulting in $R_L = 6.8571\,\Omega$ ... when hot!
- such light bulb is common for brake lights and turn indicators ...
- ... and $I_L = V_B/R_L = 1.75\,\mathrm{A} < 10\,\mathrm{A}$, the wire is thick enough, with margin! the setup is fair enough!
- **bad part:** $V_{Cu} = 900\,\mathrm{m}^3$, $m_{Cu} = 8064\,\mathrm{Mg}$ (metric tons), cost $\approx 78\,\mathrm{MUS\$}$, $R_w = 6.720\,\mathrm{M\Omega}$, and R_w is going to be the biggest problem!
- superconducting version first, with R_w later ...

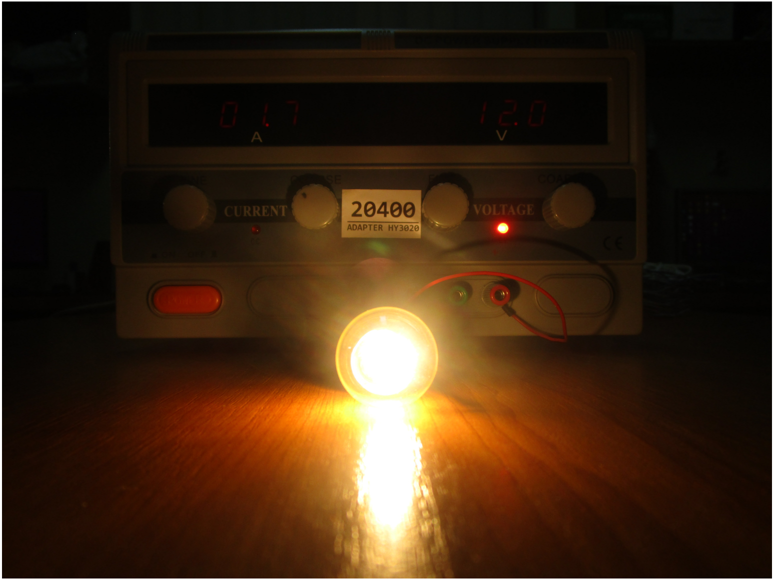
a word about incandescent light bulbs . . . at 10%



a word about light bulbs . . . at 20%



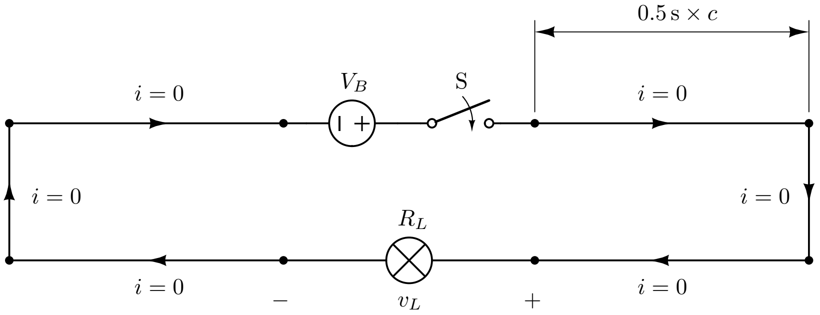
about light bulbs . . . at 100% of the rated voltage



a word about fine print: warning!

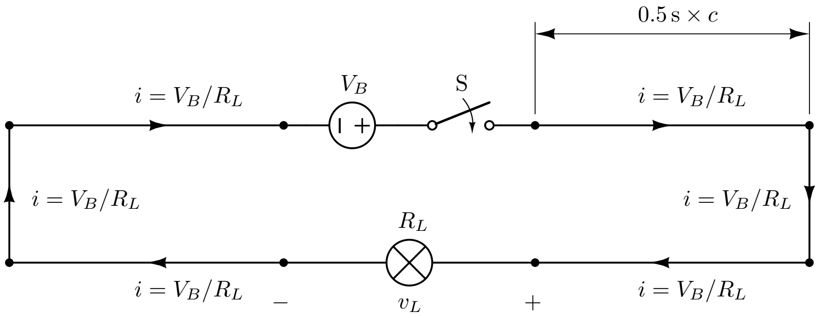
- **be careful with fine print!**
- **be careful when common notions are redefined!**
- what is our goal in communication? to convey meaning?
- the goal is to understand the system and to learn something that might be useful in analyzing other systems
- words are to convey meaning, not to misguide
- the setup turned out to be **an excellent example!**
- so, no hard feelings, let's enjoy analyzing the system and learning something out of that!
- **thanks** to all of the people who got involved!
- and, please, no hard feelings!
- **the answer is not one bit of information!**

the transient: beginning



$W_T = 0$

the transient: end



$$W_T = l\,L' \left(\frac{V_B}{R_L}\right)^2 = 1336\,\mathrm{J} = 0.371\,\mathrm{Wh}$$

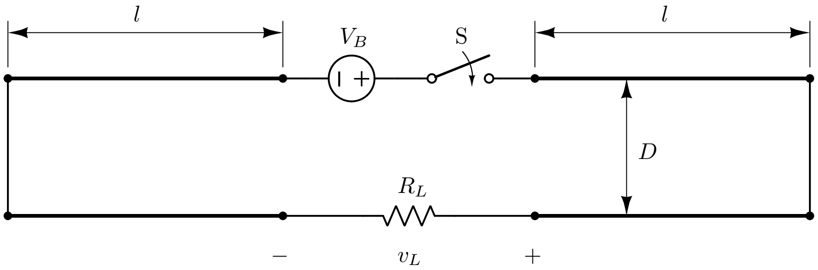
to get this in 3.33 ns you would need $P_B \approx 400\,\mathrm{GW}$ source

besides, any effect of closing the switch in 3.33 ns could reach only 1 m

the transient

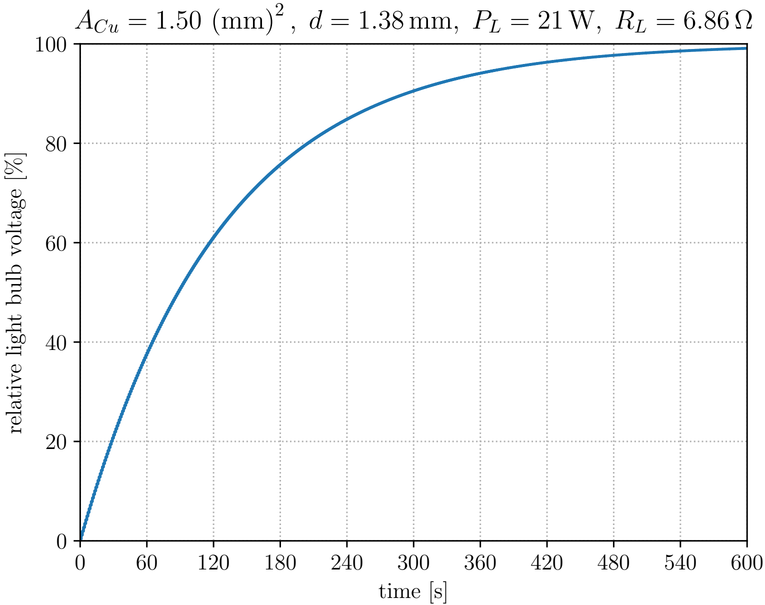
- clearly, there is a transient that lasts much longer than $\frac{1\,\mathrm{m}}{c} \approx 3.33\,\mathrm{ns}$, in the order of seconds, at least
- there is a light cone
- but, how does the current in the loop change over time and over the loop, in space?
- is it possible that initial current of the bulb, at $t = 0^+$, equals the value at the end, at $t \to \infty$, and that the transient occurs somewhere else, away from the bulb?
- it will be shown here that **even that is possible**, but with slightly modified setup and at much lower final voltage than required to light the bulb really

simplified model . . . for the most of the analysis

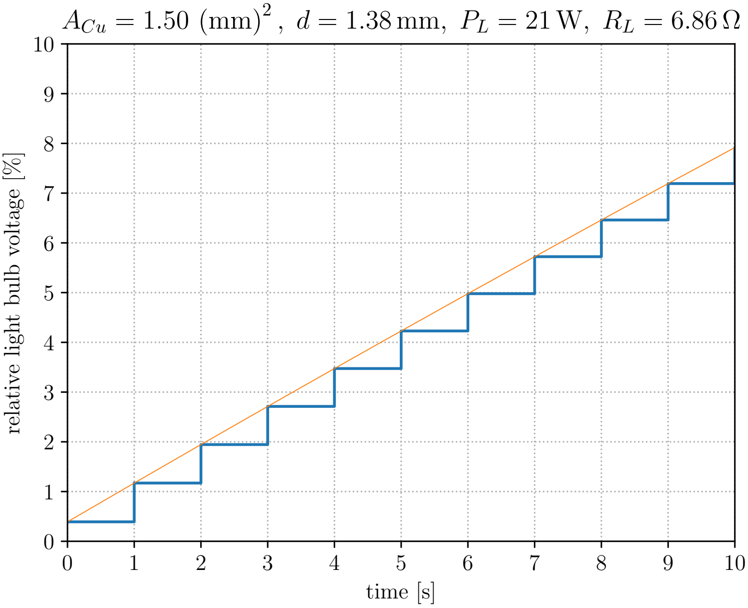


1. thick lines represent transmission lines . . .
2. the light bulb is replaced by a linear resistor

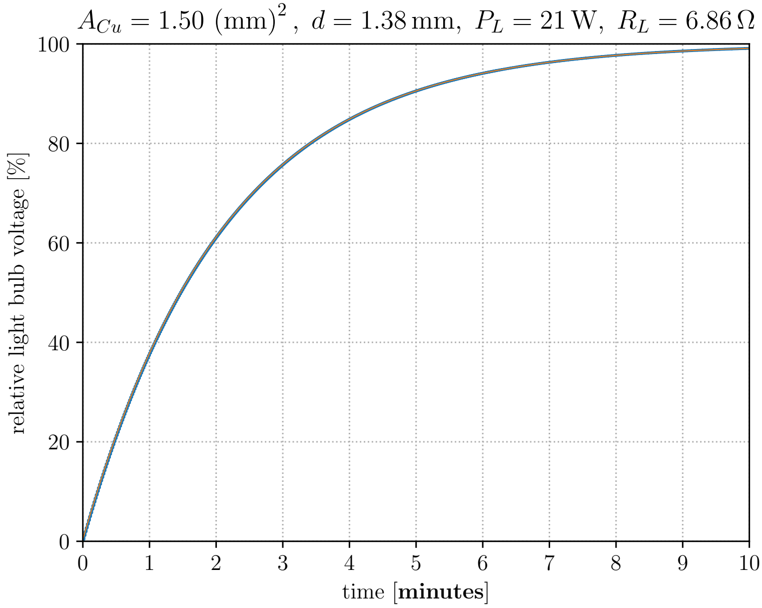
solution, $R_w = 0$



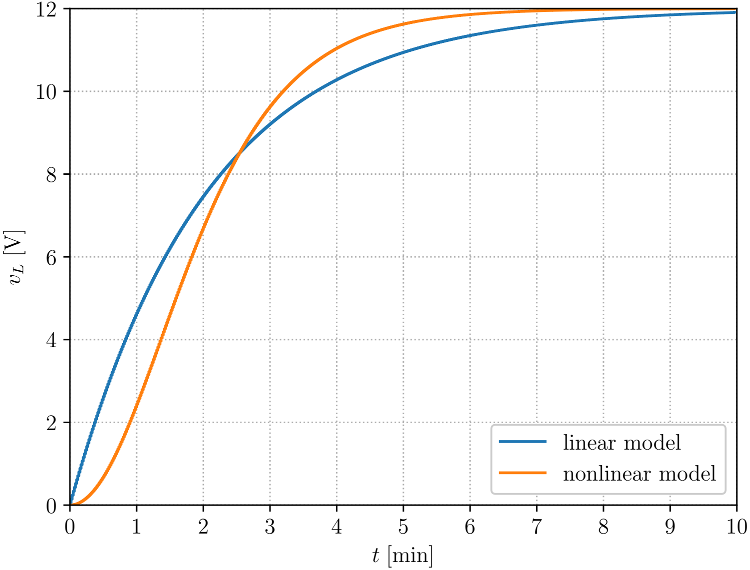
solution, $R_w = 0$



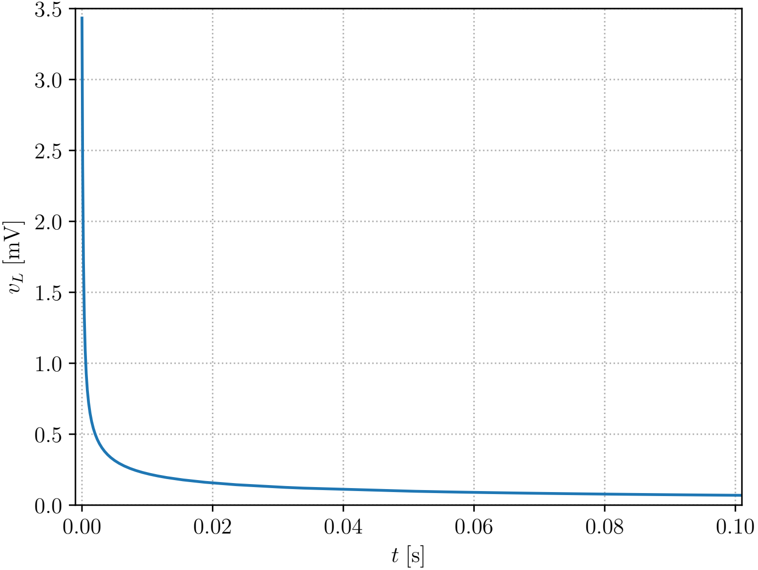
solution, $R_w = 0$



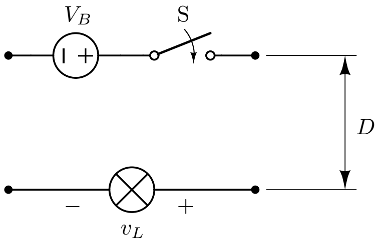
solution, $R_w = 0$, nonlinear load, light bulb



total disaster, $R_w \neq 0$, $R_L = 0.5\,\Omega$ since $V_L \approx 0$



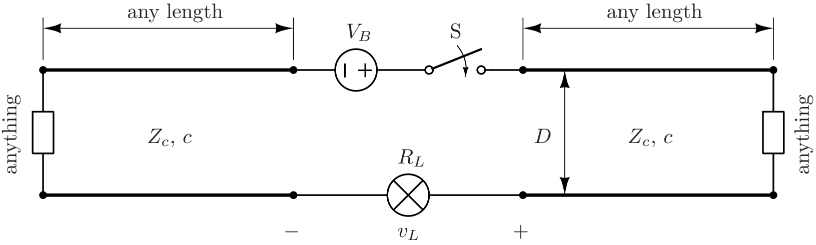
a much cheaper alternative for the setup



the same answer if you ask the question clearly: "when the first information (signal) about closing of the switch would reach the light bulb?"

answer: in $\Delta t = \dfrac{D}{c}$

the same initial voltage in many cases . . .

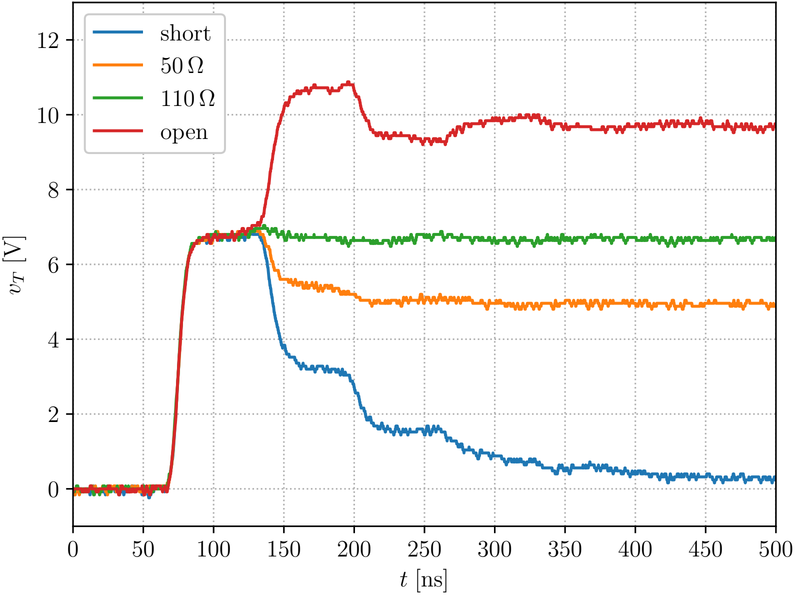


$$v_L\left(0^+\right) = \frac{R_L}{2\,Z_c + R_L}\,V_B$$

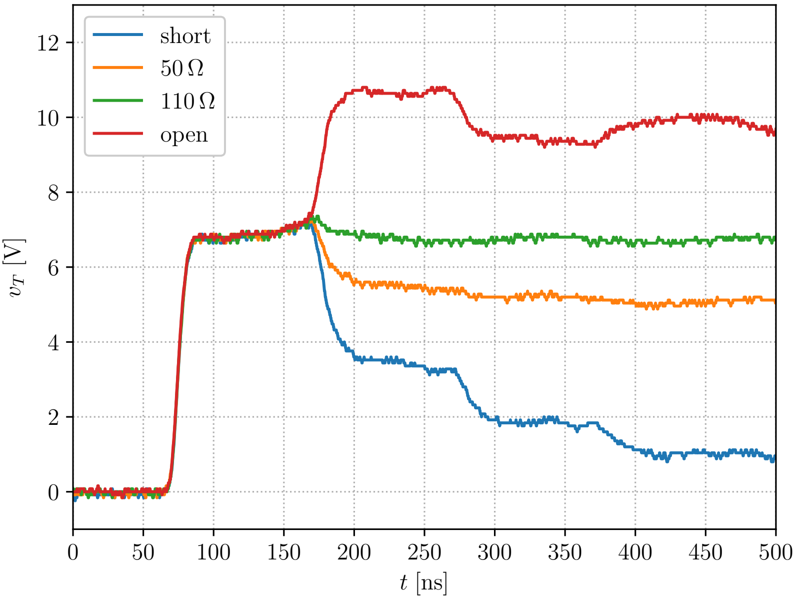
$$Z_c = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\operatorname{arccosh}\left(\frac{D}{d}\right) \approx 119.92\,\Omega\,\operatorname{arccosh}\left(\frac{D}{d}\right)$$

actually, since 2019 the constant is exactly $119.9169832\,\Omega$

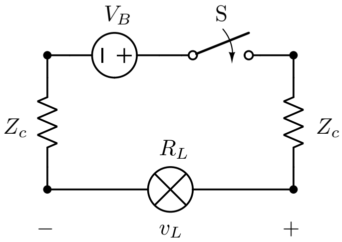
the same initial voltage … white wire (6 m)



the same initial voltage . . . blue-white wire (10 m)



equivalent circuit for the initial voltage . . .

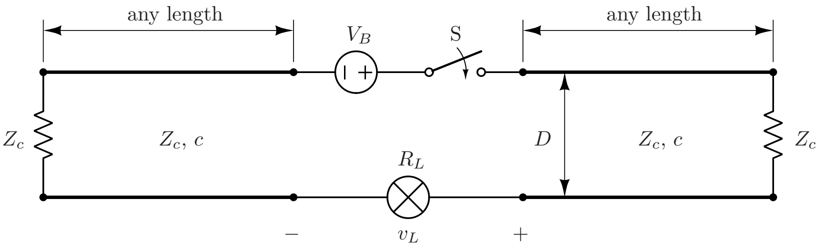


$$v_{L\,\mathrm{initial}} = \frac{R_L}{R_L + 2\,Z_c}\,V_B \approx \frac{R_L}{2\,Z_c}\,V_B$$

$$Z_c \sim 500\Omega,\; R_L \sim 10\,\Omega,\; \frac{R_L}{2\,Z_c} \sim 1\%$$

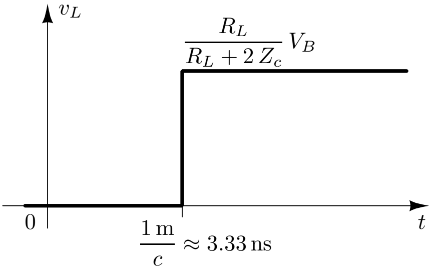
parameter choices favorable for $v_{L\,\mathrm{initial}}$

“no transient” case . . .



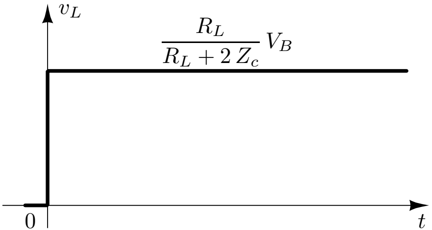
$$v_L = \frac{R_L}{R_L + 2\,Z_c}\,V_B \approx \frac{R_L}{2\,Z_c}\,V_B \sim 1\%\,V_B$$

“no transient” case . . . delayed



$$v_L = \frac{R_L}{R_L + 2\,Z_c}\,V_B \approx \frac{R_L}{2\,Z_c}\,V_B \sim 1\%\,V_B$$

"no transient" case ... actually computed



$$v_L = \frac{R_L}{R_L + 2\,Z_c}\,V_B \approx \frac{R_L}{2\,Z_c}\,V_B \sim 1\%\,V_B$$

motivation: why did I do this?

- because I feel that transmission line theory is not taught enough on the undergraduate level
- because the problem proposed by Veritasium is great to illustrate:
  - mathematical modeling
  - solving systems in electrical engineering
  - circuit analysis techniques
  - circuits with distributed parameters
  - transmission line theory
  - wave equation
  - discrete time modeling
  - analysis of nonlinear circuits
  - analysis of transmission lines in frequency domain
  - transforms, impulse response, step response ...
- because I had fun in solving the problem ($\leftarrow$ that's it)
- because I'd like to share this joy with you

why Oliver Heaviside?

because **Oliver Heaviside** invented the most of the techniques needed to solve this problem:

- Heaviside step function turns the switch on
- Heaviside response is our problem
- although I did not use the full potential of his operational calculus: the problem is simple enough, used in a part of the analysis
- Maxwell's equations, although I dd not use his vector calculus, the model is simplified using his $\downarrow$
- telegrapher's equations reduced the problem to one spatial dimension; actually to decouple: $3\mathrm{D} = 2\mathrm{D} + 1\mathrm{D}$
- coaxial cable I used in my examples

with some help across the Channel ...

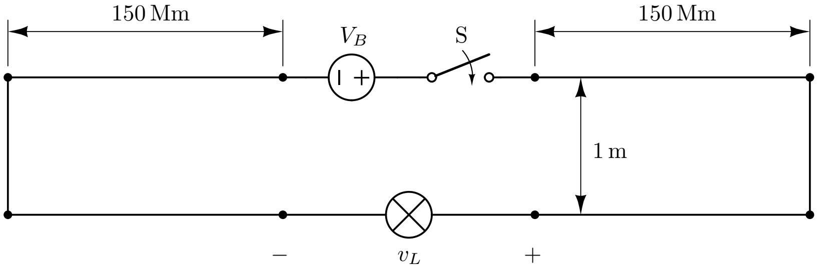
mathematics part:

- one-dimensional wave equation is going to be our model
- solved by Jean le Rond d'Alembert, ingeniously!
- but our example allows reduction to linear algebraic equations that model DC circuits

overall:

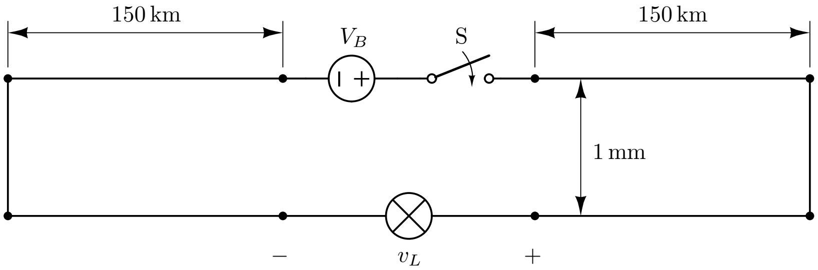
- our 21^{st} century problem is solved by 19^{th} physics and 18^{th} century mathematics
- affordable 21^{st} century computers provided nice typesetting and graphs, some simulation, communication, dissemination, resources, references ... well, this **is** important
- and the computers I use are run by free software

a word about scaling, space . . .



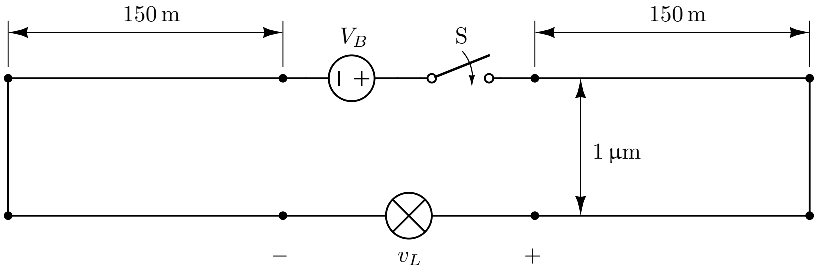
- all of the labeled dimensions are length!
- and the figure is geometry, about length
- let's put it in scale!

a word about scaling, space . . .

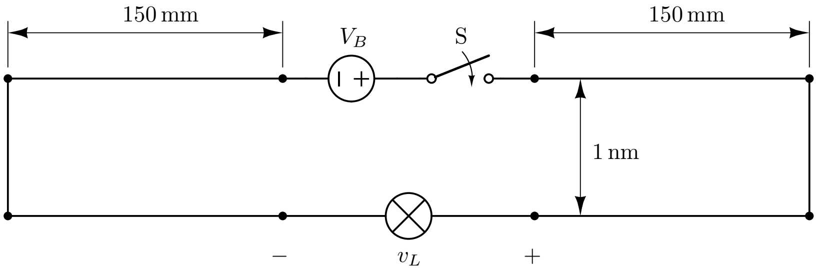


- one dimension is really pronounced!
- decoupling?
- possible to reduce the space to one dimension?

a word about scaling, space . . .

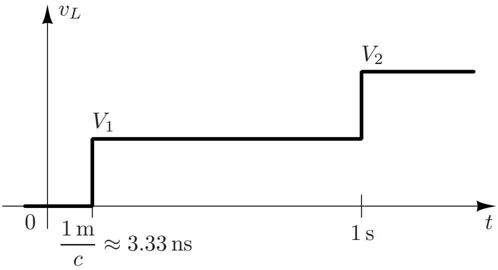


a word about scaling, space . . .



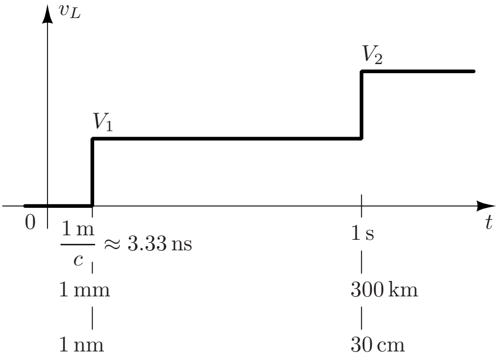
1 nm is the size of several atoms

another word about scaling, time ...



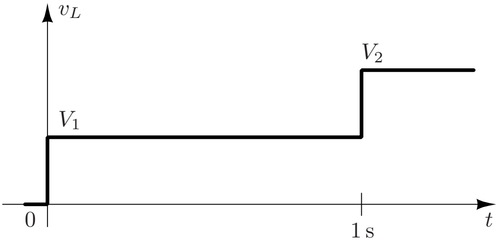
- labeled quantities are time
- presented as length
- but we can also put it in scale!

another word about scaling, time . . .



1 nm is the size of several atoms

another word about scaling, time . . .



- $3.33\,\mathrm{ns} \ll 1\,\mathrm{s}$, in this context $3.33\,\mathrm{ns} \approx 0$
- and I'll do that!

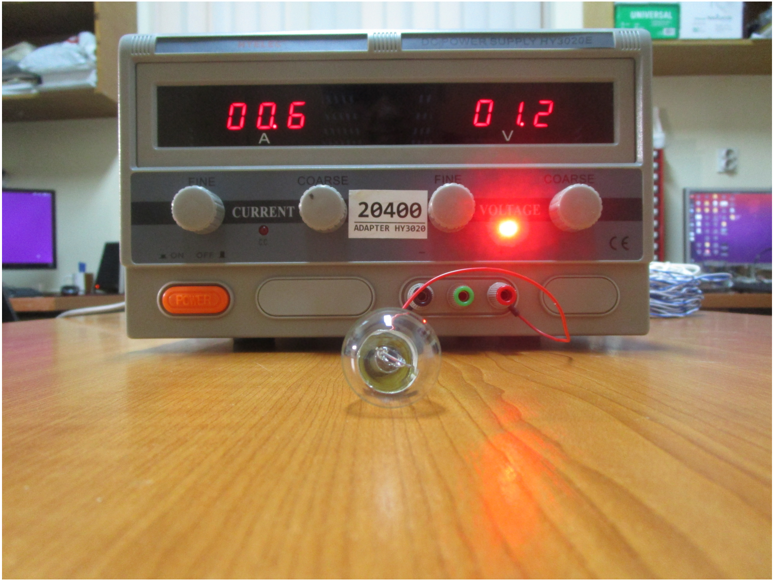
overview of scaling

- we do not execute algorithms (any more)
- we create algorithms
- we should distinguish relevant and negligible
- we are here to create mathematical models
- and this is a quantitative issue ...
- ... mapped to our mind somehow
- do you have a feeling about the problem?
- quantitative information is frequently lost in symbols
- an example: a person has A_m money on his/her account
  a) $A_m = 3.33$ US\$ $\sim 3.33\,\mathrm{ns}$
  b) A_m is one billion US\$ $\sim 1\,\mathrm{s}$
- a quantitative issue, but a big difference in the quality of life
- **do you feel the problem?**

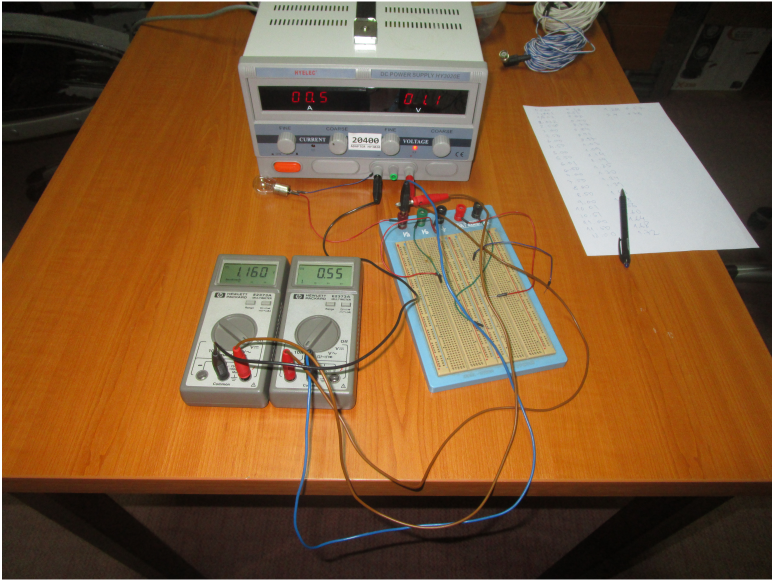
about incandescent light bulbs . . .

- incandescent light bulb . . .
- a really simple system . . .
- a wire heated up to shine . . .
- in a bulb to isolate the wire from oxygen to prevent burning
- low pressure gas to prevent explosions . . .
- dynamic system, thermal inertia, thermal conduction . . .
- black-body radiation . . .
- initiated development of quantum mechanics . . .
- did I say a really simple system?
- used as an indicator (a poor one) in our story . . .
- examples that follow, light bulbs: 12 V DC, 21 W, and 230 V AC, 60 W

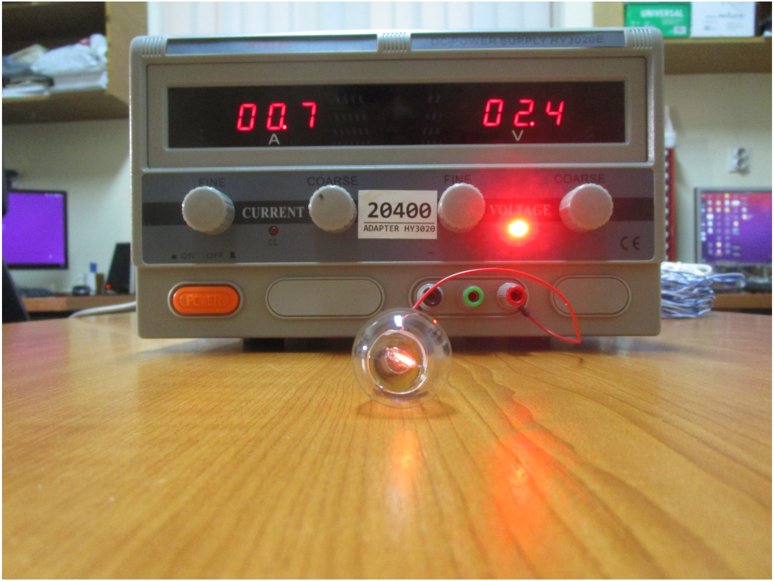
about incandescent light bulbs . . . at 10%



about light bulbs . . . at 10%

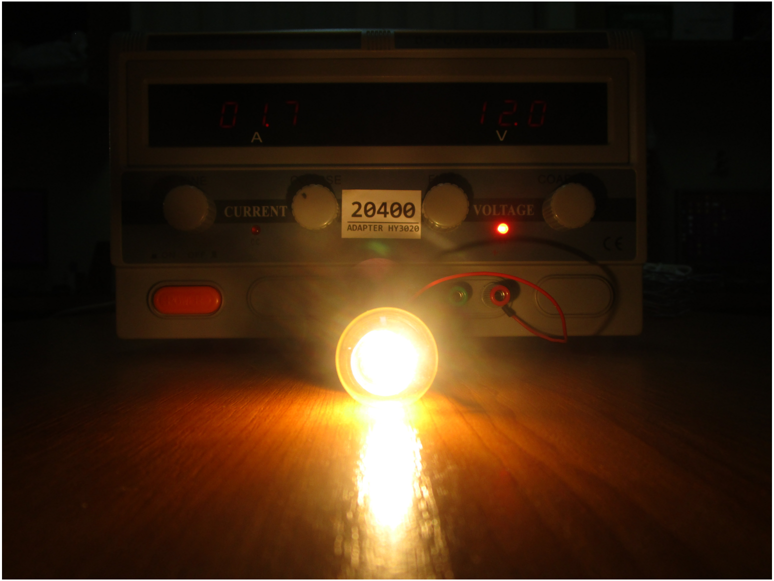


about light bulbs . . . at 20%

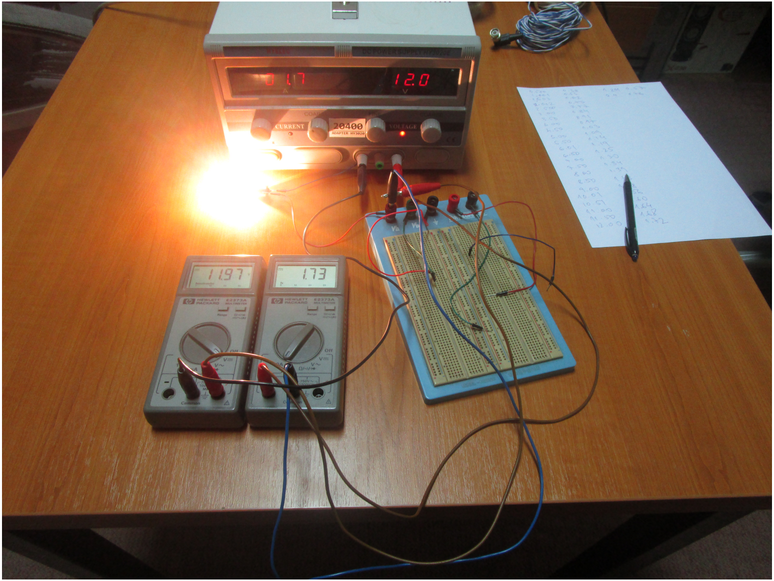


about light bulbs . . . at 20%

about light bulbs . . . at 100% of the rated voltage



about light bulbs . . . at 100% of the rated voltage

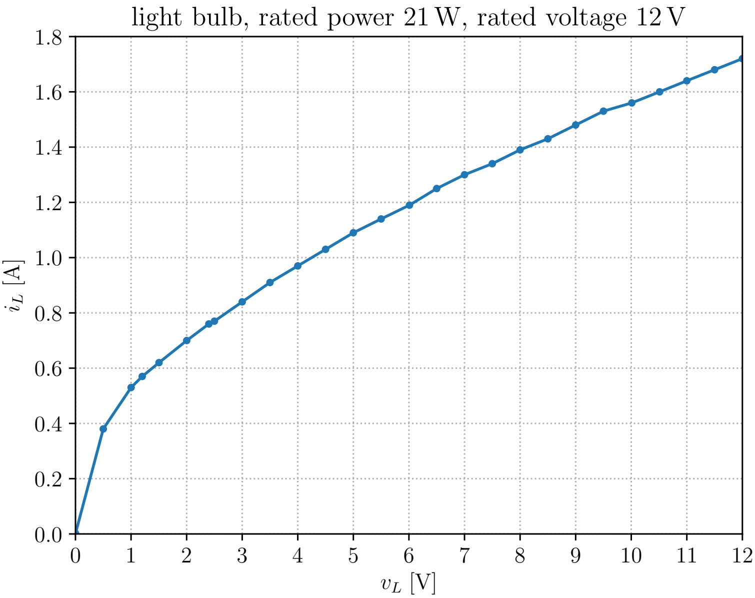


a word about AC light bulbs . . . at 10%

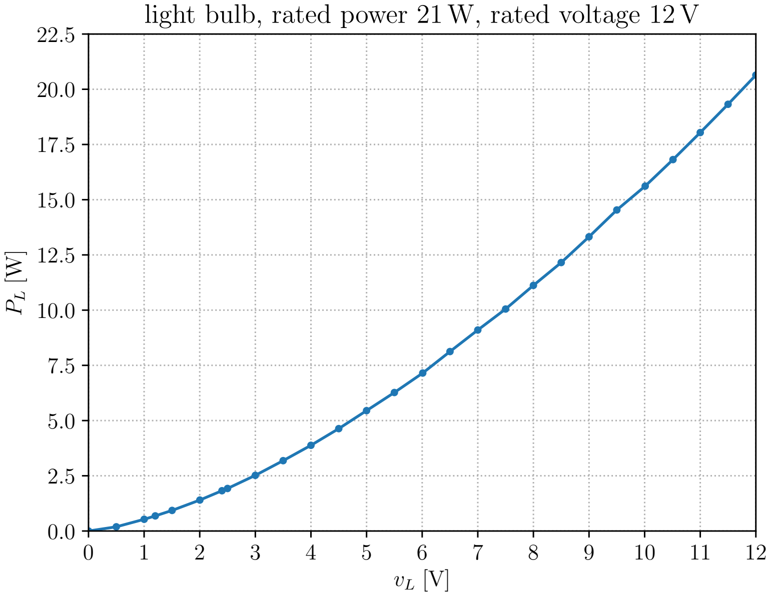
about AC light bulbs . . . at 20%

about AC light bulbs . . . at 100% of the rated voltage

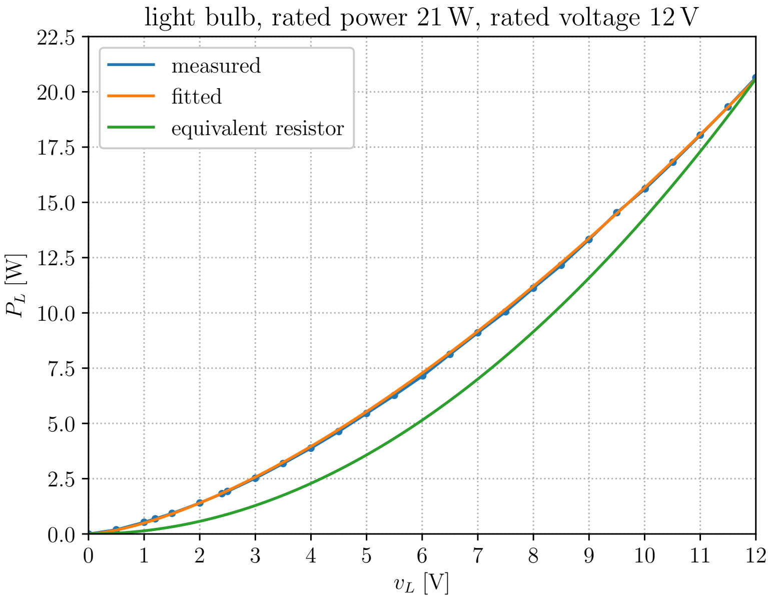
about light bulbs ... DC nonlinearity



about light bulbs . . . power



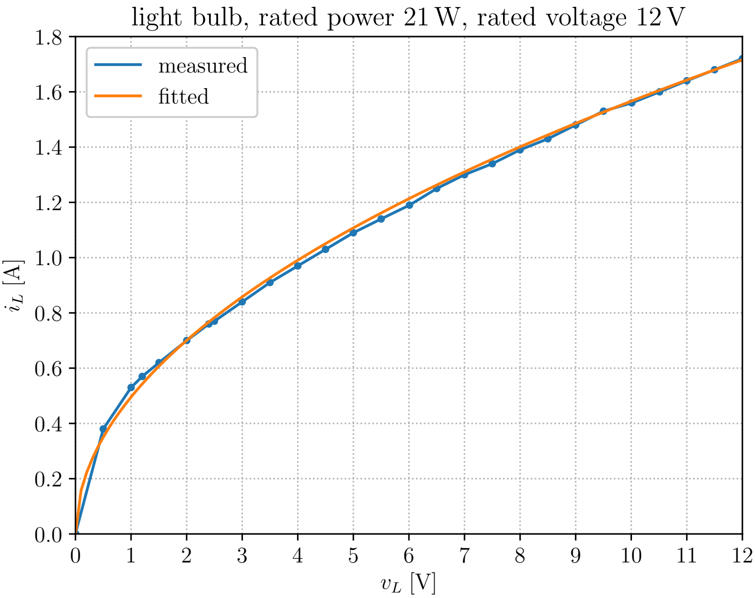
about light bulbs . . . power, $P_L \sim v_L^{\frac{3}{2}}$



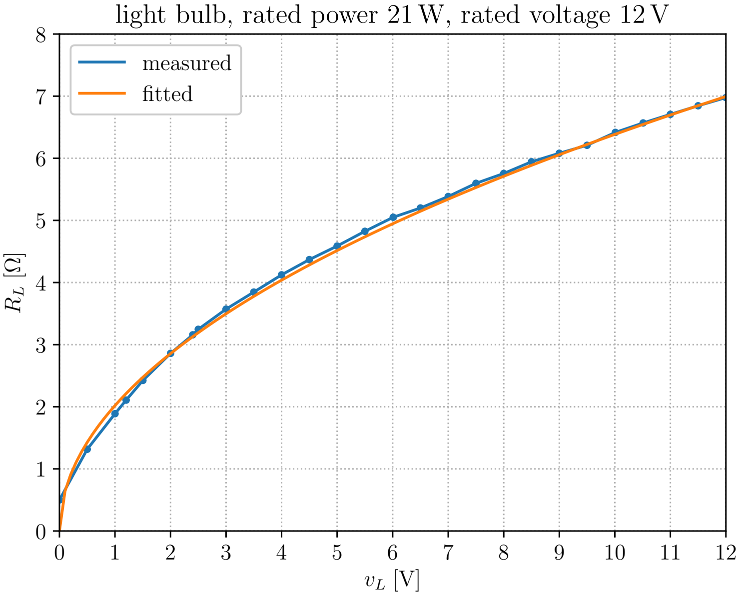
light bulbs: square root law?

- curve fitting, assumed law $i_L = I_{L0}\sqrt{\frac{v_L}{1\,\mathrm{V}}}$
- linear least squares to fit I_{L0} ...
- for the actual light bulb $I_{L0} = 0.4953\,\mathrm{A}$
- fits amazingly well!
- why?
- empirical data: $\frac{R_{L,\,\mathrm{rated}}}{R_{L,\,\mathrm{at}\ 0}} = \frac{6.857\,\Omega}{0.5\,\Omega} \approx 13.71$
- I'll stop here with the light bulb electrical model ...
- in our application: the light bulb is a DC nonlinear element
- we'll neglect the dynamics ...
- we know DC dependence of v_L and i_L and that's all we need
- ... at this point ...

about light bulbs . . . curve fitting

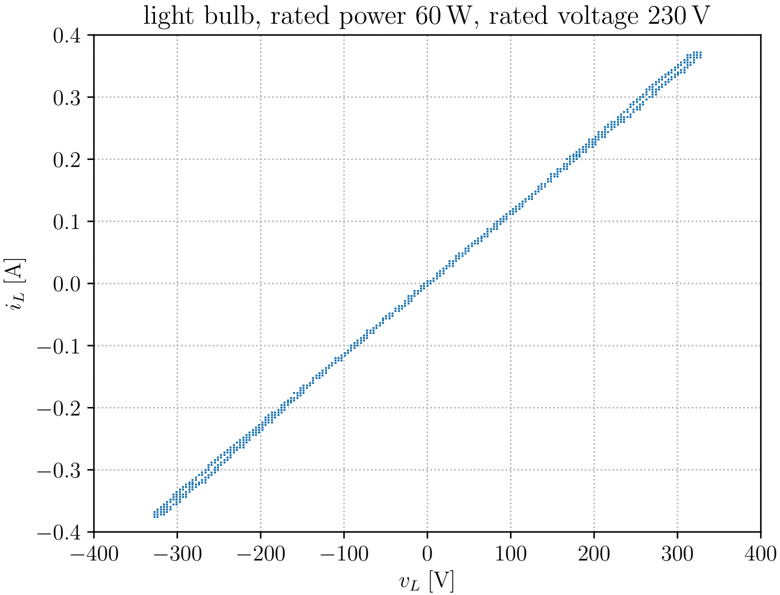


about light bulbs ... $R_L(V_L)$, $1:14$, $R_L(0) = 0.5\,\Omega$

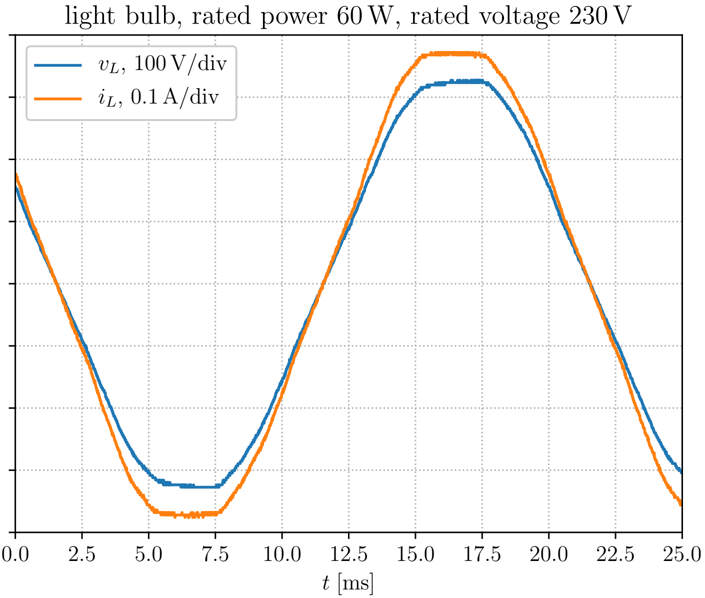


another light bulb, a bigger one, AC

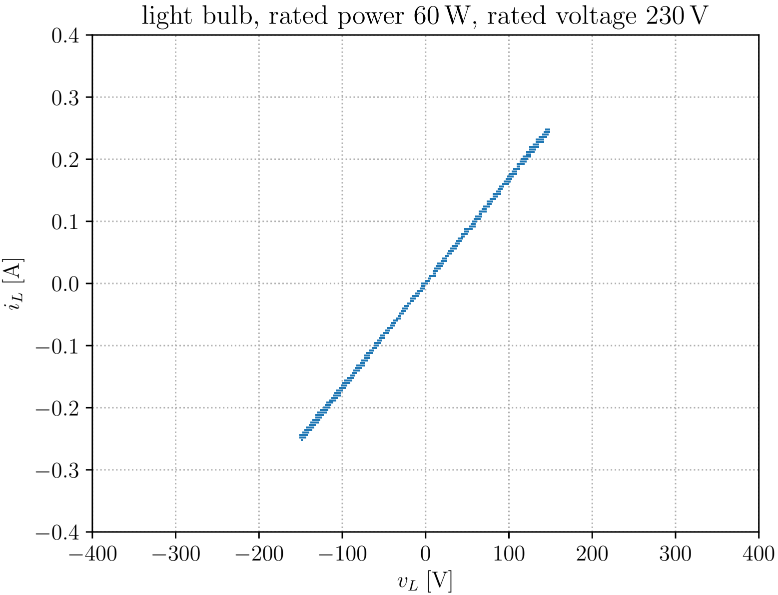
AC light bulb ... AC linear (almost)



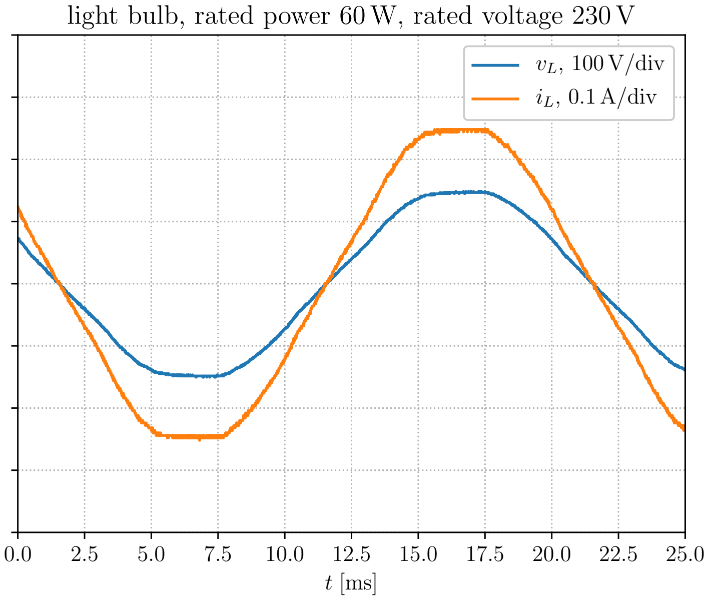
AC light bulb ... AC linear (almost)



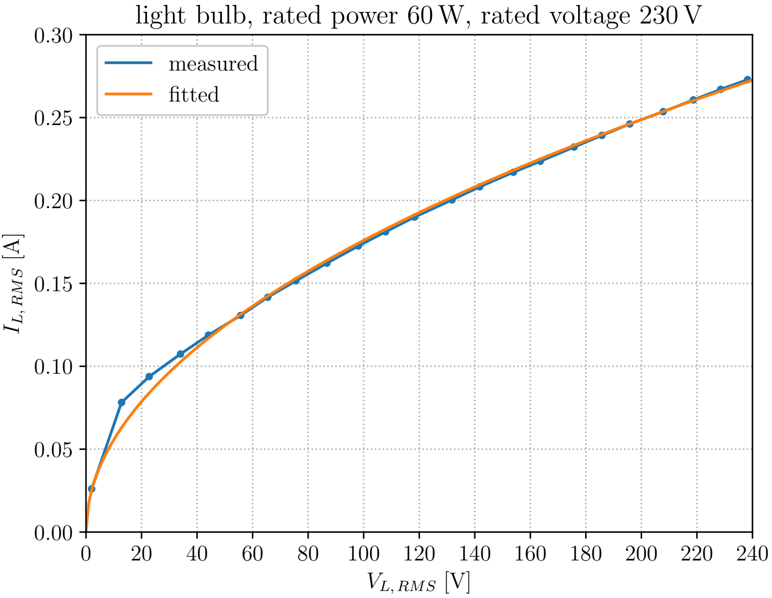
AC light bulb . . . AC linear



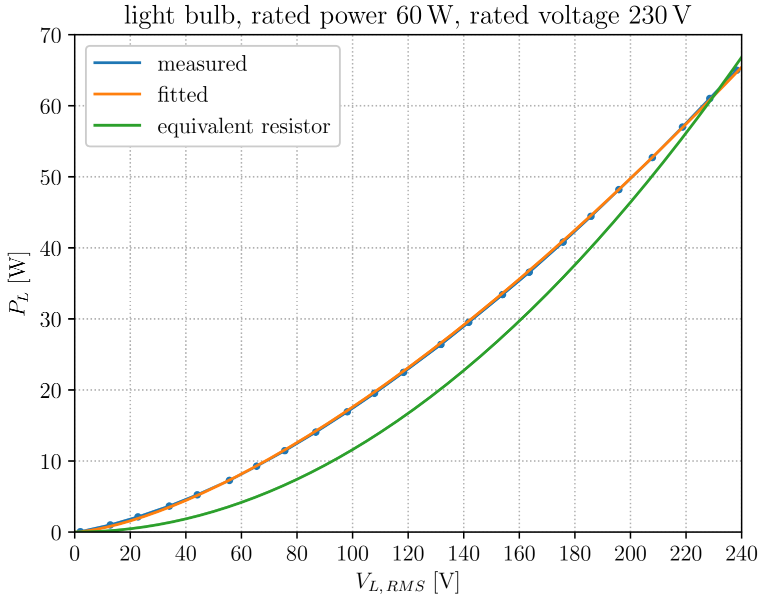
AC light bulb . . . AC linear



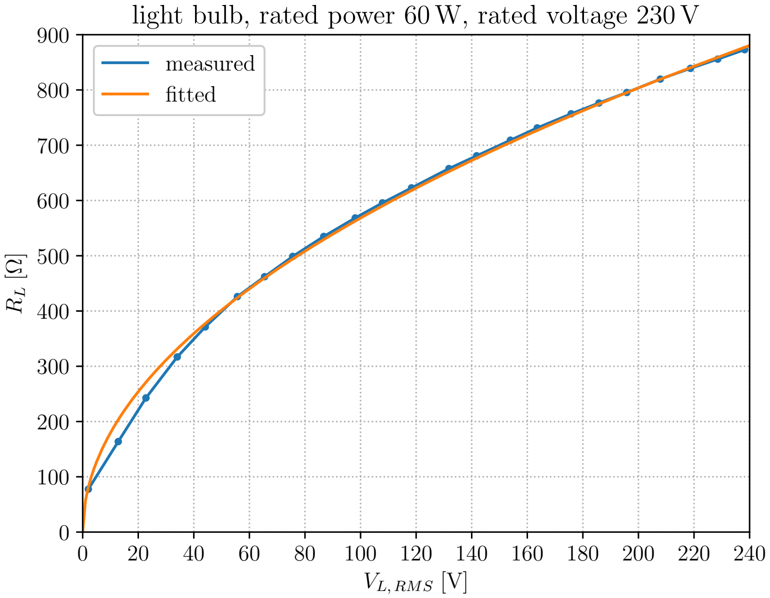
AC light bulb ... DC clearly nonlinear



AC light bulb . . . power



AC light bulb ... $R_L(V_L)$



about AC light bulbs ...

- **AC linear** element, v_L and i_L linearly dependent
- **DC nonlinear** element, $V_{L,\,RMS}$ and $I_{L,\,RMS}$ not linearly related
- again, square root law perfectly fits the data: $I_{L,\,RMS} \approx I_{L0}\sqrt{\dfrac{V_{L,\,RMS}}{1\,\mathrm{V}}}$, $I_{L0} \approx 0.0176\,\mathrm{A}$
- why?
- empirical data: $\dfrac{R_{L,\,\mathrm{rated}}}{R_{L,\,\mathrm{at}\ 0}} = \dfrac{881.67\,\Omega}{65.9\,\Omega} \approx 13.38$, again
- behavior amazingly similar to 12 V DC light bulb!

overview of light bulbs

- **somewhere between 10% and 20% of the rated voltage some light appears**; in steady state, after the filament is heated up; this also takes time
- light bulb is a **nonlinear** element, **DC nonlinear**
- on the other hand, light bulb is a **linear** element, **AC linear** ...
- significant variation of DC resistance over the applied voltage (RMS value), typically $R_L(0) : R_L(V_{\text{rated}}) = 1 : 14$
- light bulb is a **dynamic** element, filters out AC variations at 50 Hz or 60 Hz, almost perfectly
- dynamics dominantly thermal (important for AC linearity, no power quality problems, current and voltage linearly dependent over $\sim 20\,\text{ms}$, harmonic pollution not generated)
- a really interesting system! simple, but complex!
- **at first, we'll simplify: light bulb $\rightarrow$ linear resistor**

more words about light bulbs ...

- light bulb is a poor indicator, especially when it comes to 3.3 ns as a time to indicate
- significant dependence of luminosity on the voltage applied
- significant dynamic response, slow system!
- when the light bulb is on? do we need a fine print?
- are all the light bulbs in the universe on, due to the cosmic microwave background radiation?
- except for those in perfect Faraday cages without internal electromagnetic sources
- and this started all the of the problems ...
- and caused all of the benefits: interest in the topic!
- when the light bulb is on turned out to be a complex question!
- **thanks Andres!**

Maxwell's equations . . . from a T-shirt

And God said . . .

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

. . . and then there was light.

Maxwell's equations . . . from another T-shirt

. . . or if you like it better in the integral form . . .

$$\oiint_{\partial V} \vec{D} \cdot d\vec{S} = \iiint_V \rho \, dV$$

$$\oiint_{\partial V} \vec{B} \cdot d\vec{S} = 0$$

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

. . . which looks much easier . . .

whatever: formulated by Oliver Heaviside

Maxwell's equations . . .

- why did I mention God in Maxwell's equations?
- because you frequently need God to solve them!
- in **some** cases even humans can solve them
- in some cases we apply numerical methods and computers
- in **some** cases we simplify them neglecting some effects . . .
- . . . that should be negligible . . .
- and this is a **quantitative** issue . . .
- essentially: a great achievement of 19^{th} century physics
- unification theory
- birth of theoretical physics, $\frac{\partial \vec{D}}{\partial t}$ out of equations . . .
- $c = \frac{1}{\sqrt{\varepsilon_0\, \mu_0}}$: light is an electromagnetic phenomenon?
- basis for special relativity . . .
- present formulation by Oliver Heaviside

Gustav Kirchhoff and circuit theory

- Kirchhoff's circuit laws are formulated by Gustav Kirchhoff in 1845, while he was a student, before Maxwell's equations
- in essence, simplification and reduction of Maxwell's equations in cases where applicable
- many applicable cases: all lumped parameter electric circuits!
- in this analysis there is no space, three variables are gone, the circuit is a "material point"
- well, not really a point, there are nodes and branches that form loops, as well as circuit components inside that "point"
- partial differential equations reduced to ordinary differential equations over time variable
- definitely applicable to DC circuits, in that case even time variable is gone, just algebraic equations remain
- ubiquitous in electrical engineering, used wherever possible, really frequently, really useful!

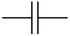
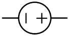
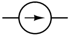
Kirchhoff's current law

- continuity equation, charge conservation $\vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t}$
- integral form, $\displaystyle\oiint_{\partial V} \vec{J} \cdot d\vec{S} = -\iiint_V \frac{\partial \rho}{\partial t}\, dV$
- define currents as $\displaystyle i_k \triangleq \iint_{S_k} \vec{J} \cdot d\vec{S}$
- assume $\displaystyle\iiint_V \frac{\partial \rho}{\partial t}\, dV \approx 0$ for "nodes"
- for any closed surface $\displaystyle\partial V = \bigcup_k S_k$ we get
- KCL: $\displaystyle\boxed{\sum_k i_k = 0}$
- simple equation: linear, algebraic, homogeneous

Kirchhoff's voltage law

- Faraday's law of induction, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- integral form, $\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$
- define voltages $v_k \triangleq \int_{l_k} \vec{E} \cdot d\vec{l}$
- assume $\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \approx 0$ over "loops"
- for any closed loop $\partial S = \bigcup_k l_k$ we get
- KVL: $\boxed{\sum_k v_k = 0}$
- simple equation: linear, algebraic, homogeneous

constitutive relations

- Kirchhoff's laws do not mutually relate voltages and currents, neither care about circuit components, they model effects caused by element interconnections
- elements are described by constitutive relations (CRs): they may cause some trouble, but not as Maxwell's equations
- resistor, : $v = R\,i$; linear, algebraic, homogeneous
- capacitor, , $i = C\,\frac{dv}{dt}$, linear, differential, homogeneous
- inductor, , $v = L\,\frac{di}{dt}$, linear, differential, homogeneous
- voltage source, , $v = e_g\,(t)$; linear, nonhomogeneous
- current source, , $i = i_g\,(t)$; linear, nonhomogeneous
- R, L, C: linear homogeneous CRs, some have $\frac{d}{dt}$, "dynamic"
- voltage and current source: linear, but not homogeneous
- there are also nonlinear elements, like diodes, light bulbs ...

circuit theory

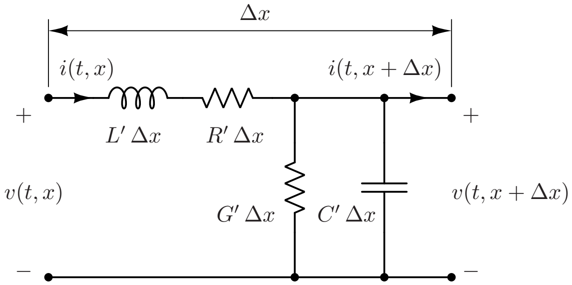
- circuit theory is based on Kirchhoff's laws
- to solve a circuit one half of the equations originate from Kirchhoff's laws, the other half from constitutive relations
- about equations and solving circuits
- essentially, applied mathematics
- behavior of classes of circuits (linear, nonlinear, resistive, dynamic, 1st order, 2nd order, ...)
- circuit synthesis, like filters ...
- „lumped parameter“ circuits: there is no space for space in the model (electrical material point)
- everything is close together enough
- detailed modeling avoided (imagine home wiring model)
- solving Maxwell's equations avoided

EM field theory versus electric circuit theory

- **electromagnetic field theory:**
  - deals with fields: electric $\vec{E}$ and $\vec{D}$ and magnetic $\vec{H}$ and $\vec{B}$
  - bundles electric and magnetic field with partial differential equations (in vector form by Oliver Heaviside)
  - there is space in the model
  - hard to describe the system analytically (like home wiring)
  - hard to solve the equations
- **electric circuit theory:**
  - deals with voltages and currents, v and i
  - spatial dimensions are neglected, encapsulated in the circuit graph, in the interconnections
  - Kirchhoff's laws: linear homogeneous algebraic equations, simple
  - much easier to solve the model
  - frequently used, I make my living using it
  - field carries energy ...
  - encapsulated in voltages and currents here

transmission line theory: circuits with distributed parameters (something in between)

let the space be one dimensional … transmission lines …



again, by Oliver Heaviside!

telegrapher's equations

telegrapher's equations:

$$\frac{\partial v(t,\, x)}{\partial x} = -L' \frac{\partial i(t,\, x)}{\partial t} - R'\, i(t,\, x)$$

$$\frac{\partial i(t,\, x)}{\partial x} = -C' \frac{\partial v(t,\, x)}{\partial t} - G'\, v(t,\, x)$$

- partial differential equations over $v(t, x)$ and $i(t, x)$
- by Oliver Heaviside, 1876
- linear, homogeneous
- symmetry over $v(t, x)$ and $i(t, x)$
- voltages and currents **are mutually related**
- when you solve for $v(t, x)$ you also get $i(t, x)$ and vice versa

telegrapher's equations

solved over $v(t, x)$ and $i(t, x)$:

$$\frac{\partial^2 v(t,x)}{\partial x^2} - L' C' \frac{\partial^2 v(t,x)}{\partial t^2} = (R' C' + G' L') \frac{\partial v(t,x)}{\partial t} + G' R' v(t,x)$$
$$\frac{\partial^2 i(t,x)}{\partial x^2} - L' C' \frac{\partial^2 i(t,x)}{\partial t^2} = (R' C' + G' L') \frac{\partial i(t,x)}{\partial t} + G' R' i(t,x)$$

- hyperbolic partial differential equations
- complete symmetry over $v(t, x)$ and $i(t, x)$
- but $v(t, x)$ and $i(t, x)$ **are still related!**
- if you know $i(t, x)$, you would know $v(t, x)$, and vice versa
- techniques to solve? transformations?
- well, at this point we would keep in time domain, without any transformations … later on phasors would enter the stage …

overview of the methods

- analysis of macroscopic electromagnetic systems ...
- three methods covered:
  1. Maxwell's equations, field equations
  2. electric circuit theory
  3. transmission line theory
- make it as simple as possible, but not simpler than that ...
- after the geometry of the problem had been analyzed ...
- transmission line theory + circuit theory are chosen to address the problem ...
- so, let's solve the problem!

lossless case . . . will be our case in time domain

neglect losses, substitute $R' = 0$, $G' = 0$

$$\frac{\partial v(t,\,x)}{\partial x} = -L'\,\frac{\partial i(t,\,x)}{\partial t}$$

$$\frac{\partial i(t,\,x)}{\partial x} = -C'\,\frac{\partial v(t,\,x)}{\partial t}$$

and we get famous wave equation

$$\frac{\partial^2 v(t,\,x)}{\partial x^2} - L'\,C'\,\frac{\partial^2 v(t,\,x)}{\partial t^2} = 0$$

$$\frac{\partial^2 i(t,\,x)}{\partial x^2} - L'\,C'\,\frac{\partial^2 i(t,\,x)}{\partial t^2} = 0$$

some new variables: Z_c and c

$$c = \frac{1}{\sqrt{L' \, C'}}$$

$$Z_c \triangleq \sqrt{\frac{L'}{C'}}$$

$$\frac{\partial^2 v(t,\, x)}{\partial x^2} - \frac{1}{c^2} \, \frac{\partial^2 v(t,\, x)}{\partial t^2} = 0$$

$$\frac{\partial^2 i(t,\, x)}{\partial x^2} - \frac{1}{c^2} \, \frac{\partial^2 i(t,\, x)}{\partial t^2} = 0$$

$$\frac{\partial v(t,\, x)}{\partial x} = -\frac{Z_c}{c} \, \frac{\partial i(t,\, x)}{\partial t}$$

$$\frac{\partial i(t,\, x)}{\partial x} = -\frac{1}{c \, Z_c} \, \frac{\partial v(t,\, x)}{\partial t}$$

wave equation and its solutions ...

wave equation, one dimension in space, over a general variable y

$$\frac{\partial^2 y(x,\, t)}{\partial x^2} - \frac{1}{c^2}\,\frac{\partial^2 y(x,\, t)}{\partial t^2} = 0$$

solution, ingenious (I'd never guess it):

$$y(x,\, t) = y_{\rightarrow}(x - c\,t) + y_{\leftarrow}(x + c\,t)$$

please note: $y_{\rightarrow}(\bullet)$ and $y_{\leftarrow}(\bullet)$ are functions of one variable!

$x - c\,t$ and $x + c\,t$ bundle the space and time together

physical dimension of that variable is length here

I'll repack the solution for my convenience ... we record waveforms with time as an independent variable, so let the time be the base variable! I tend to think in the terms I see!

solutions!

$$v(t, x) = v_{\rightarrow}\left(t - \frac{x}{c}\right) + v_{\leftarrow}\left(t + \frac{x}{c}\right)$$

$$i(t, x) = \frac{1}{Z_c} v_{\rightarrow}\left(t - \frac{x}{c}\right) - \frac{1}{Z_c} v_{\leftarrow}\left(t + \frac{x}{c}\right)$$

please note: $v_{\rightarrow}(t_{\rightarrow})$ and $v_{\leftarrow}(t_{\leftarrow})$ are **single** variable functions, its physical dimension is time here

and these variables are in turn functions of two variables:

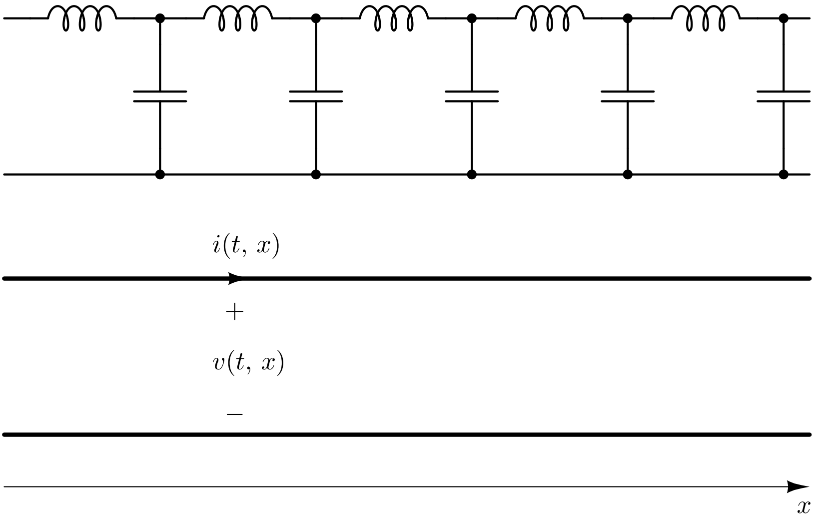
$t_{\rightarrow} = t - \frac{x}{c}$ and $t_{\leftarrow} = t + \frac{x}{c}$

and there is a trade-off between space and time:

$x = t_{\rightarrow} + c\,t$, x goes forward, and $x = t_{\leftarrow} - c\,t$, x goes backward

time and space bundled together

“initial conditions”



initial conditions of a line at $t = t_0$

the initial conditions are $v(t_0, x)$ and $i(t_0, x)$ for $0 \leq x \leq l$

$$v(t_0,\, x) = v_{\rightarrow}\left(t_0 - \frac{x}{c}\right) + v_{\leftarrow}\left(t_0 + \frac{x}{c}\right)$$

$$i(t_0,\, x) = \frac{1}{Z_c}\, v_{\rightarrow}\left(t_0 - \frac{x}{c}\right) - \frac{1}{Z_c}\, v_{\leftarrow}\left(t_0 + \frac{x}{c}\right)$$

after some elementary linear algebra:

$$v_{\rightarrow}\left(t_0 - \frac{x}{c}\right) = \frac{1}{2}\left(v(t_0,\, x) + Z_c\, i(t_0,\, x)\right)$$

$$v_{\leftarrow}\left(t_0 + \frac{x}{c}\right) = \frac{1}{2}\left(v(t_0,\, x) - Z_c\, i(t_0,\, x)\right)$$

at $t = t_0$ for $0 \leq x \leq l$

integration constants (DC components)?

let's get back to the relations between the voltage and the current across the line ...

$$\frac{\partial v(t,\,x)}{\partial x} = -\frac{Z_c}{c}\,\frac{\partial i(t,\,x)}{\partial t}$$

$$\frac{\partial i(t,\,x)}{\partial x} = -\frac{1}{c\,Z_c}\,\frac{\partial v(t,\,x)}{\partial t}$$

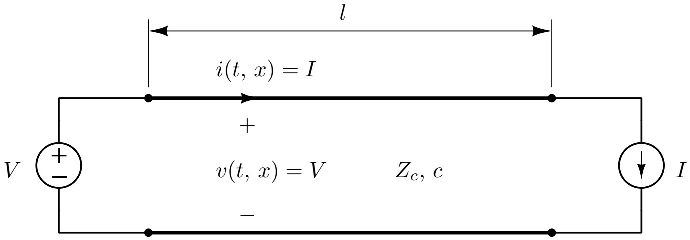
that result in:

$$v\,(t,\,x) = v_{\rightarrow}\left(t - \frac{x}{c}\right) + v_{\leftarrow}\left(t + \frac{x}{c}\right)$$

$$i\,(t,\,x) = \frac{1}{Z_c}\,v_{\rightarrow}\left(t - \frac{x}{c}\right) - \frac{1}{Z_c}\,v_{\leftarrow}\left(t + \frac{x}{c}\right)$$

integration constants?

integration constants are included!



$$V = v_{\rightarrow} + v_{\leftarrow}$$

$$I = \frac{v_{\rightarrow}}{Z_c} - \frac{v_{\leftarrow}}{Z_c}$$

$$\Rightarrow$$

$$v_{\rightarrow} = \frac{1}{2}\,(V + Z_c\, I)$$

$$v_{\leftarrow} = \frac{1}{2}\,(V - Z_c\, I)$$

a hypothetical example

parameter values made up to get numbers easy to compute with, very far from realistic values:

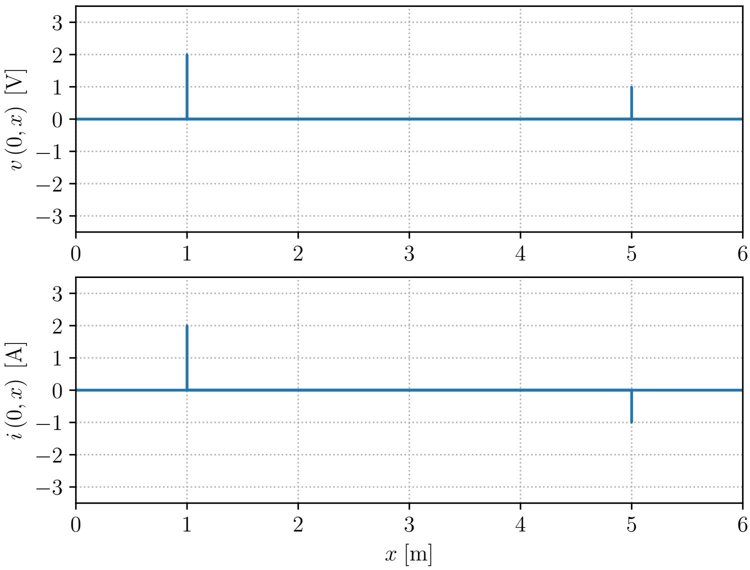
$Z_c = 1\,\Omega$

$c = 1\,\frac{\mathrm{m}}{\mathrm{s}}$

$l = 6\,\mathrm{m}$

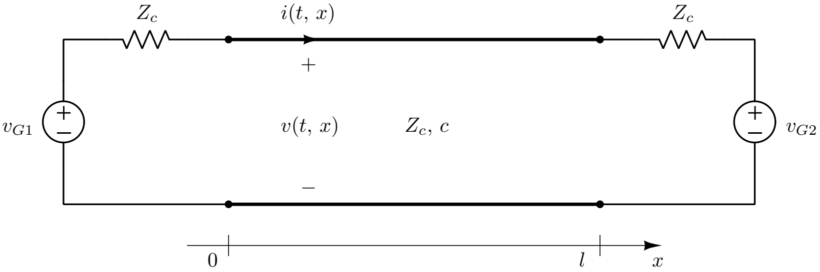
and that's all we need for the transmission line electrical model

solution: initial conditions at $t = 0$



is this feasible?

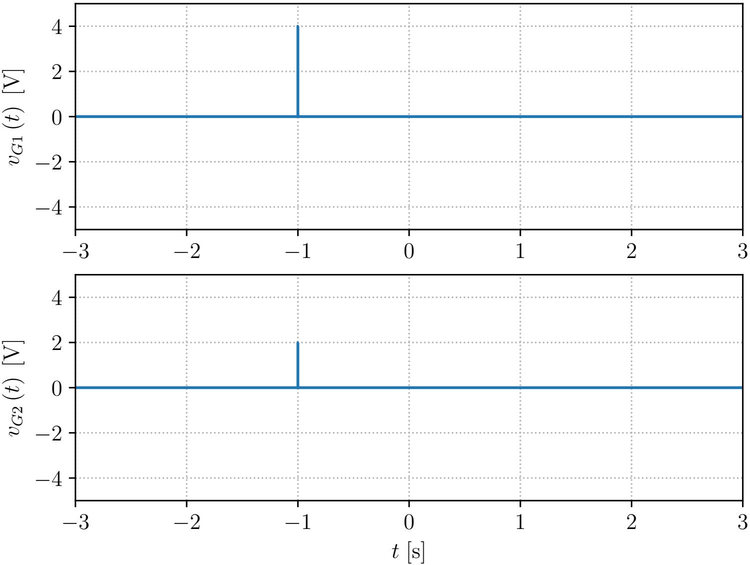
yes, it is!



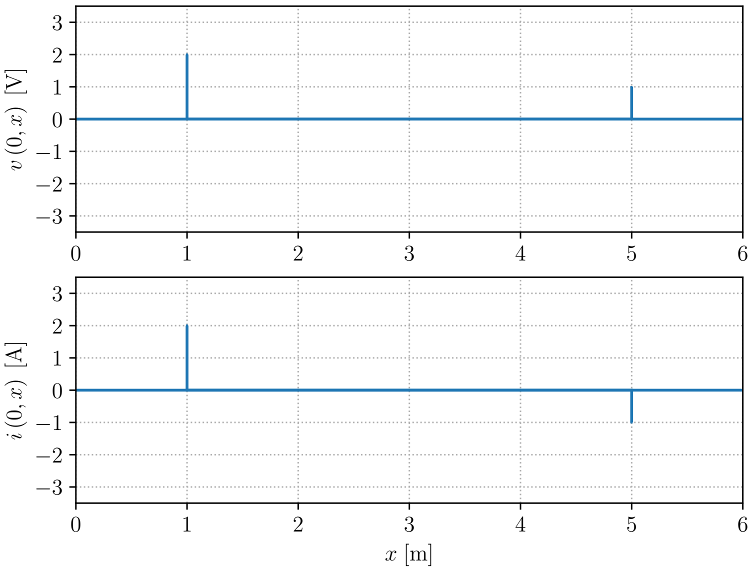
in theory this is easy; in practice, there would be some problems to synchronize the generators, but solvable ...

a mathematical note: if you care about differentiability, smooth up the functions a little bit, generalized functions ...

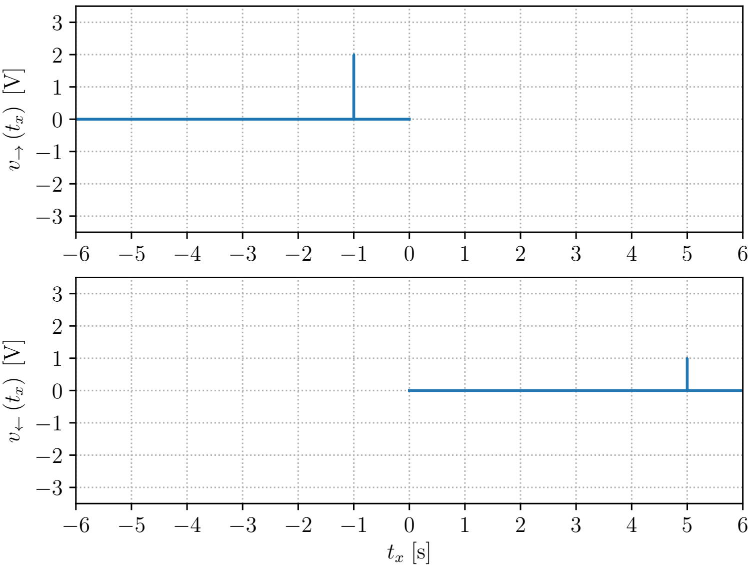
voltages of the generators



back to the solution: initial conditions at $t = 0$



initial conditions, wave functions at $t = 0$



initial conditions, wave functions at $t = 0$

- please remember, initial conditions encode what already is on the line
- t_x is a dummy variable
- for $v_{\rightarrow}$ let us follow $t_x = -1\,\mathrm{s}$
- for $v_{\rightarrow}$ the argument is $t_x = t - \frac{x}{c}$
- the $v_{\rightarrow}$ pulse would move along the line according to $x = 1\,\frac{\mathrm{m}}{\mathrm{s}}\,(1\,\mathrm{s} + t)$
- for $v_{\leftarrow}$ let us follow $t_x = 5\,\mathrm{s}$
- for $v_{\leftarrow}$ the argument is $t_x = t + \frac{x}{c}$
- the $v_{\leftarrow}$ pulse would move along the line according to $x = 1\,\frac{\mathrm{m}}{\mathrm{s}}\,(5\,\mathrm{s} - t)$

initial conditions, wave functions at $t = 0$

- $v_{\rightarrow}$ encodes past at $x = 0$, what has been sent to the line
- $v_{\leftarrow}$ encodes future at $x = 0$, what is already on the line and will be received
- both functions cover $\Delta t = \frac{l}{c}$ in time:
  $v_{\rightarrow}$ in the past,
  $v_{\leftarrow}$ in the future
- classical length based assumed solutions are more intuitive for specifying initial conditions, since the initial conditions are length based
- measurements are time based, you fix the position and measure what is going on in time
- in this presentation, I favored measurements ...
- since measurements judge everything ...

another way to look at initial conditions

- $v_f\left(t_0, x\right) \triangleq \frac{1}{2}\left(v\left(t_0, x\right)+Z_c\, i\left(t_0, x\right)\right)$
- $v_f(t, x)$ is the wave that travels forward
- $v_b\left(t_0, x\right) \triangleq \frac{1}{2}\left(v\left(t_0, x\right)-Z_c\, i\left(t_0, x\right)\right)$
- $v_b(t, x)$ is the wave that travels backward
- actual meanings of "forward" and "backward" are determined by the reference direction for the current
- finally, connection with $v_{\rightarrow}\left(t_x\right)$ and $v_{\leftarrow}\left(t_x\right)$:
  $v_{\rightarrow}\left(t_0-\frac{x}{c}\right)=v_f\left(t_0,\, x\right)$
  $v_{\leftarrow}\left(t_0+\frac{x}{c}\right)=v_b\left(t_0,\, x\right)$
- for $v_f\left(t_0,\, x\right)$ just rescale x to time, shift to t_0 and flip over t_0 to get $v_{\rightarrow}\left(t_x\right)$
- for $v_b\left(t_0,\, x\right)$ just rescale x to time, shift to t_0, no flipping to get $v_{\leftarrow}\left(t_x\right)$

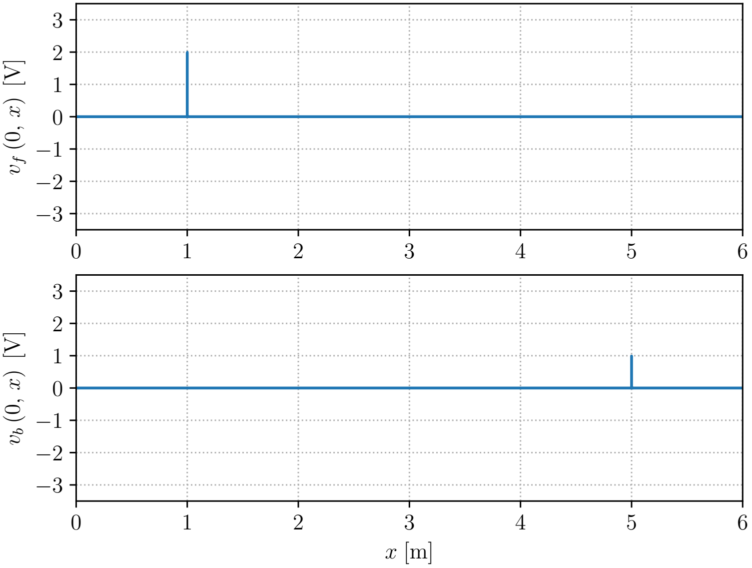
wave functions and physical reality, finally

- **you can exchange space for time and vice versa:**
- $v_{\rightarrow}\left(t-\frac{x}{c}\right)=v_f\left(t,\,x\right)$
- $v_{\leftarrow}\left(t+\frac{x}{c}\right)=v_b\left(t,\,x\right)$

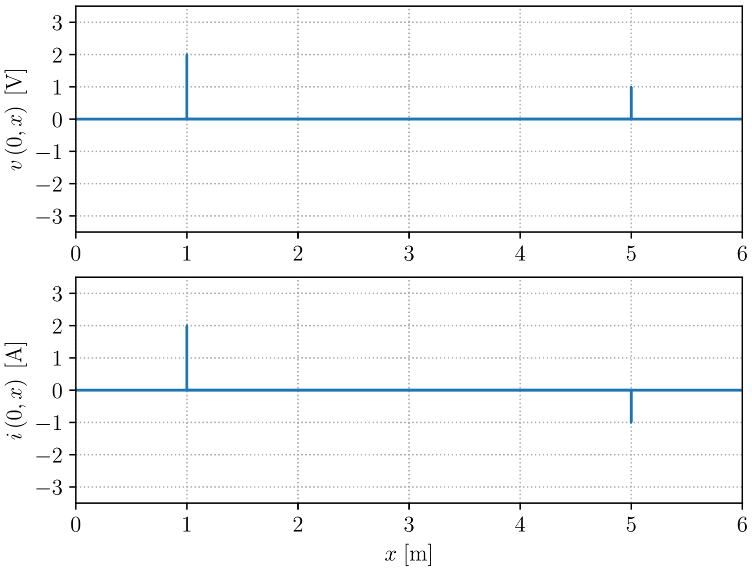
- **measurable quantities:**
- $v\left(t,\,x\right)=v_f\left(t,\,x\right)+v_b\left(t,\,x\right)$
- $i\left(t,\,x\right)=\frac{1}{Z_c}\,\left(v_f\left(t,\,x\right)-v_b\left(t,\,x\right)\right)$

- **forward and backward:**
- $v_f\left(t,\,x\right)=\frac{1}{2}\,\left(v\left(t,\,x\right)+Z_c\,i\left(t,\,x\right)\right)$
- $v_b\left(t,\,x\right)=\frac{1}{2}\,\left(v\left(t,\,x\right)-Z_c\,i\left(t,\,x\right)\right)$

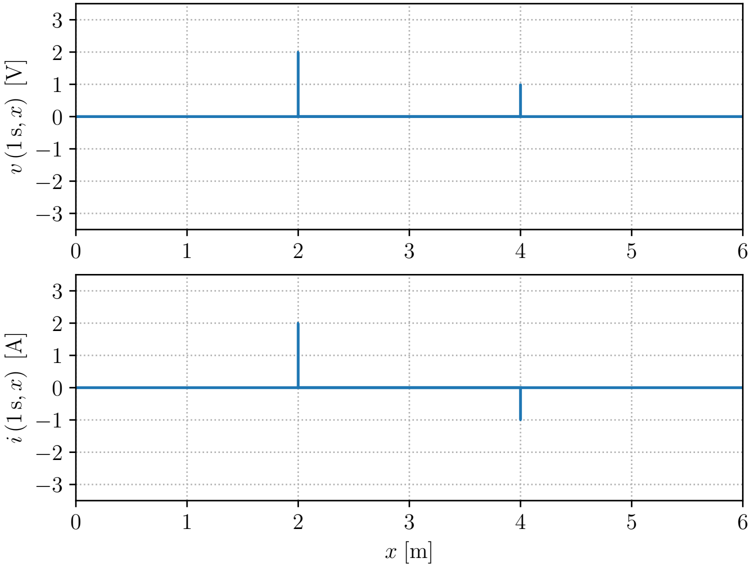
decomposition at $t = 0$



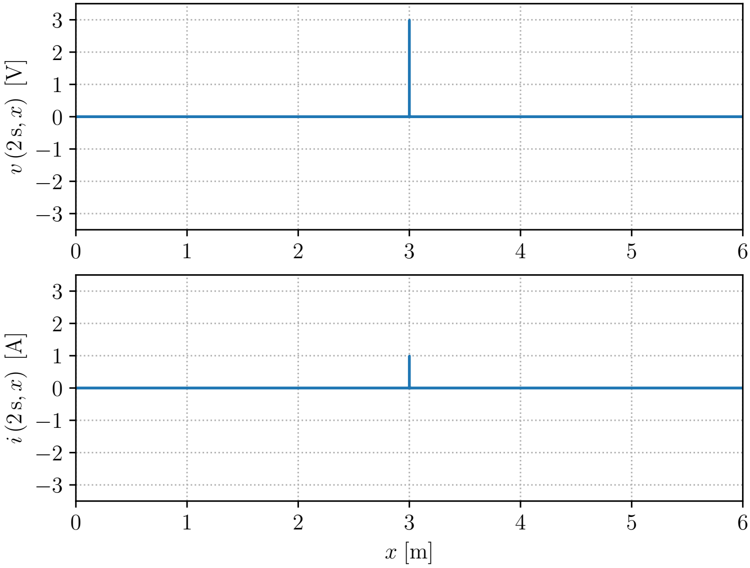
again and again, initial conditions at $t = 0$



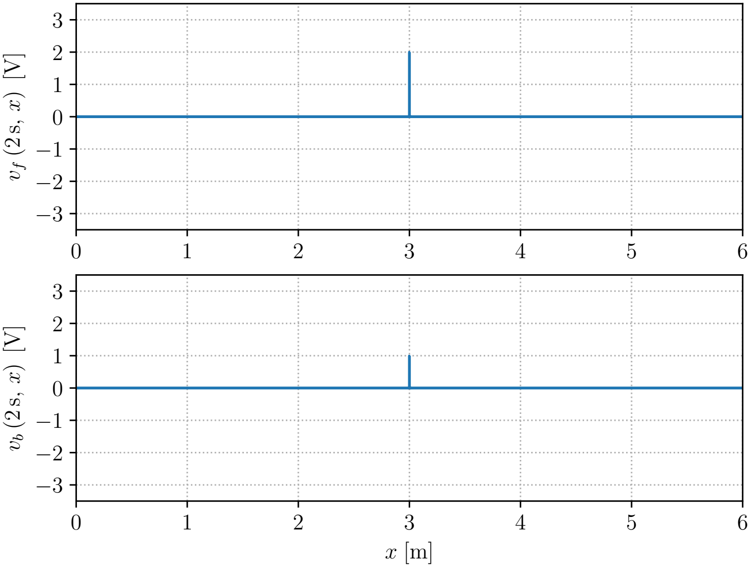
solution: evolution, $t = 1\,\mathrm{s}$



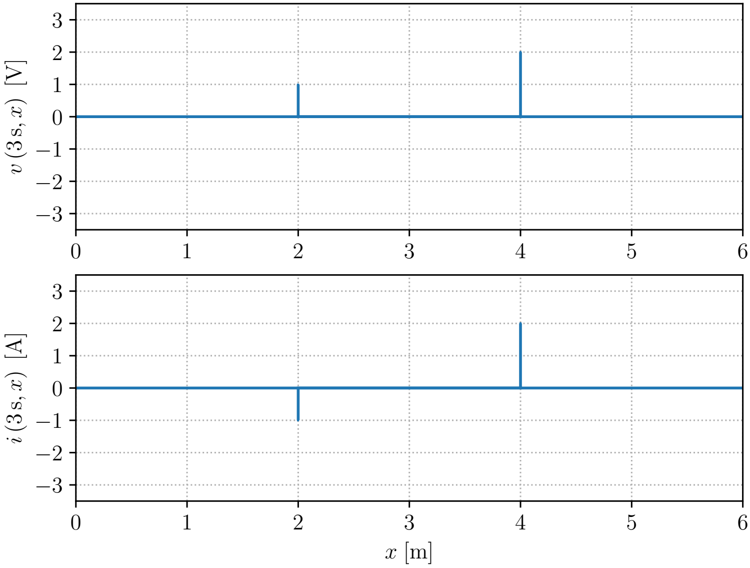
solution: evolution, $t = 2\,\mathrm{s}$



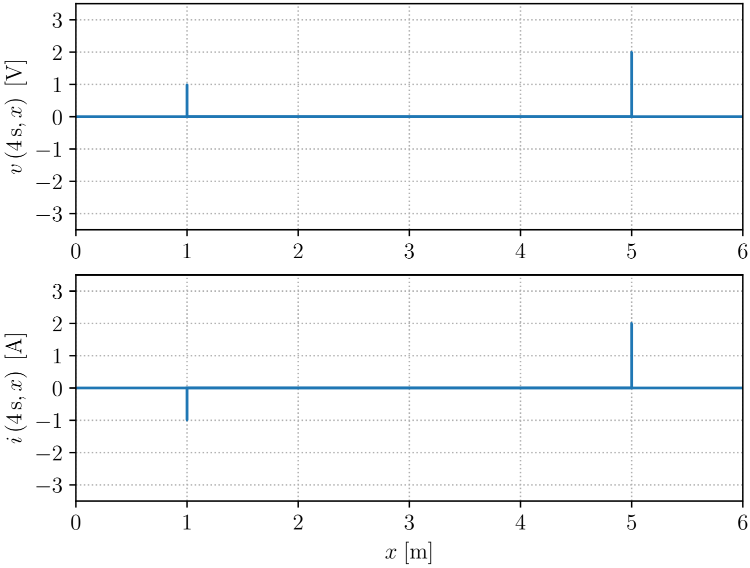
decomposition at $t = 2\,\mathrm{s}$



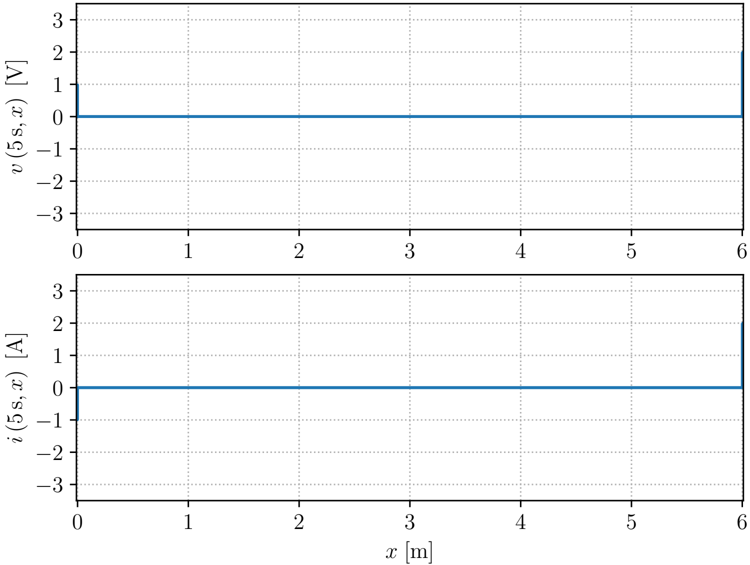
solution: evolution, $t = 3\,\mathrm{s}$



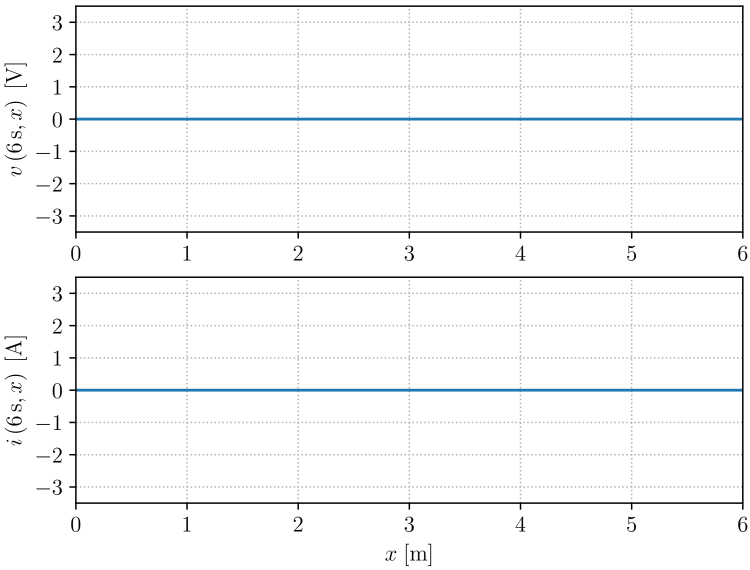
solution: evolution, $t = 4\,\mathrm{s}$



solution: pulses received, $t = 5\,\mathrm{s}$, full duplex



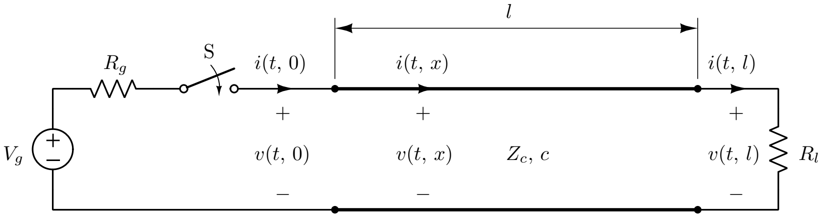
solution: pulses died out, $t = 6\,\mathrm{s}$



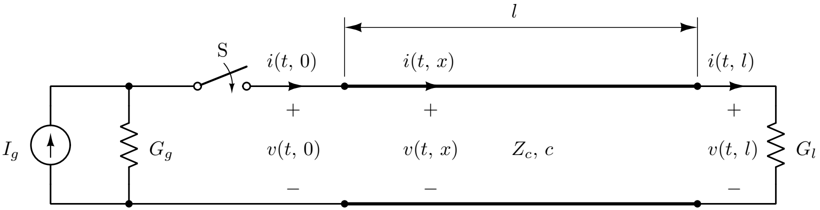
it takes some time to gain intuition . . .

- not intuitive?
- to gain feeling, just do more examples . . .
- and if you are not sure that you are right, do some experiments, the experiment would judge!
- verify along the way
- "practice makes perfect"
- that's what we are going to do now
- and a nice video from the superposition principle

step response setup



step response setup, current-based version



boundary conditions at $x = 0$

- voltage based: $v\,(t,\,0) + R_g\,i\,(t,\,0) = V_g$
- we should also cover for $i\,(t,\,0) = I_g$
- general form: $a_0\,v\,(t,\,0) + b_0\,i\,(t,\,0) = c_0$
- Thevenin's source: $a_0 = 1,\, b_0 = R_g,\, c_0 = V_g$
- ideal voltage source: $a_0 = 1,\, b_0 = 0,\, c_0 = V_g$
- Norton's source: $a_0 = G_g,\, b_0 = 1,\, c_0 = I_g$
- ideal current source: $a_0 = 0,\, b_0 = 1,\, c_0 = I_g$
- everything linear is covered, for $0 \le R_g \le +\infty$

boundary conditions at $x = l$

- passive, in contrast to the conditions at $x = 0$
- reference for the current changed, gets inside ...
- voltage based: $v(t, l) - R_l \, i(t, l) = 0$
- we should also cover for $i(t, 0) = 0$
- general form: $a_l \, v(t, l) + b_l \, i(t, l) = 0$
- passive, $c_l = 0$
- Thevenin's equivalent: $a_l = 1$, $b_l = -R_l$
- end shorted: $a_0 = 1$, $b_0 = 0$
- Norton's equivalent: $a_0 = G_g$, $b_0 = 1$
- end opened: $a_0 = 0$, $b_0 = 1$
- everything linear is covered, for $0 \leq R_l \leq +\infty$

three special cases at $x = l$: open, short, and Z_c

- equations:
  1. $v(t, l) = v_f(t, l) + v_b(t, l)$
  2. $i(t, l) = \frac{1}{Z_c}(v_f(t, l) - v_b(t, l))$
  3. $a_l\, v(t, l) + b_l\, i(t, l) = 0$
- goals:
  1. eliminate $v(t, l)$ and $i(t, l)$
  2. express $v_b(t, l)$ in terms of $v_f(t, l)$
- in cases:
  1. short: $v(t, l) = 0$
  2. open: $i(t, l) = 0$
  3. matched: $v(t, l) = Z_c\, i(t, l)$
- results:
  1. short: $v_b(t, l) = -v_f(t, l)$
  2. open: $v_b(t, l) = v_f(t, l)$
  3. matched: $v_b(t, l) = 0$

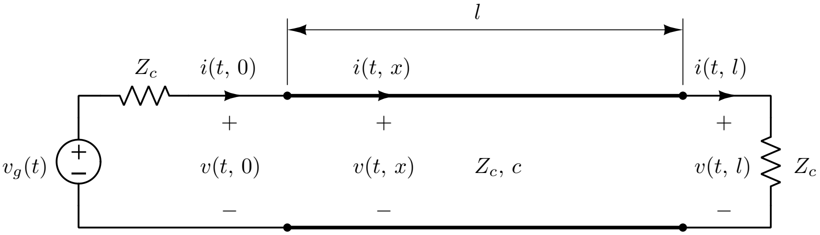
reflection at $x = l$, general R_l value

- equations:
  1. $v\,(t,\, l) = v_f\,(t,\, l) + v_b\,(t,\, l)$
  2. $Z_c\, i\,(t,\, l) = v_f\,(t,\, l) + v_b\,(t,\, l)$
  3. $v\,(t,\, l) - R_l\, i\,(t,\, l) = 0$
- goals:
  1. eliminate $v\,(t,\, l)$ and $i\,(t,\, l)$
  2. express $v_b\,(t,\, l)$ in terms of $v_f\,(t,\, l)$
- solution: $\boxed{v_b\,(t,\, l) = \dfrac{R_l - Z_c}{R_l + Z_c}\, v_f\,(t,\, l)}$
- $\Gamma_l \triangleq \dfrac{v_b\,(t,\, l)}{v_f\,(t,\, l)} = \dfrac{R_l - Z_c}{R_l + Z_c}$
- special cases:
  1. short $R_l = 0$: $\Gamma_l = -1$
  2. open: $R_l \to \infty$: $\Gamma_l = 1$
  3. matched: $R_l = Z_c$: $\Gamma_l = 0$

reflection at $x = 0$, general R_g value

- equations:
  1. $v(t, 0) = v_f(t, 0) + v_b(t, 0)$
  2. $Z_c\, i(t, 0) = v_f(t, 0) + v_b(t, 0)$
  3. $v(t, 0) + R_g\, i(t, 0) = V_g$
- goals:
  1. eliminate $v(t, 0)$ and $i(t, 0)$
  2. express $v_f(t, 0)$ in terms of $v_b(t, 0)$
- solution: $\boxed{v_f(t, 0) = \dfrac{Z_c}{R_g + Z_c} V_g + \dfrac{R_g - Z_c}{R_g + Z_c} v_b(t, 0)}$
- $\Gamma_g \triangleq \dfrac{v_f(t, 0)}{v_b(t, 0)} = \dfrac{R_g - Z_c}{R_g + Z_c}$
- for $R_g = Z_c$: $\Gamma_g = 0$, $v_f(t, 0) = \dfrac{1}{2} V_g$

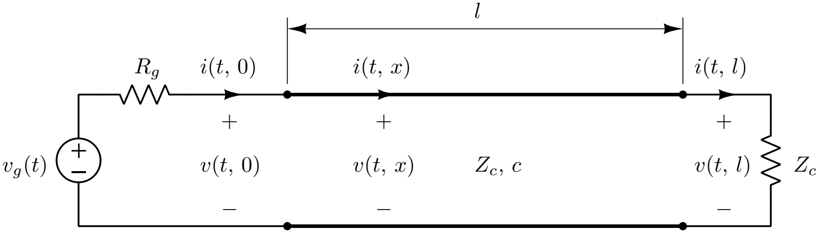
matched transmission line



$$v\,(t,\,l) = \frac{1}{2}\,v_g\left(t - \frac{l}{c}\right)$$

$$\Delta t \triangleq \frac{l}{c}$$

half matched transmission line



$$v\left(t,\, l\right) = \frac{Z_c}{Z_c + R_g} v_g \left(t - \frac{l}{c}\right)$$

works for $R_g = 0$: $v\left(t,\, l\right) = v_g \left(t - \frac{l}{c}\right)$

$$\Delta t \triangleq \frac{l}{c}$$

who cares about the mirror when there is no light?

step response: cause and effect (lattice diagram)

$t/\Delta t$	$v_f(t,\,0)$	$v_b(t,\,0)$	$v_f(t,\,l)$	$v_b(t,\,l)$
0^-	0	0	0	0
0^+	$V_{f,\,1}$	0	0	0
1^-	$V_{f,\,1}$	0	0	0
1^+	$V_{f,\,1}$	0	$V_{f,\,1}$	$V_{b,\,1}$
2^-	$V_{f,\,1}$	0	$V_{f,\,1}$	$V_{b,\,1}$
2^+	$V_{f,\,2}$	$V_{b,\,1}$	$V_{f,\,1}$	$V_{b,\,1}$
3^-	$V_{f,\,2}$	$V_{b,\,1}$	$V_{f,\,1}$	$V_{b,\,1}$
3^+	$V_{f,\,2}$	$V_{b,\,1}$	$V_{f,\,2}$	$V_{b,\,2}$
4^-	$V_{f,\,2}$	$V_{b,\,1}$	$V_{f,\,2}$	$V_{b,\,2}$
4^+	$V_{f,\,3}$	$V_{b,\,2}$	$V_{f,\,2}$	$V_{b,\,2}$
...	...	...	...	...
...	...	...	...	...

other notations possible ... and will be used later on ...

bouncing rules ...

1. $V_{f,1} = \alpha_g V_g$
2. $V_{b,k} = \Gamma_l V_{f,k}, \; k \in \{1, 2 \ldots\}$
3. $V_{f,k+1} = \alpha_g V_g + \Gamma_g V_{b,k} = \alpha_g V_g + \Gamma_g \Gamma_l V_{f,k}, \; k \in \{1, 2 \ldots\}$

overall, the iteration is

1. $V_{f,k+1} = A + b V_{f,k}$
2. $A \triangleq \alpha_g V_g$
3. $b \triangleq \Gamma_g \Gamma_l$

expand it, geometric series?

closed form solution

$$V_{f,1} = A$$

$$V_{f,2} = A + b\,V_{f,1} = A\;(1+b)$$

$$V_{f,3} = A + b\,V_{f,2} = A + b\,A\;(1+b) = A\;\left(1+b+b^2\right)$$

$$V_{f,4} = A + b\,V_{f,3} = \ldots = A\;\left(1+b+b^2+b^3\right)$$

$$\ldots$$

$$V_{f,k} = A\sum_{j=1}^{k-1} b^j$$

$$V_{f,k} = A\,\frac{1-b^k}{1-b}$$

final value? $\dfrac{A}{1-b}$?

final value?

$$\Gamma_l \triangleq \frac{R_l - Z_c}{R_l + Z_c} = \frac{\rho_l - 1}{\rho_l + 1},\ \rho_l \triangleq \frac{R_l}{Z_c},\ 0 \le \rho_l \le +\infty,\ -1 \le \Gamma_l \le +1$$

$$\Gamma_g \triangleq \frac{R_g - Z_c}{R_g + Z_c} = \frac{\rho_g - 1}{\rho_g + 1},\ \rho_g \triangleq \frac{R_g}{Z_c},\ 0 \le \rho_g \le +\infty,\ -1 \le \Gamma_g \le +1$$

$$-1 \le \Gamma_g \Gamma_l \le +1$$

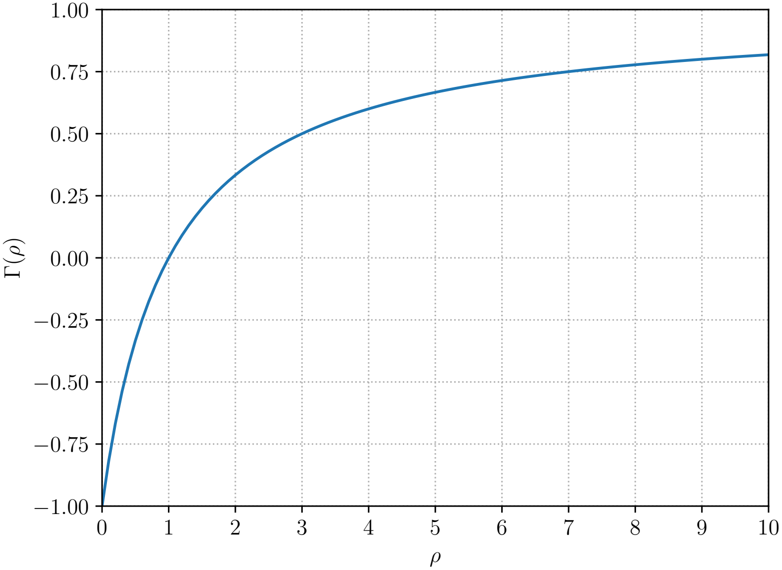
$$-1 \le b \le +1$$

avoiding extremes where both of R_l and R_g are either 0 or $+\infty$ (four combinations), $-1 < b < +1$, and then:

$$V_{f,\infty} = \frac{A}{1-b} = \frac{1}{2} \frac{R_l + Z_c}{R_l + R_g} V_g$$

thanks, Maxima! you can get it here

just to gain some intuition: $\Gamma(\rho) = (\rho + 1) \,/\, (\rho - 1)$



final value!

$$V_{f,\infty} = \frac{1}{2}\,\frac{R_l + Z_c}{R_l + R_g}\,V_g$$

$$V_{b,\infty} = \frac{1}{2}\,\frac{R_l - Z_c}{R_l + R_g}\,V_g$$

$$v\,(\infty,\, x) = V_\infty = V_{f,\infty} + V_{b,\infty} = \frac{R_l}{R_l + R_g}\,V_g$$

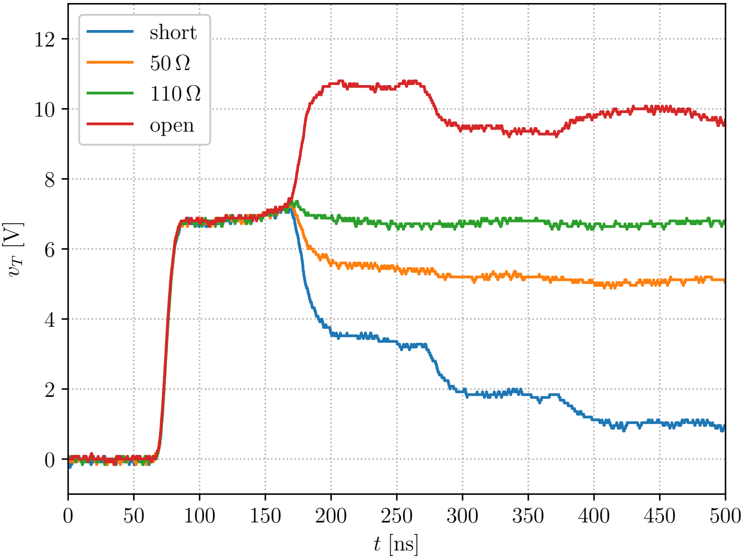
$$i\,(\infty,\, x) = \frac{1}{Z_c}\,\left(V_{f,\infty} - V_{b,\infty}\right) = \frac{1}{R_l + R_g}\,V_g$$

well, expected, really!

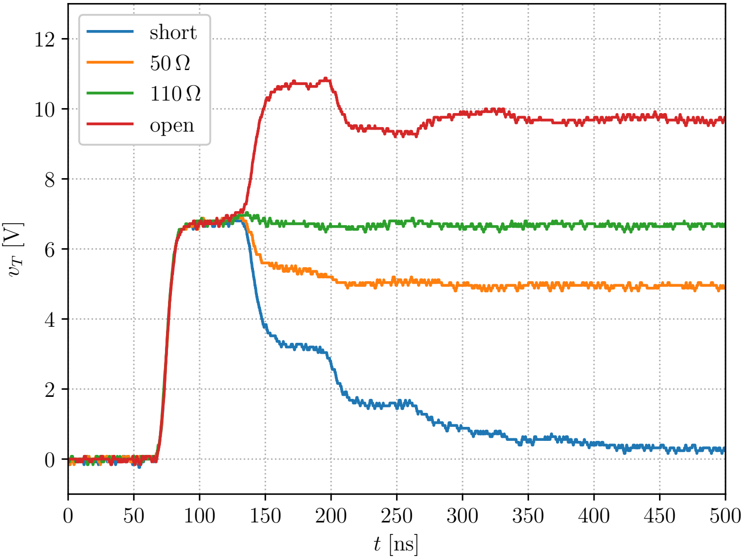
please note: two constants as a superposition of two waves!

cables: $10\,\mathrm{m}$, $6\,\mathrm{m}$, respectively, for both $Z_c \approx 110\,\Omega$

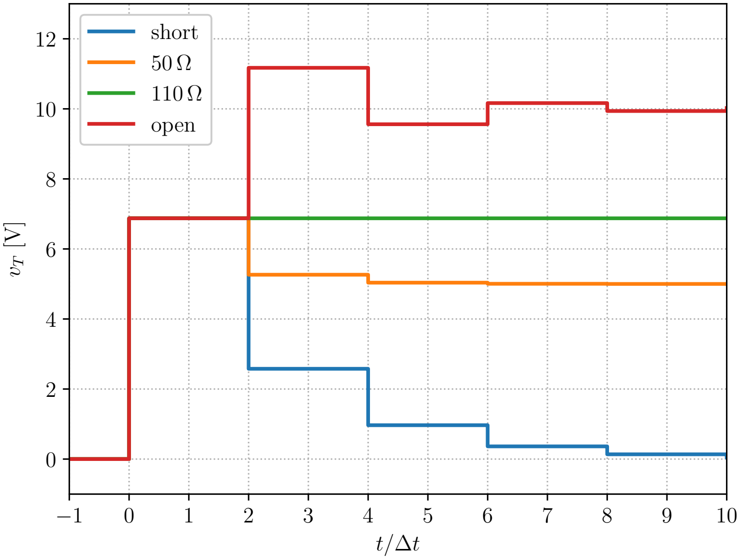
blue-white: $v\,(t,\,0)$, $l = 10\,\mathbf{m}$, $c = 192.31\,\frac{\mathbf{Mm}}{\mathbf{s}}$, $V_g = 10\,\mathbf{V}$, $R_g = 50\,\Omega$, R_l varies from 0 to ∞



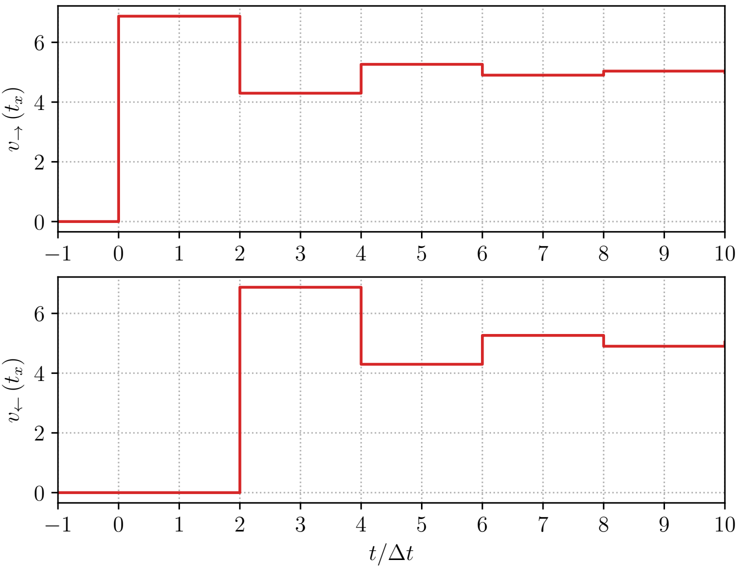
white: $v\,(t,\,0)$, $l = 6\,\mathbf{m}$, $c = 181.82\,\frac{\mathbf{Mm}}{\mathbf{s}}$, $V_g = 10\,\mathbf{V}$, $R_g = 50\,\Omega$, R_l **varies from** 0 **to** ∞



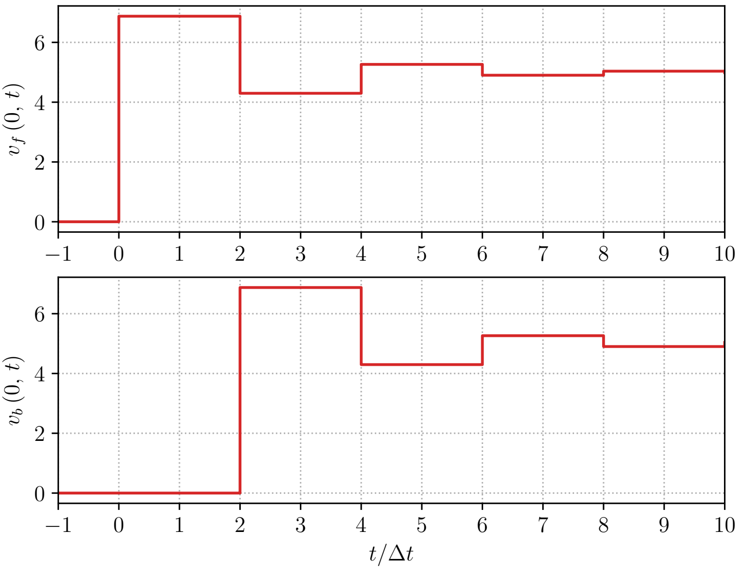
in theory: $v(t, 0)$, $\Delta t = l/c$,
$V_g = 10\,\mathbf{V}$, $R_g = 50\,\Omega$, R_l **varies from** 0 **to** ∞



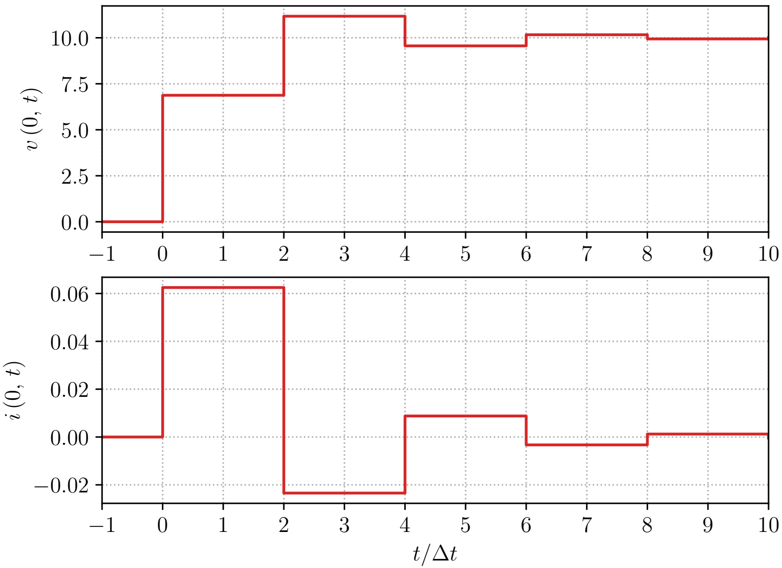
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l \to \infty$



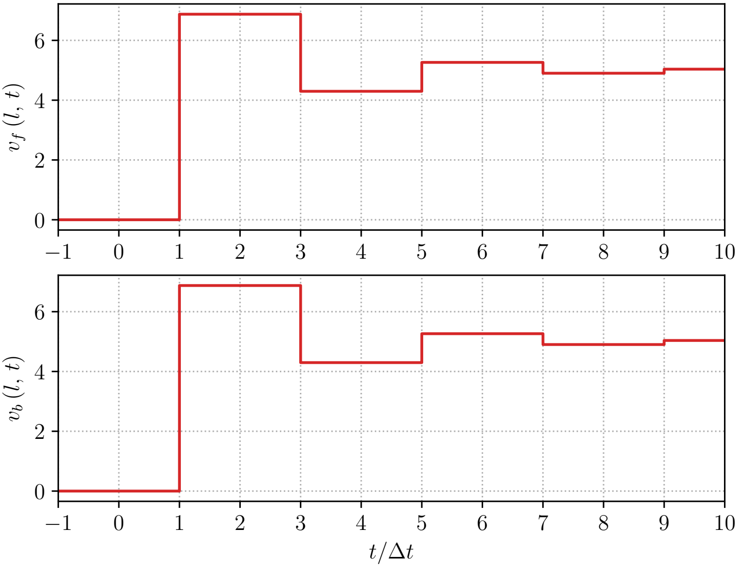
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l \to \infty$



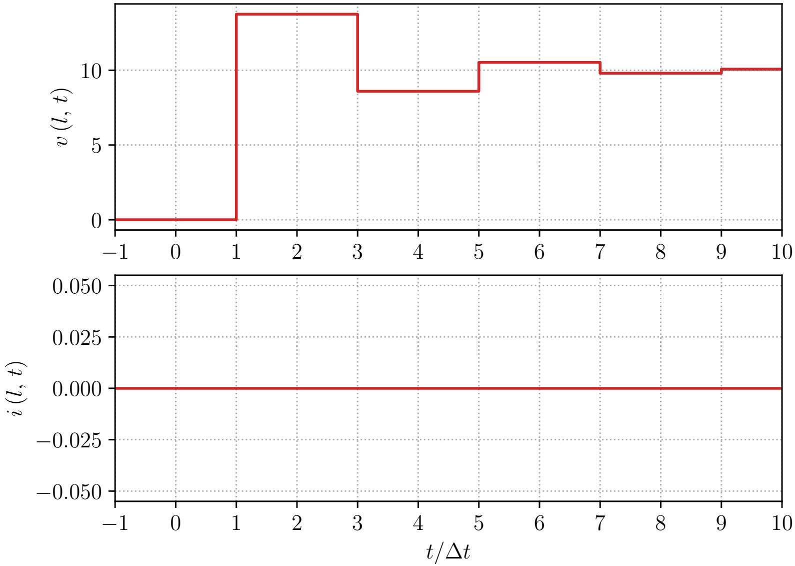
$$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l \to \infty$$



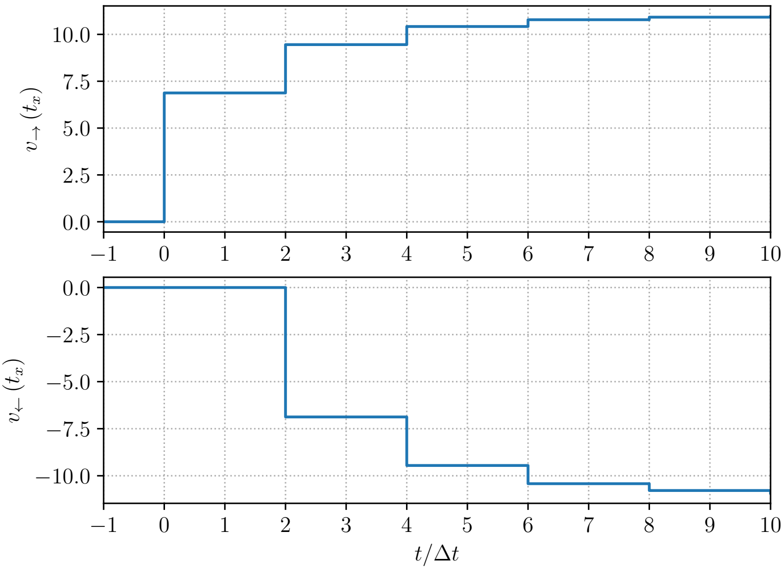
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l \to \infty$



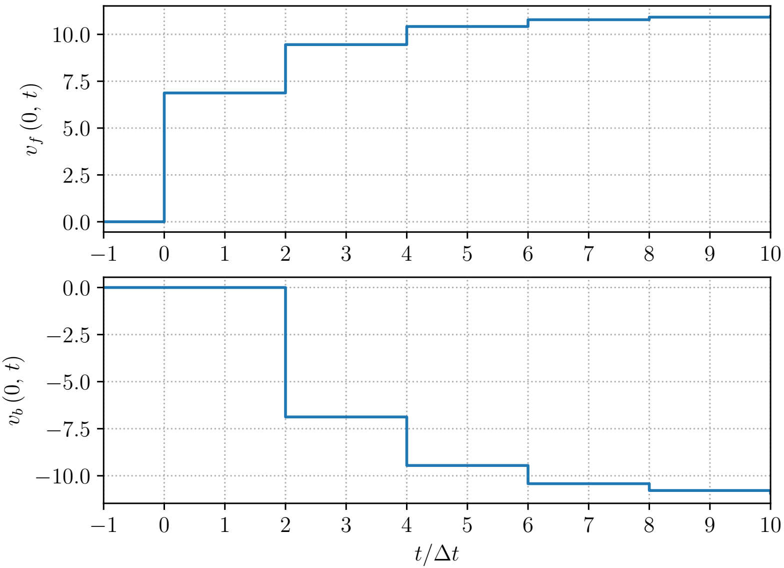
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l \to \infty$



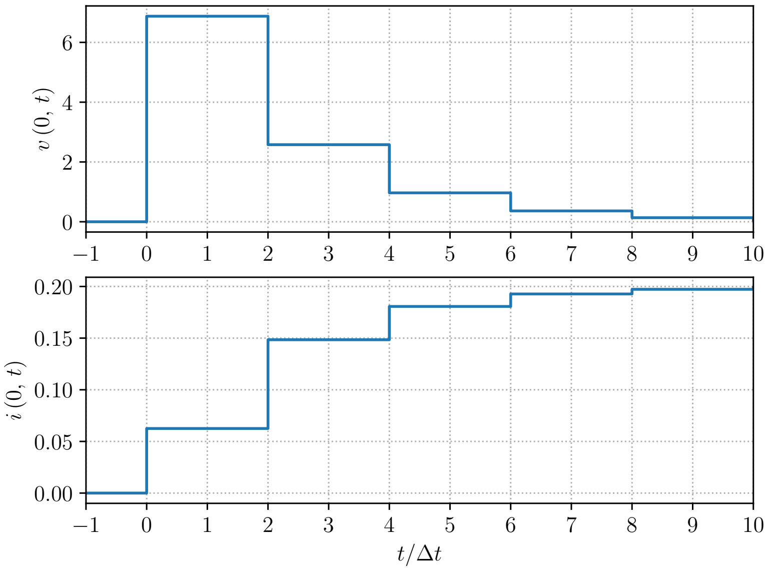
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 0$



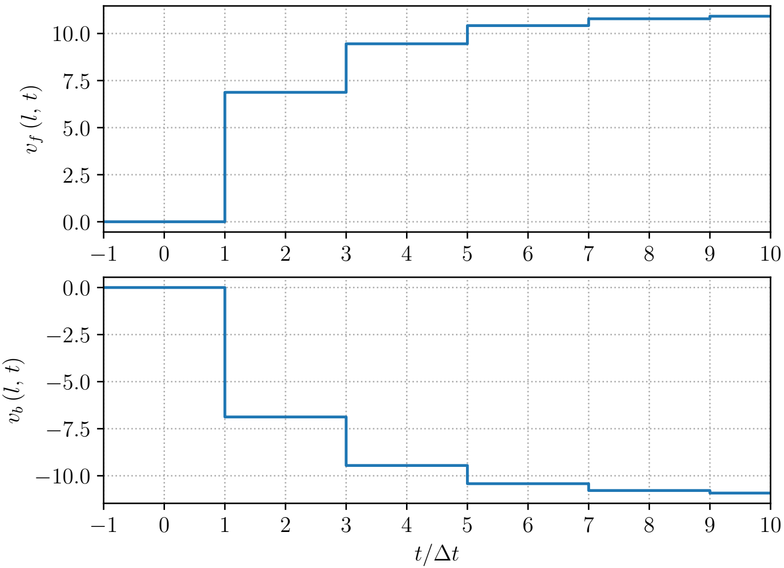
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 0$



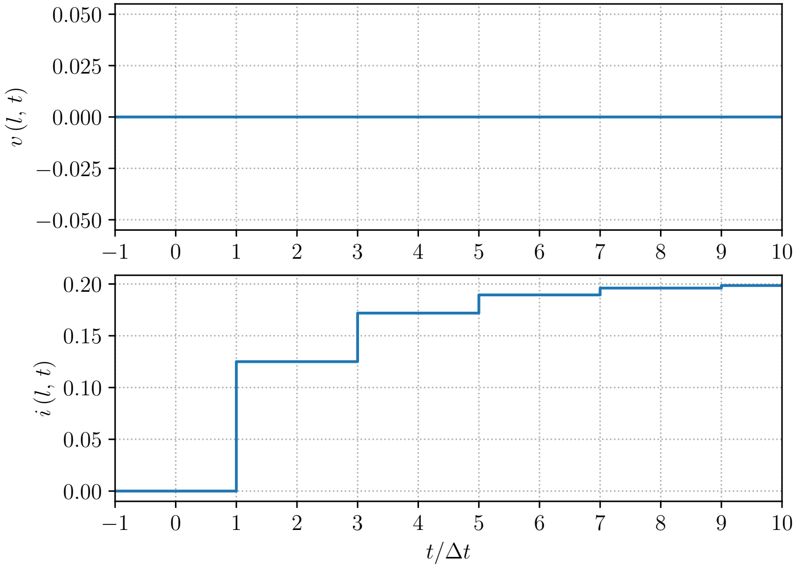
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 0$



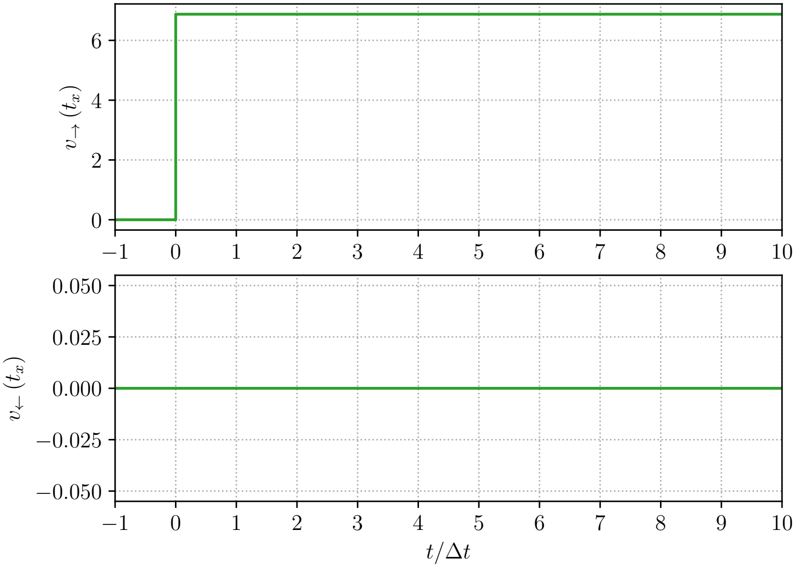
$V_g = 10\,\text{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 0$



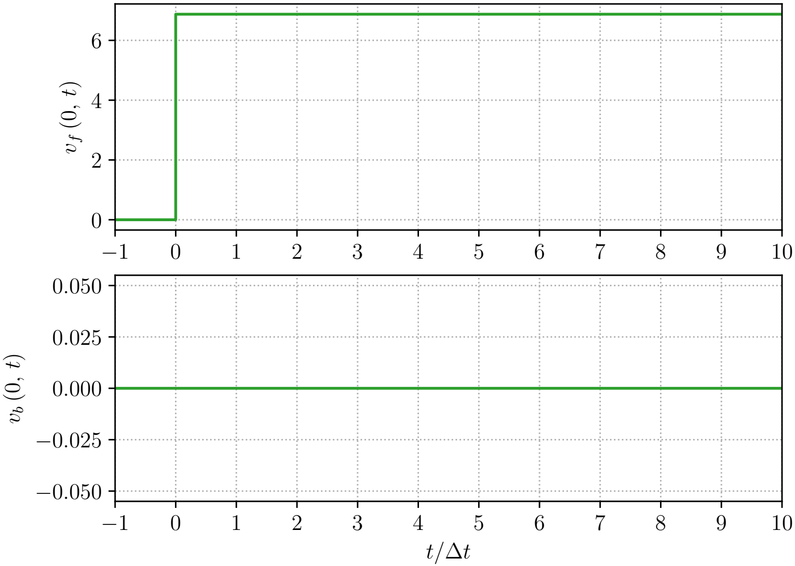
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 0$



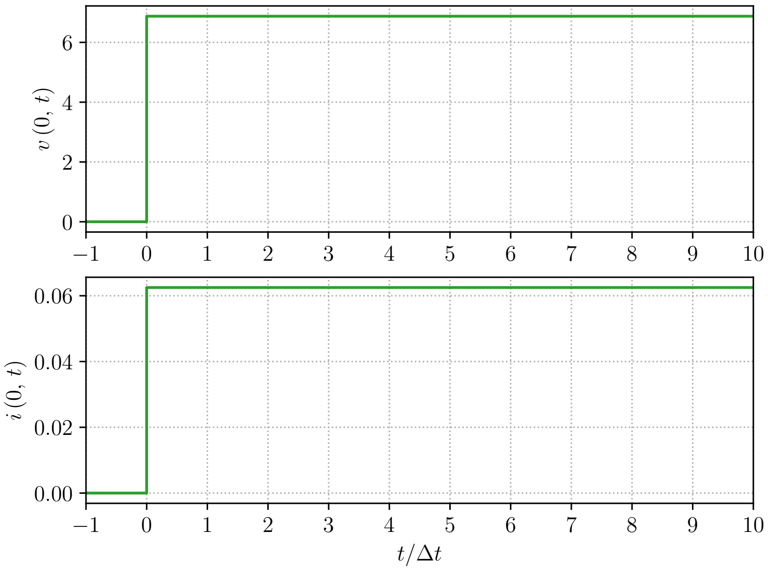
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 110\,\Omega$



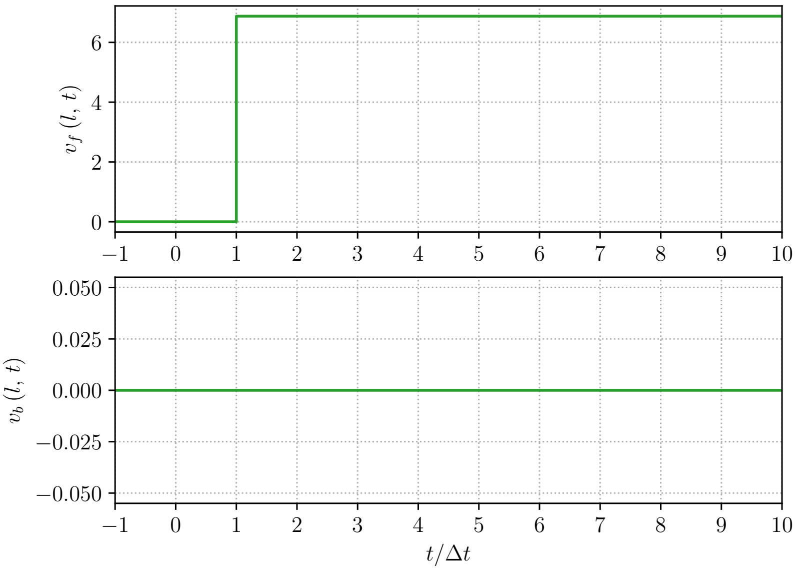
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 110\,\Omega$



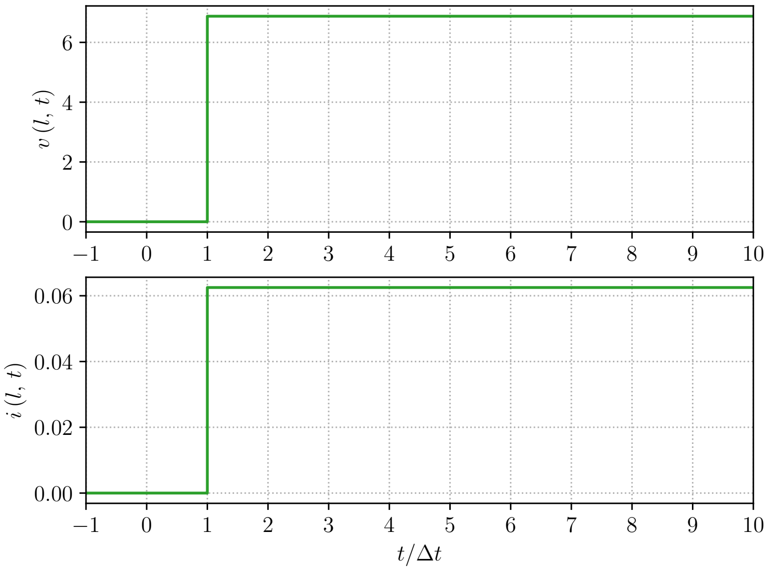
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 110\,\Omega$



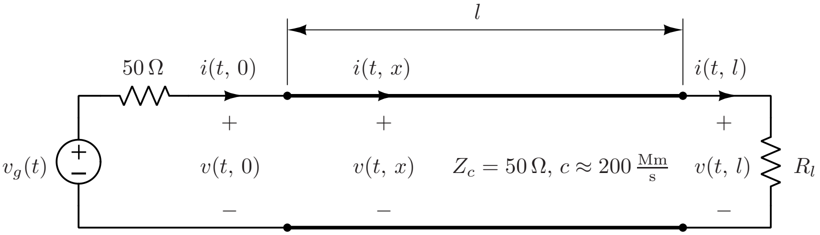
$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 110\,\Omega$



$V_g = 10\,\mathrm{V},\ R_g = 50\,\Omega,\ Z_c = 110\,\Omega,\ R_l = 110\,\Omega$



some more experiments: setup

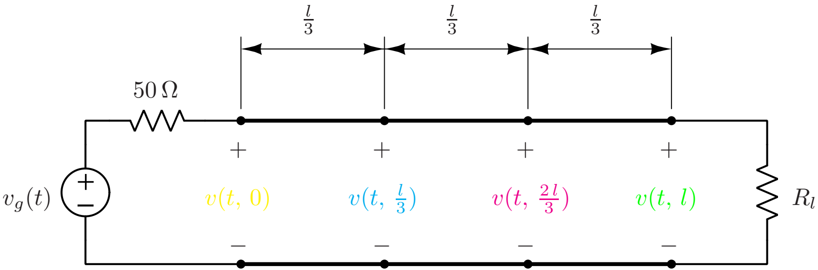


everything that can happen happens in two steps, $2\,\Delta t = 2\,\frac{l}{c}$

coaxial cable (by **Oliver Heaviside**, 1880, British patent No. 1,407) in three segments, to measure spatial (over x) effects

first, some time-domain reflectometry

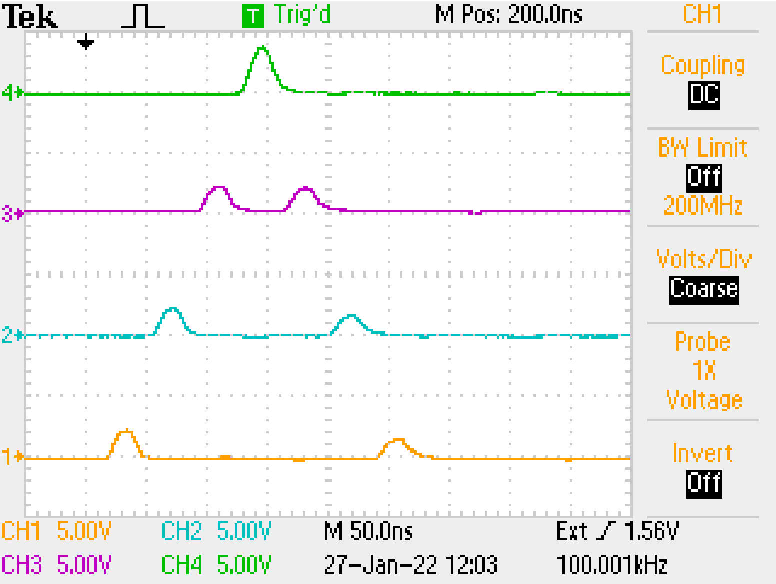
some more experiments: setup



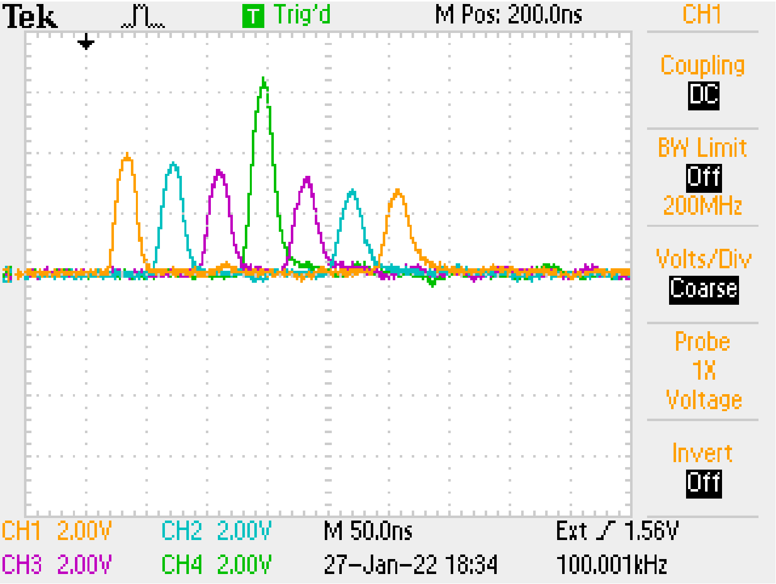
- four channels, four points:
  1. beginning, at $x = 0$, yellow
  2. after $\frac{l}{3}$, cyan
  3. after $\frac{2\,l}{3}$, magenta
  4. at the end, at l, green
- three coaxial cable segments, $\frac{l}{3} \approx 7.5\,\mathrm{m}$ each
- $Z_c = 50\,\Omega,\ c \approx 200\,\frac{\mathrm{Mm}}{\mathrm{s}}$

some more experiments: setup, pulse response

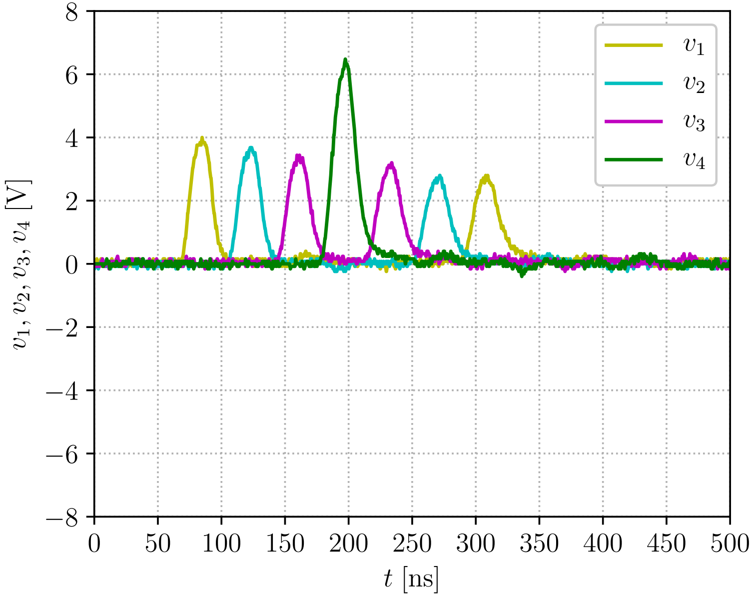
some more experiments: pulse, open



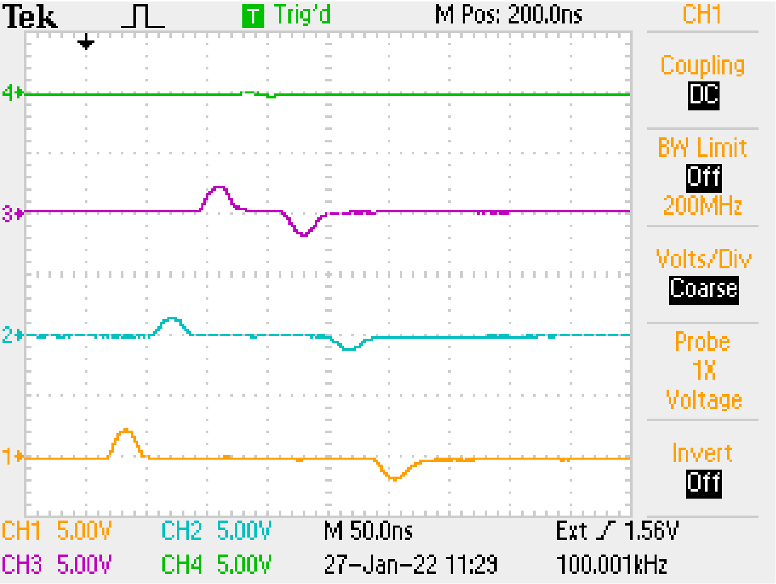
some more experiments: pulse, open



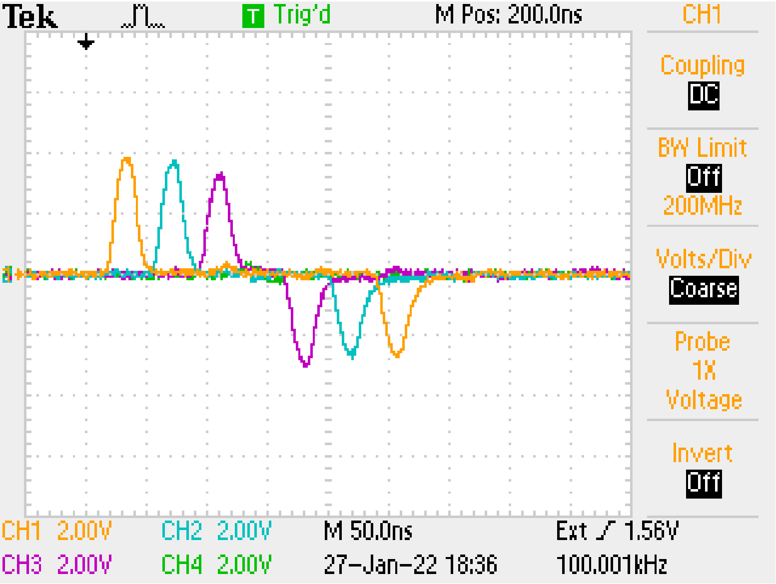
some more experiments: pulse, open



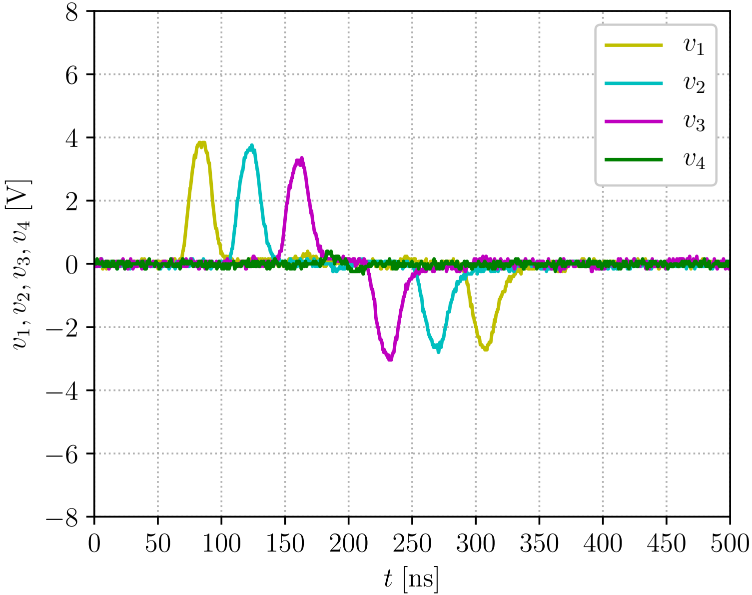
some more experiments: pulse, short



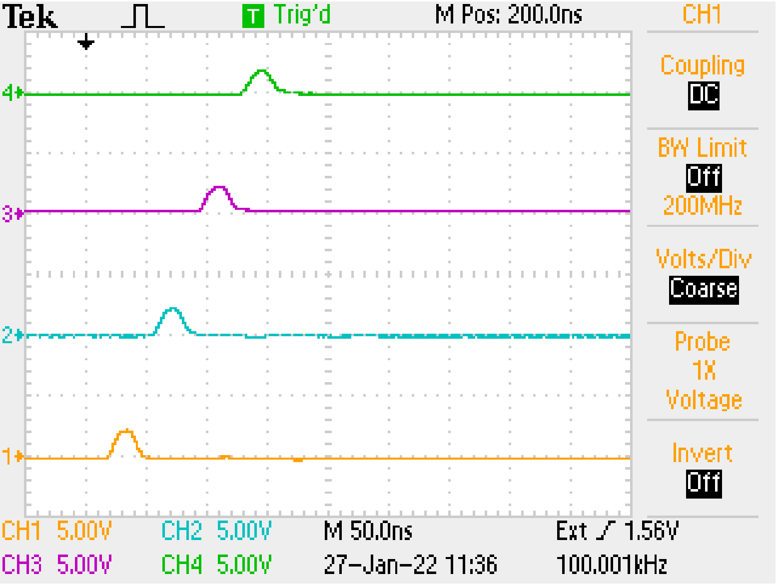
some more experiments: pulse, short



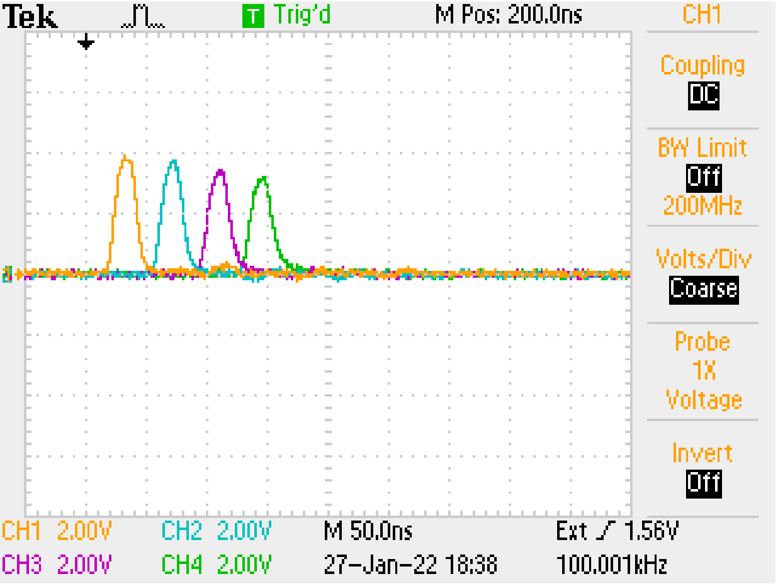
some more experiments: pulse, short



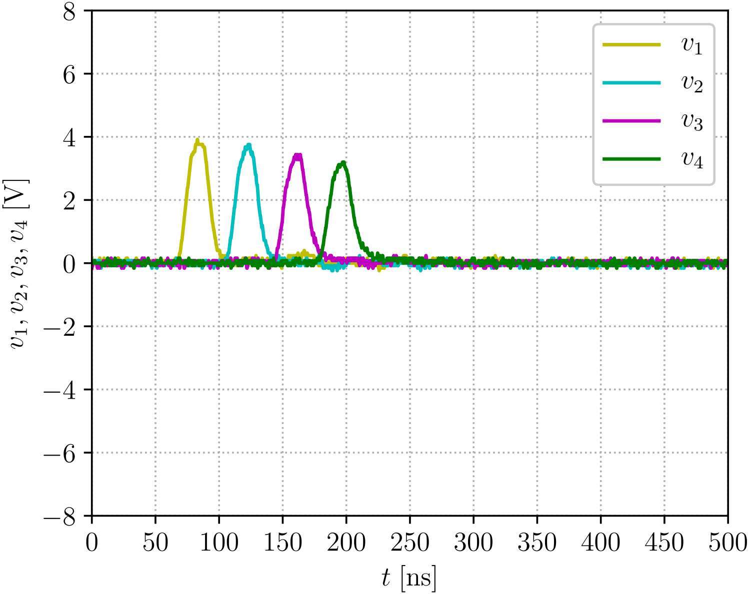
some more experiments: pulse, matched, $R_l = 50\,\Omega$



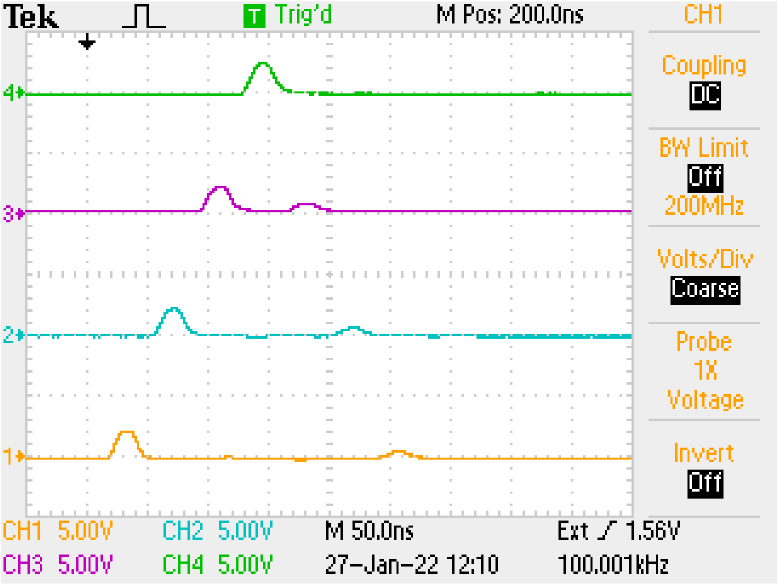
some more experiments: pulse, matched, $R_l = 50\,\Omega$



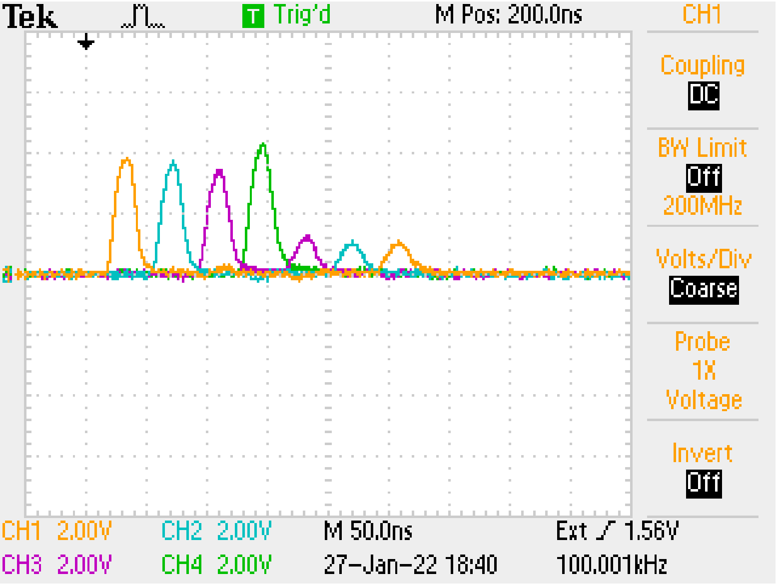
some more experiments: pulse, matched, $R_l = 50\,\Omega$



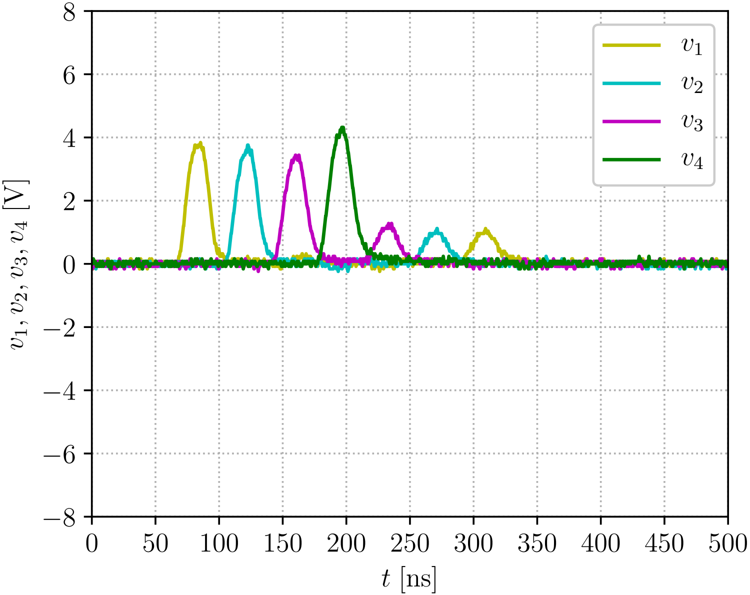
some more experiments: pulse, $R_l = 110\,\Omega$



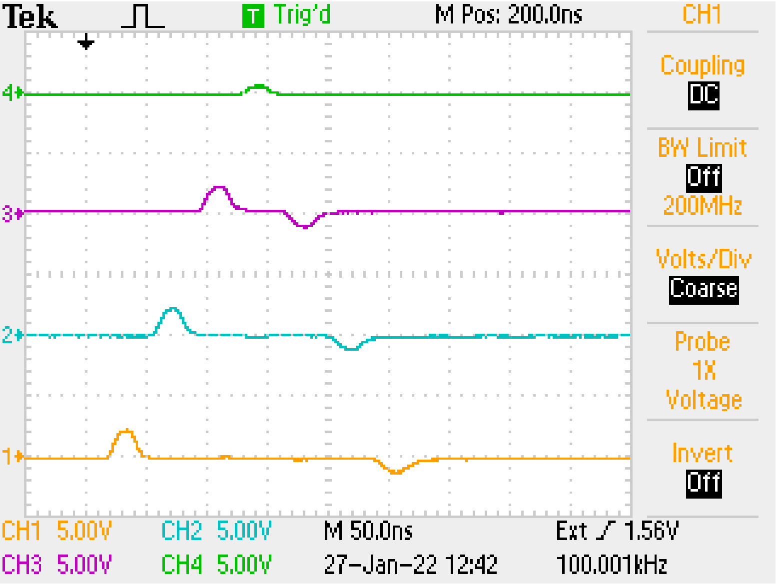
some more experiments: pulse, $R_l = 110\,\Omega$



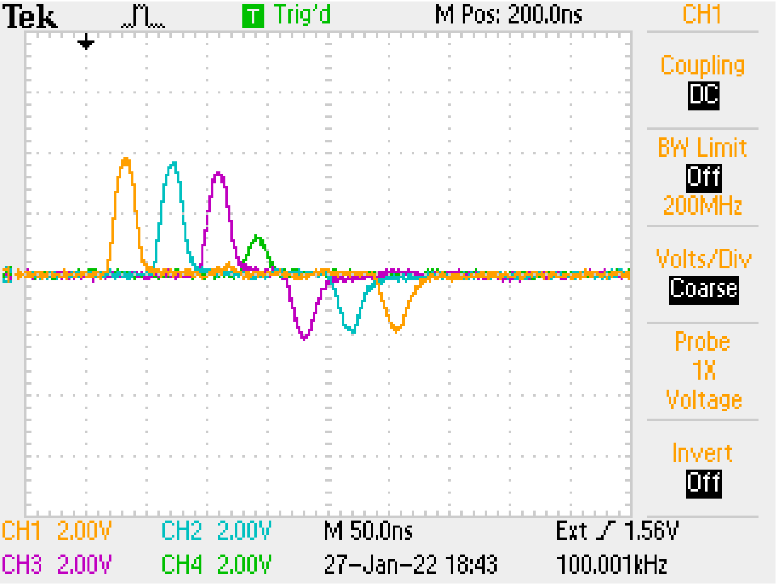
some more experiments: pulse, $R_l = 110\,\Omega$



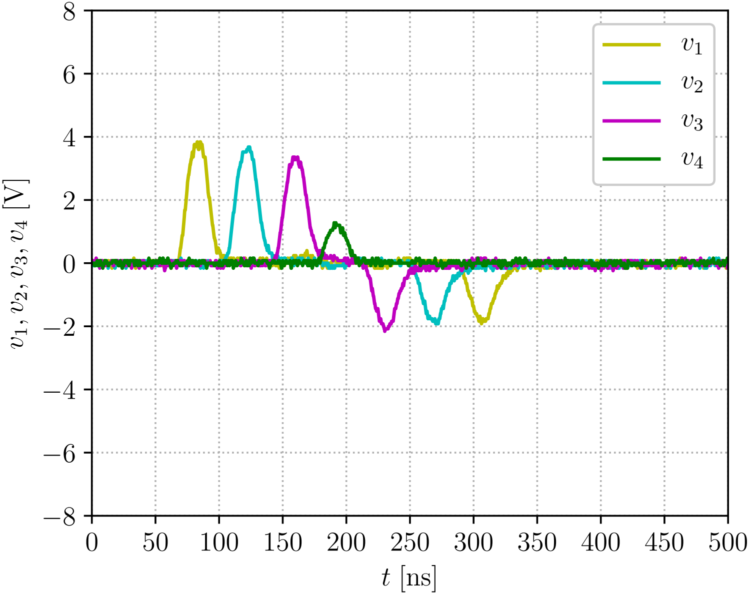
some more experiments: pulse, $R_l = 10\,\Omega$



some more experiments: pulse, $R_l = 10\,\Omega$

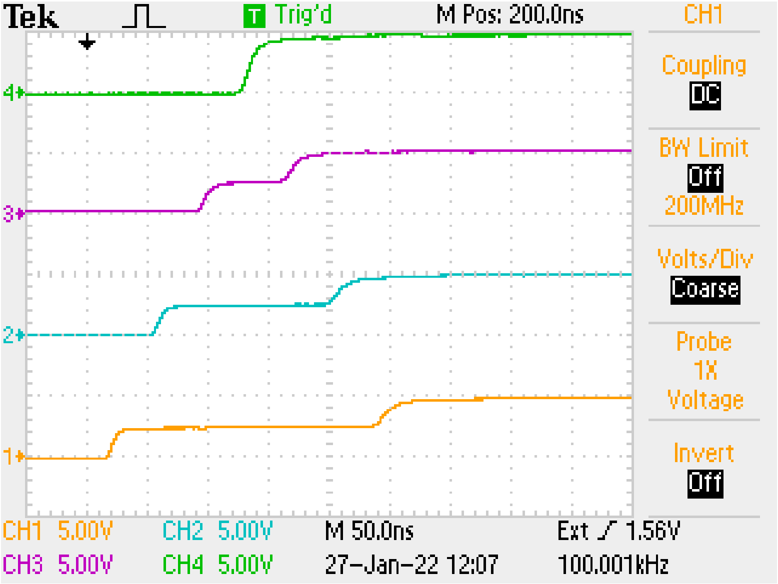


some more experiments: pulse, $R_l = 10\,\Omega$

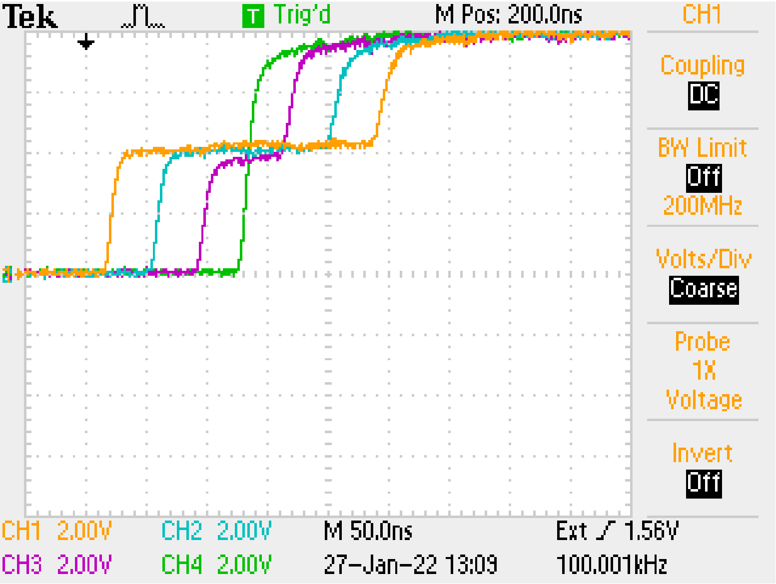


some more experiments: setup, step response

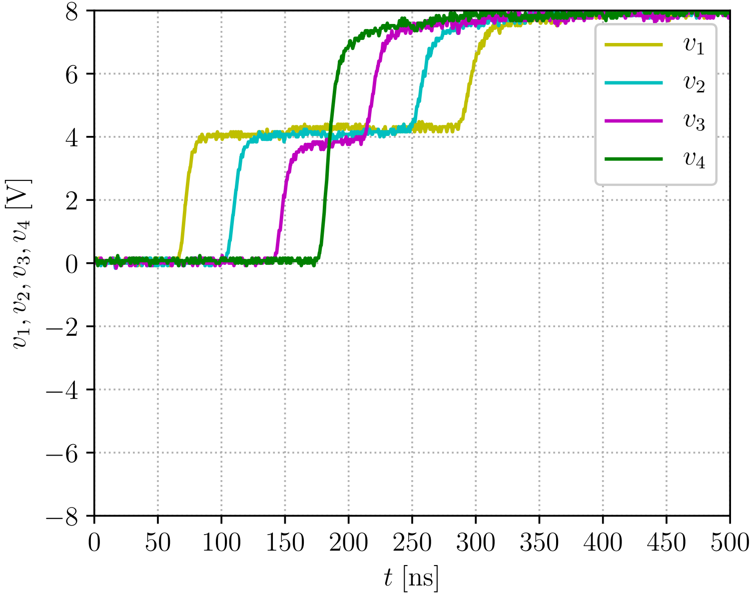
some more experiments: step, open



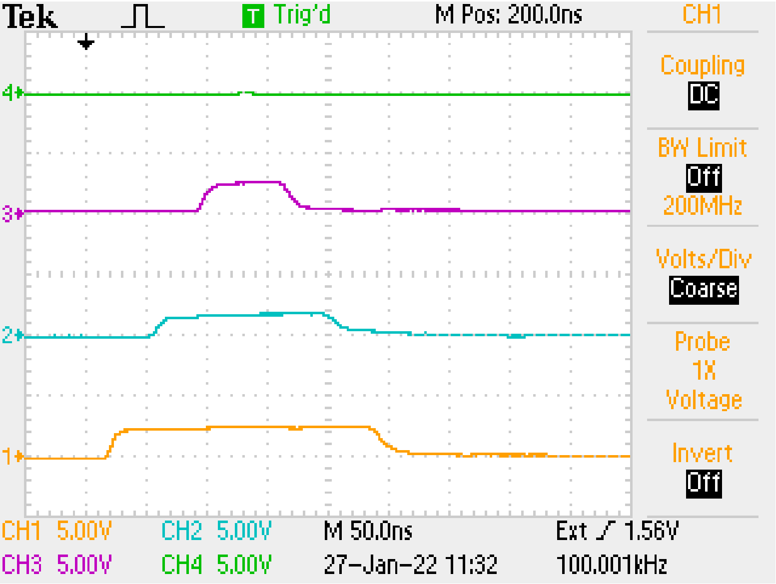
some more experiments: step, open



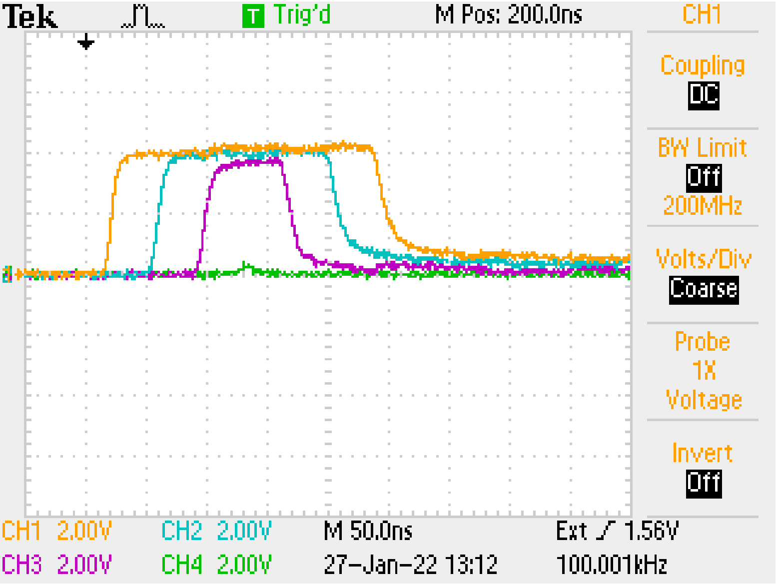
some more experiments: step, open



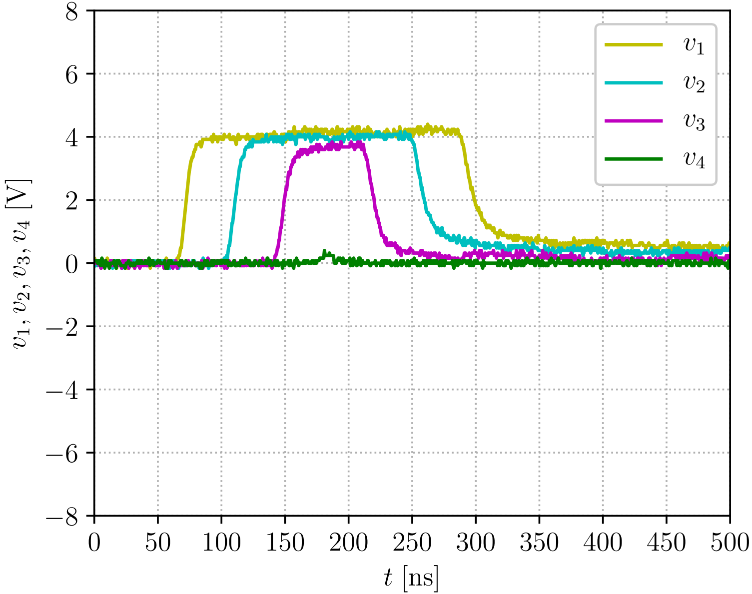
some more experiments: step, short



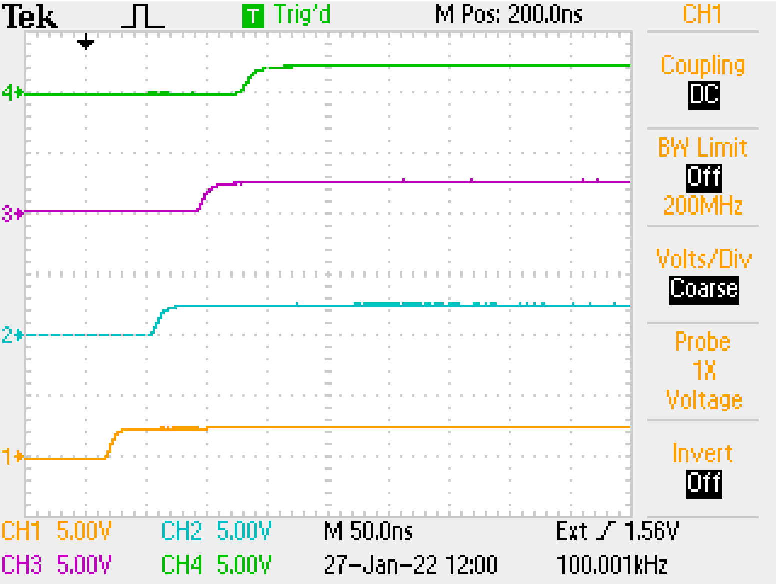
some more experiments: step, short



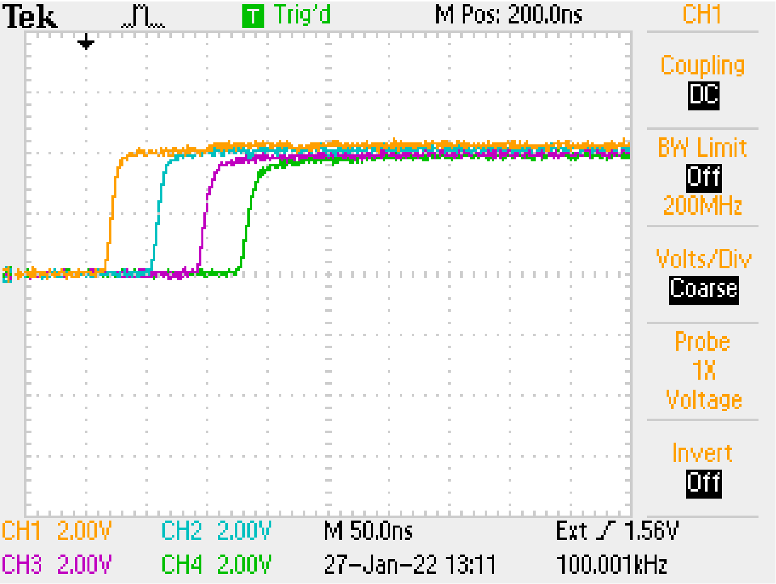
some more experiments: step, short



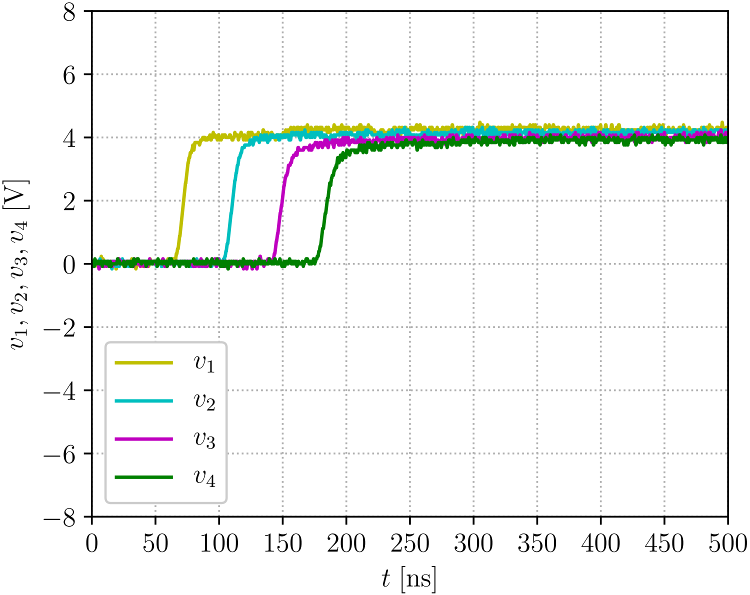
some more experiments: step, matched, $R_l = 50\,\Omega$



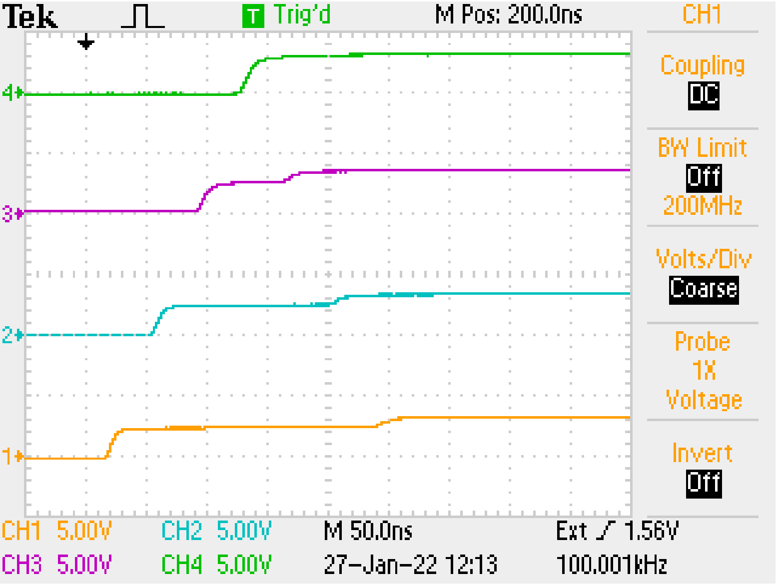
some more experiments: step, matched, $R_l = 50\,\Omega$



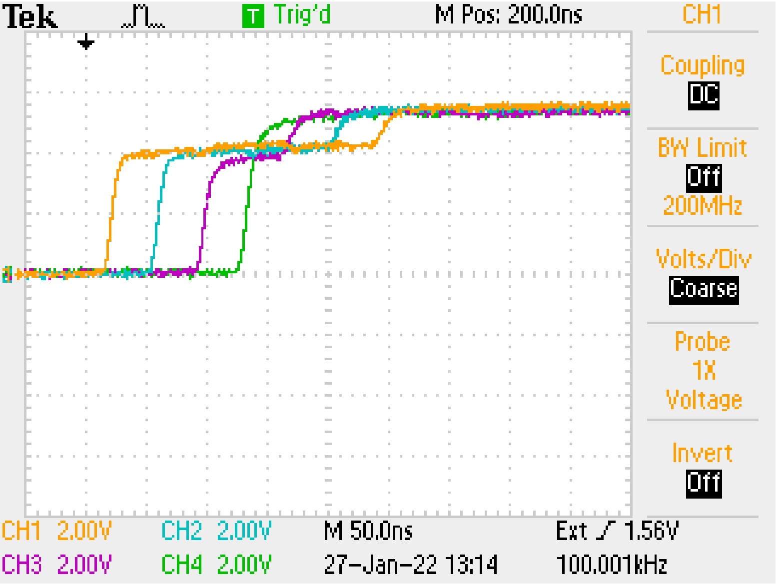
some more experiments: step, matched, $R_l = 50\,\Omega$



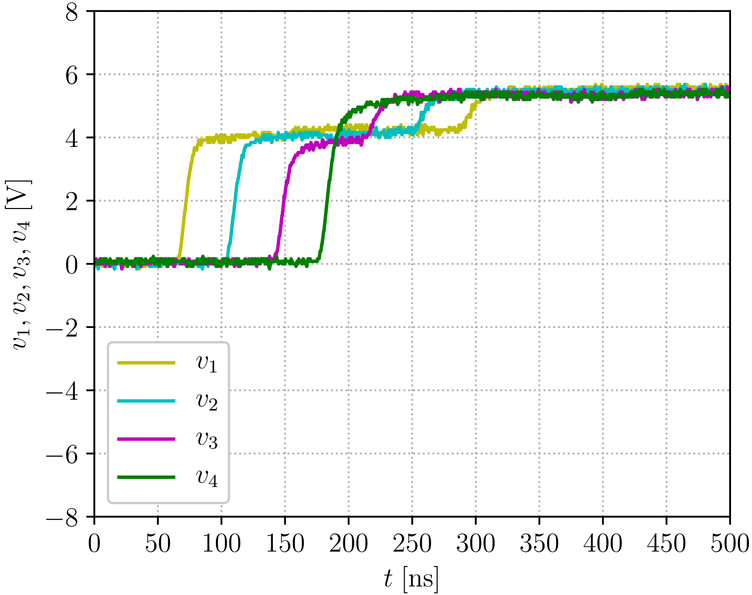
some more experiments: step, $R_l = 110\,\Omega$



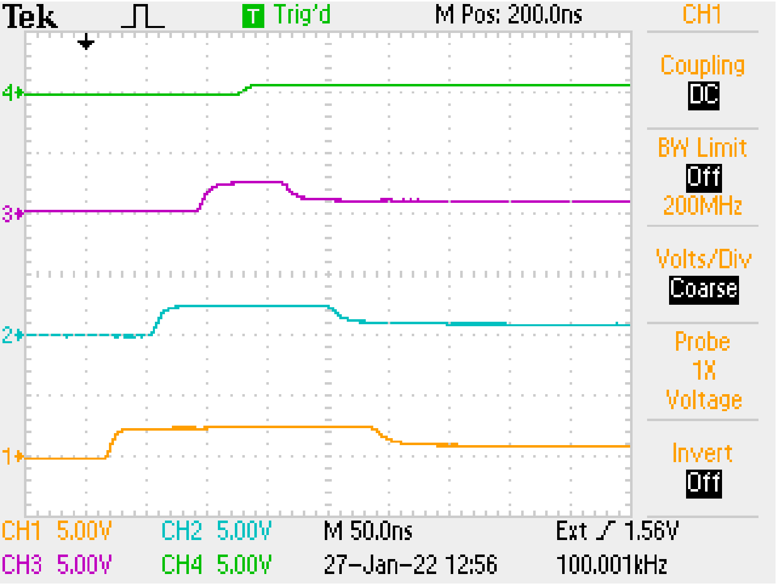
some more experiments: step, $R_l = 110\,\Omega$



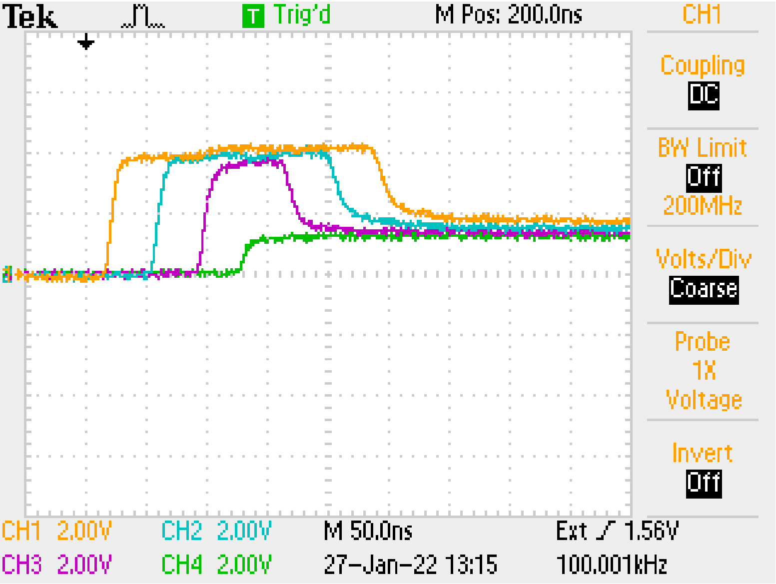
some more experiments: step, $R_l = 110\,\Omega$



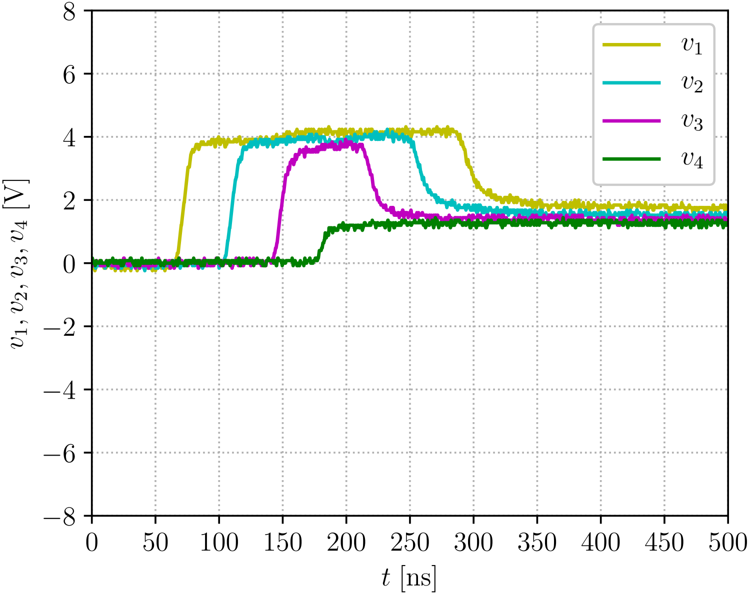
some more experiments: step, $R_l = 10\,\Omega$



some more experiments: step, $R_l = 10\,\Omega$



some more experiments: step, $R_l = 10\,\Omega$



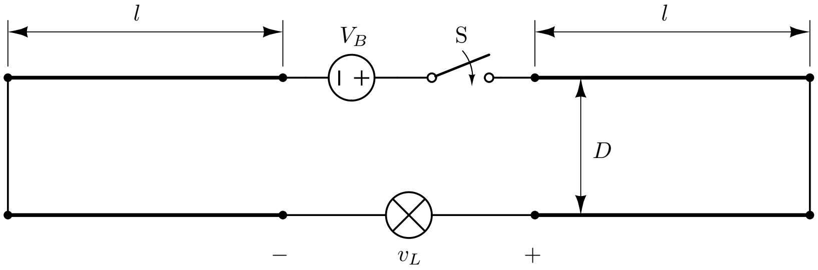
overview of lossless transmission lines

- lossless case of telegrapher's equations
- wave equation ...
- solution in the form of two waves ...
- that travel the opposite directions ...
- the waves determined by the interplay of the voltage and the current on the line
- initial conditions ...
- and boundary conditions ...
- and reflections ...
- lots of experiments ...
- to gain intuition ...

back to the problem that initiated this ...

- okay, we refreshed some circuit theory and some transmission line theory ...
- it is clear that "the light bulb is on" is not really a firm scientific criterion
- but this is not a big deal: how the transient would like?
- we should be able to compute that!
- the setup model suffers from the wire superconductivity: otherwise it is okay, assumptions are reasonable, sort of
- let us assume that we have some super copper to conduct our current, other parameters do not need makeup
- and let us compute the transient, finally!

the model: transmission line



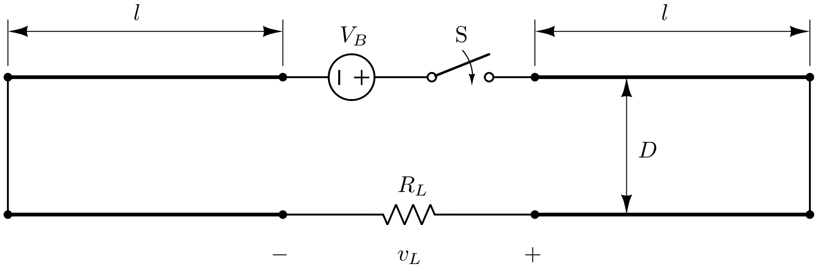
$D = 1\,\mathrm{m}$, $l = 150\,\mathrm{Mm}$

clearly $l \gg D$, transmission line is a suitable model

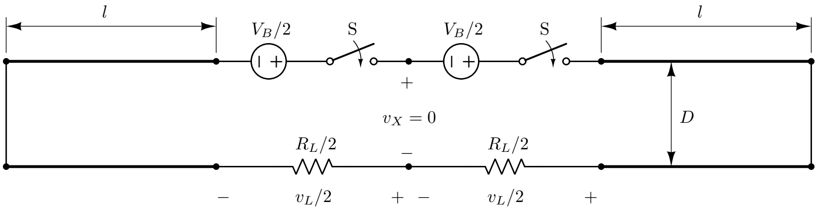
propagation time over $1\,\mathrm{m}$ neglected, circuit theory

overall: circuit theory + transmission line theory

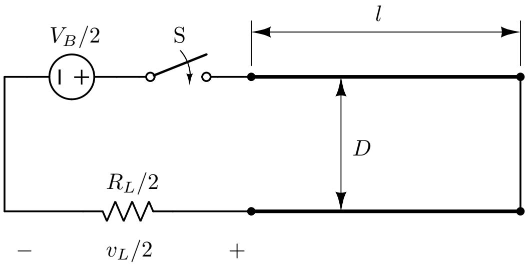
the model: linear load



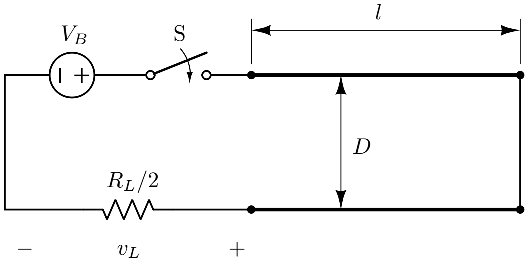
the model: bisection



the model: reduction to a half



the model: the use of linearity; be careful!

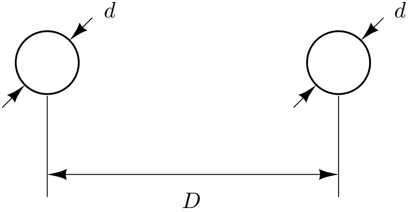


and the solution would be the same! does **not** apply if the circuit is not linear, and only one nonlinear element is sufficient to make the circuit nonlinear! **be careful!**

electrical parameters of the transmission line, Z_c and c?

for c, it is clear; for Z_c, the wire data are missing, d; this **affects** the transient!

the model: transmission line parameters



$$L' = \frac{\mu_0}{\pi}\,\operatorname{arccosh}\left(\frac{D}{d}\right)$$

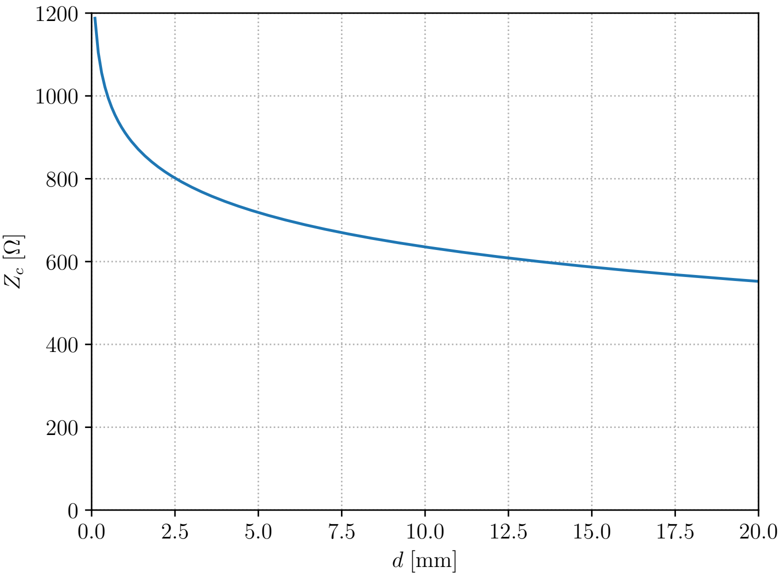
$$Z_c = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\operatorname{arccosh}\left(\frac{D}{d}\right)$$

$$C' = \frac{\pi\,\varepsilon_0}{\operatorname{arccosh}\left(\frac{D}{d}\right)}$$

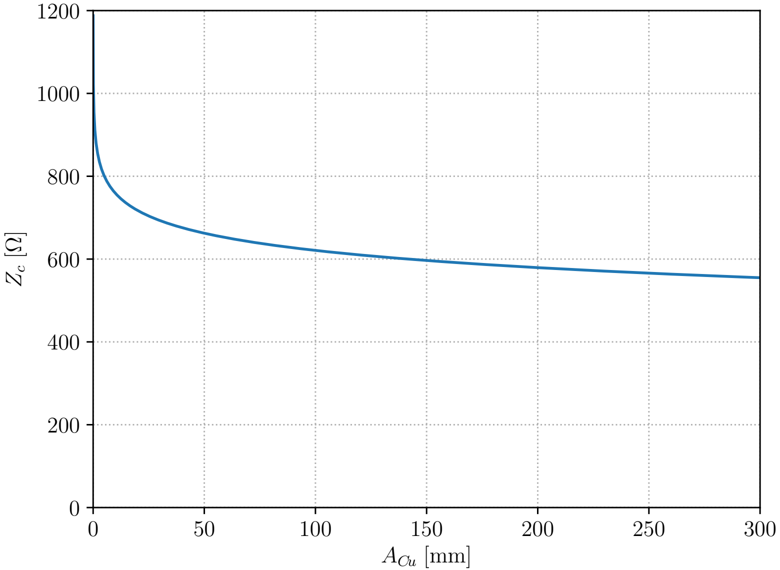
$$c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}}$$

fine print: skin effect, complete here, “hollow conductor” … about $\frac{1}{30}$ of L' might be questioned, only inductance affected, up to $100\,\mathrm{nH/m}$

$Z_c\,(d)$, $D = 1\,\mathrm{m}$



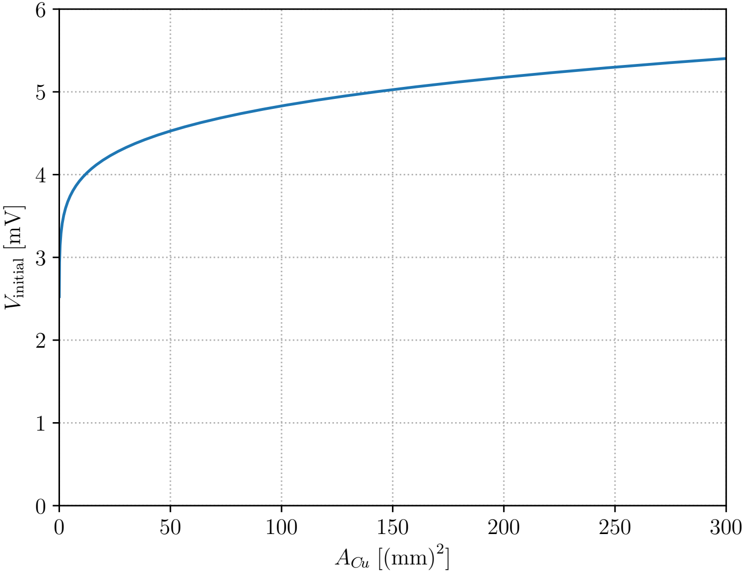
$Z_c\,(A_{Cu}),\; D = 1\,\mathrm{m}$



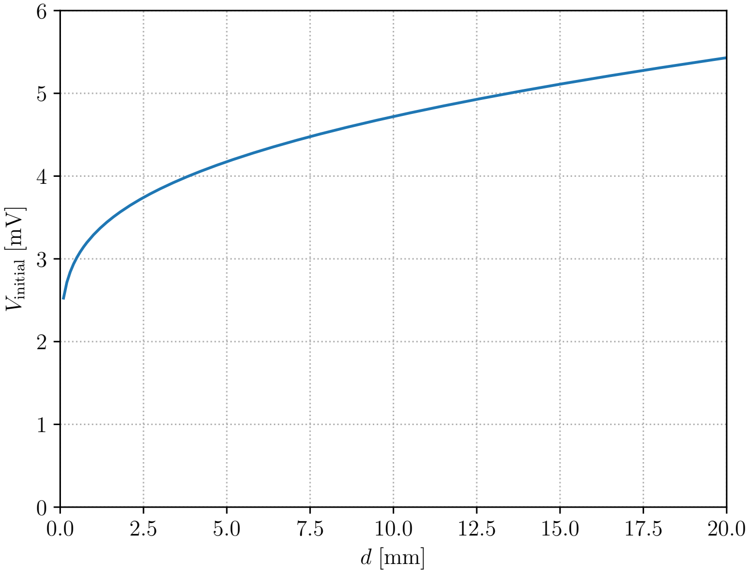
our example . . .

- $V_B = 12\,\mathrm{V}$
- light bulb, $V_{\text{rated}} = 12\,\mathrm{V}$, $P_{\text{rated}} = 21\,\mathrm{W}$
- $I_{\text{rated}} = 1.75\,\mathrm{A}$, $R_{\text{rated}} = 6.86\,\Omega$, $R_{\text{initial}} \approx 0.5\,\Omega$
- d? I have chosen $A_{Cu} = 1.5\ (\mathrm{mm})^2$, it can stand $10\,\mathrm{A}$, $d = 1.382\,\mathrm{mm}$, fair enough
- $Z_c = 872.68\,\Omega$
- just to note: $\dfrac{R_{\text{initial}}}{2\,Z_c} = 0.0286\%$, not 1%, more than 30 times less than that!
- $V_{\text{initial}} \approx 3.44\,\mathrm{mV}$!
- but this would make the light bulb shine, according to the meaning redefined by fine print!
- scale: $3\,\mathrm{mV} \ll 12\,\mathrm{V}$, 4000 times!

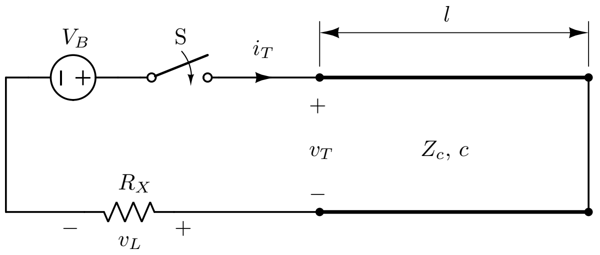
initial voltage versus A_{Cu}



initial voltage versus d



the model: final



- and now we have to solve the model!
- do you expect me to solve partial differential equations?
- the only thing we care about is the light bulb voltage, v_L
- so, let's focus to the only thing we care about, v_T, since $v_L = V_B - v_T$ when the switch is on!
- we need to get v_T, and to get that we need to relate v_T and i_T somehow, "constitutive relation" of a lumped parameter equivalent …

let's do some math … at $x = 0$!

$$v_T\,(t,\,0) = v_f\,(t,\,0) + v_b\,(t,\,0)$$

$$Z_c\,i_T\,(t,\,0) = v_f\,(t,\,0) - v_b\,(t,\,0)$$

let's get rid of $v_f\,(t,\,0)$

$$v_T\,(t,\,0) - Z_c\,i_T\,(t,\,0) = 2\,v_b\,(t,\,0)$$

and

$$v_b\,(t,\,0) = \Gamma_l\,v_f\,(t - 2\,\Delta t,\,0)$$

so

$$\boxed{v_T\,(t,\,0) - Z_c\,i_T\,(t,\,0) = 2\,\Gamma_l\,v_f\,(t - 2\,\Delta t,\,0)}$$

where

$$\boxed{v_f\,(t,\,0) = Z_c\,i_T\,(t,\,0) + \Gamma_l\,v_f\,(t - 2\,\Delta t,\,0)}$$

discrete time model, general

$2\,\Delta t$ indexing here, the time is sampled in $2\,\Delta t$ steps ...

$$v_{T,\,n} \triangleq v_T\left((2\,n\,\Delta t)^+\,,\,0\right)$$

$$v_{T,\,n+1} \triangleq v_T\left((2\,\left(n+1\right)\,\Delta t)^+\,,\,0\right)$$

$$\Delta t \triangleq \frac{l}{c}$$

let us generalize the notation, ...

$$\text{variable}_n \triangleq \text{variable}\left((2\,n\,\Delta t)^+\,,\,0\right)$$

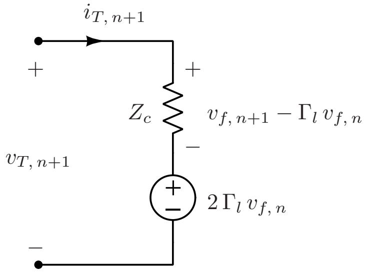
$$\text{variable}_{n+1} \triangleq \text{variable}\left((2\,\left(n+1\right)\,\Delta t)^+\,,\,0\right)$$

$$\text{variable} \in \{i_T,\,v_f\}$$

initial condition: $v_{f,\,-1} = v_f\left((-2\,\Delta t)^+\,,\,0\right) = v_f\left(0^-,\,0\right) = 0$

remember, $v_{f,\,0} = v_f\left(0^+,\,0\right)$

and an equivalent circuit ...

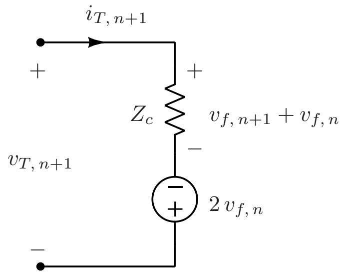


after highly sophisticated math, we finally reached a DC circuit!

three special cases:

1. short, $\Gamma_l = -1$ (the one we care about)
2. open, $\Gamma_l = 1$
3. matched $\Gamma_l = 0$

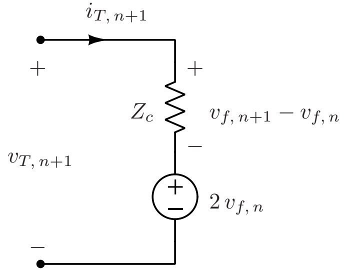
discrete time model, termination shorted, $\Gamma_l = -1$



DC steady state? $n \to \infty$, $v_{f,\,n+1} = v_{f,\,n} = v_{f\infty}$,
$v_{f,\,n+1} + v_{f,\,n} = 2\,v_{f,\,\infty}$

result: $v_{T,\,\infty} = 0$

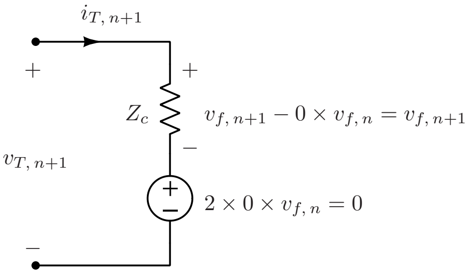
discrete time model, termination open, $\Gamma_l = 1$



DC steady state? $n \to \infty$, $v_{f,\,n+1} = v_{f,\,n} = v_{f\infty}$, $v_{f,\,n+1} - v_{f,\,n} = 0$

result: $i_{T,\,\infty} = 0$

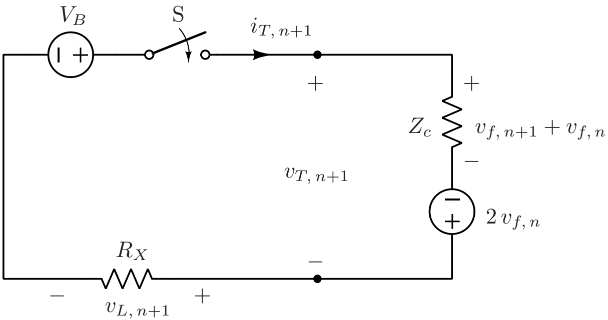
discrete time model, termination Z_c, $\Gamma_l = 0$



complete loss of memory

result: $v_{T,\infty} = Z_c\, i_{T,\infty}$

finally: the discrete time model to solve



- after so many equations and high math we ended up with a DC circuit to solve!
- $v_{f,\,0} = 0$, $R_X = R_L/2$

already solved, actually!

- actually, already solved in a general case before
- there is a closed form solution, exponential
- regarding importance, we'll solve it again here ...
- actual motive is different, to include $R_X(v_L)$
- topologically trivial: there is only one loop
- electrically trivial: resistive circuit; well, almost resistive, a sequence of resistive circuits
- the only problem is $v_{f,n}$, the past
- and that problem is a mathematical problem, not electrical, it is treated as an independent voltage source

let's solve for $v_{f,n}$ ...

$$V_B + 2\,v_{f,n} = (Z_c + R_X)\; i_{T,n+1}$$

$$v_{f,n+1} + v_{f,n} = Z_c\, i_{T,n+1}$$

eliminate $i_{T,n+1}$, solve for $v_{f,n+1}$

$$v_{f,n+1} = \frac{Z_c}{Z_c + R_X}\, V_B + \frac{Z_c - R_X}{Z_c + R_X}\, v_{f,n}$$

$$v_{f,n+1} = A + b\, v_{f,n}$$

$$A = \frac{Z_c}{Z_c + R_X}\, V_B,\; b = \frac{Z_c - R_X}{Z_c + R_X}$$

$A\,(R_X)$ and $b\,(R_X)$

all the time $v_{f,0} = 0$

for fixed R_X this is a geometric series ...

$$v_{f,1} = A$$

$$v_{f,2} = A + b\,v_{f,1} = A\,(1+b)$$

$$v_{f,3} = A + b\,v_{f,2} = A\left(1+b+b^2\right)$$

$$v_{f,4} = A + b\,v_{f,3} = A\left(1+b+b^2+b^3\right)$$

$$\dots$$

$$v_{f,n} = A + b\,v_{f,3} = A\left(1+b+b^2+b^3+\ldots+b^{n-1}\right)$$

$$v_{f,n} = A\,\frac{1-b^n}{1-b}$$

$$\boxed{v_{f,n} = \frac{Z_c}{2\,R_X}\,V_B\,\left(1-\left(\frac{Z_c-R_X}{Z_c+R_X}\right)^n\right)}$$

closed form expression for v_L

after we have $v_{f,n}$ the rest is to solve a small linear resistive circuit:

$$\boxed{v_{L,n+1} = V_B \left(1 - \frac{Z_c}{Z_c + R_X} \left(\frac{Z_c - R_X}{Z_c + R_X}\right)^n\right)}$$

double check for $n = 0$:

$$v_{L,1} = V_B \left(1 - \frac{Z_c}{Z_c + R_X}\right) = \frac{R_X}{Z_c + R_X} V_B$$

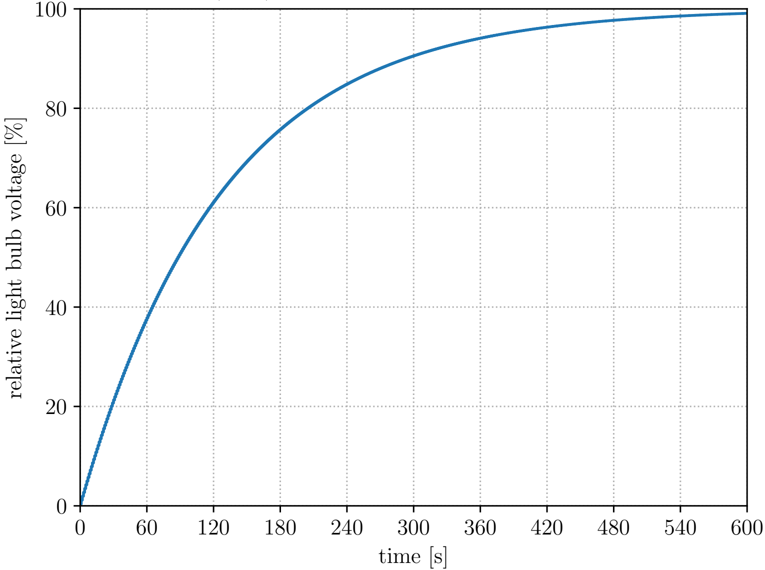
double check for $n \to \infty$:

$$v_{L,\infty} = V_B$$

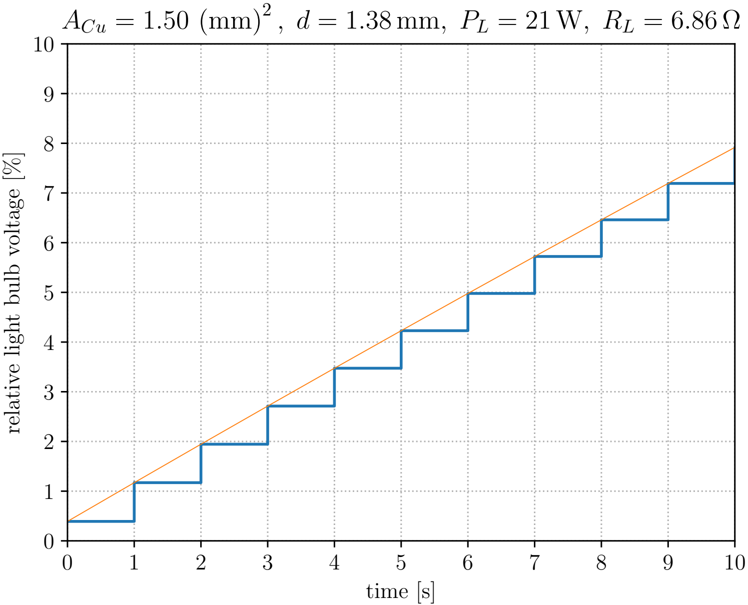
yup! okay!

let's plug the figures in . . .

$A_{Cu} = 1.50\ (\mathrm{mm})^2,\ d = 1.38\,\mathrm{mm},\ P_L = 21\,\mathrm{W},\ R_L = 6.86\,\Omega$



smooth curve?



reduction to exponential form . . .

$$v_{L,\,n+1} = V_B \left(1 - \frac{Z_c}{Z_c + R_X} \left(\frac{Z_c - R_X}{Z_c + R_X}\right)^n\right)$$

$n = \frac{t}{2\,\Delta t}$

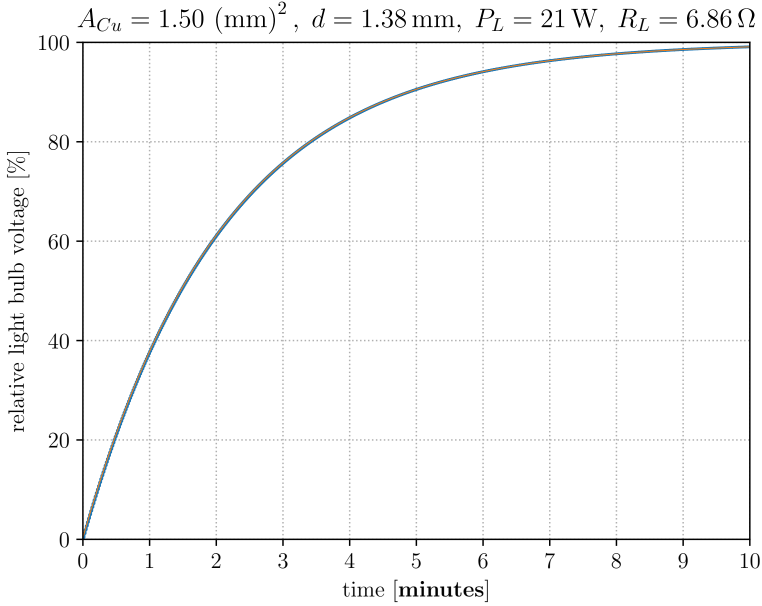
$\Delta t = \frac{l}{c}$

$b^n = e^{\ln(b^n)} = e^{n\,\ln b} = e^{\frac{t}{2\,\Delta t}\,\ln\left(\frac{Z_c - R_X}{Z_c + R_X}\right)} = e^{-\frac{t}{\tau}}$ for $Z_c > R_L$

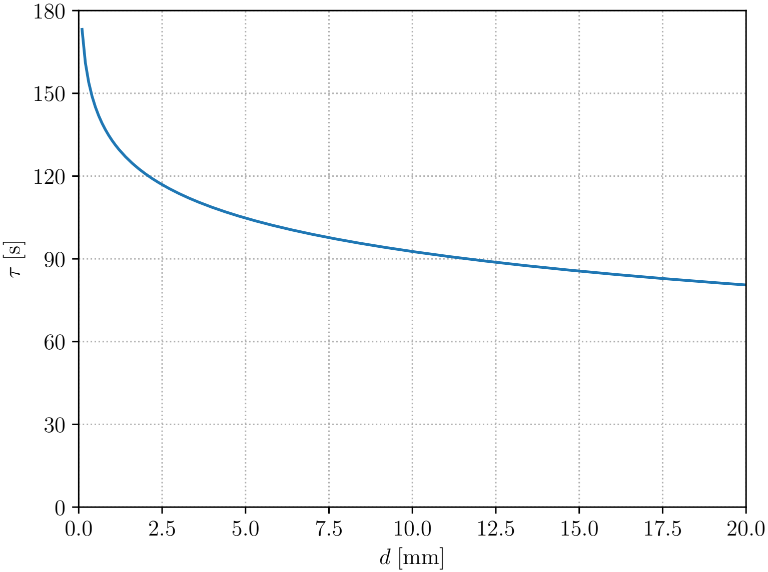
$$\boxed{\tau \triangleq -\frac{2\,\Delta t}{\ln\left(\frac{Z_c - R_X}{Z_c + R_X}\right)} = -\frac{2\,\frac{l}{c}}{\ln\left(\frac{2\,Z_c - R_L}{2\,Z_c + R_L}\right)}}$$ for $R_L < 2\,Z_c$

$$\boxed{v_L\,(t) \approx V_B \left(1 - \frac{Z_c}{Z_c + R_X}\, e^{-\frac{t}{\tau}}\right)}$$ . . . and that's it!

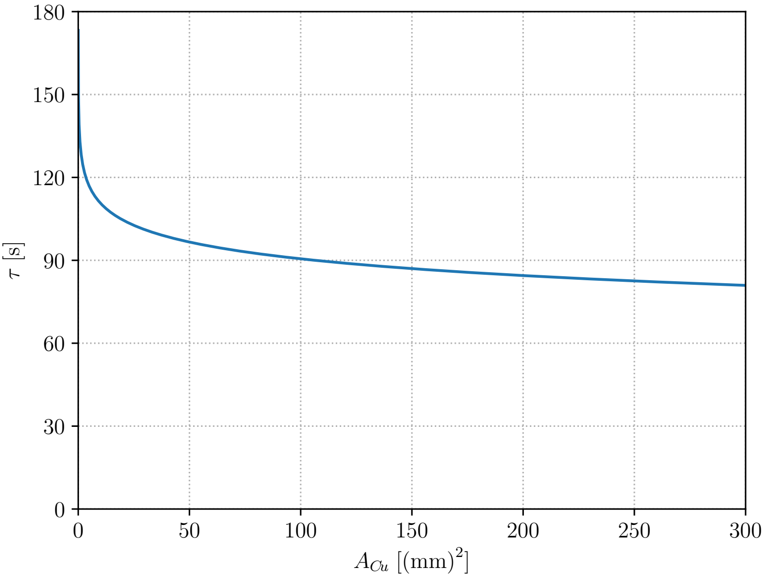
smooth curve . . . big picture



the process is actually really slow ... $\tau(d)$



the process is really slow . . . $\tau\,(A_{Cu})$



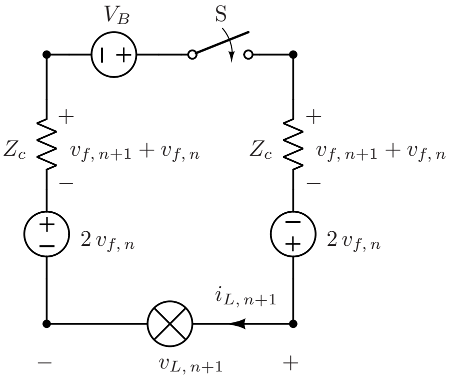
overview of the solution

- we walked a long way to get here ...
- a review of methods and models ...
- method chosen: transmission line theory + circuit theory
- closed form solution!
- the system model is not realistic, there is a wire resistance!
- we'll fix for that later ...
- but the analysis was fun!
- **scale, care about scale!**
- quantity affects quality, **plug in figures sometimes!**
- at the end, just a DC circuit analysis!
- new methods of teaching ... only the DC circuit ...
- initial voltage, at $t = 0^+$ turned out to be minor, negligible
- transient turned out to be really slow!
- **quite a different impression** than $\frac{1\,\mathrm{m}}{c} \approx 3.33\,\mathrm{ns}$?

modeling of nonlinear effects . . .

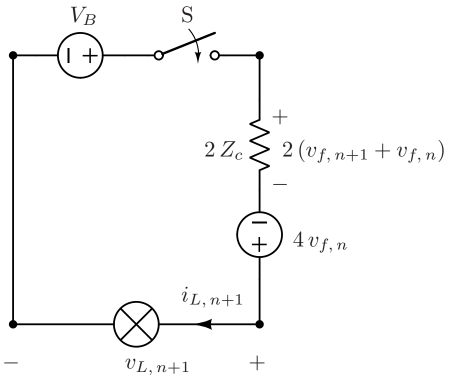
- $v_L(i_L)$ is a nonlinear function, analysis gets complicated . . .
- physical process is $R_L(T)$ . . . and we need to get $T(t)$ somehow, we need a thermal model . . .
- I do not have a model of that process . . . though it is possible to get it, but it would be too much . . .
- everything is clocked at 1 s, the light bulb is small, transient should be fast enough . . .
- besides, how general the thermal model would be? worth to invest time?
- an approach: treat the light bulb as a nonlinear resistor; no thermal inertia, just the DC model, resistive, responds instantly
- $v_L \approx 4 \frac{\mathrm{V}}{\mathrm{A}^2}\, i_L^2$, actual number is 4.076544679336893
- algebraic and (sort of) simple! let's program it!
- in real life it won't be like that, but neither much different!

nonlinear equivalent circuit



- careful with bisection when the load is nonlinear!
- $i_L = k_i \sqrt{v_L}$
- or better, $v_L = k_v\, i_L^2$, $k_v = 1/k_i^2$

reduced nonlinear equivalent circuit

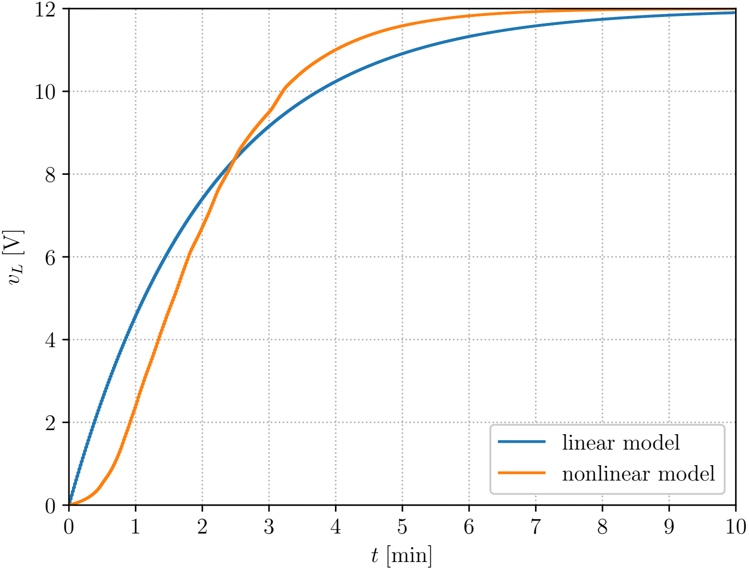


- $V_B + 4\,v_{f,\,n} = 2\,Z_c\,i_{L,\,n+1} + k_v\,i^2_{L,\,n+1}$
- simple, quadratic equation ...
- $i_{L,\,n+1} = \dfrac{-Z_c + \sqrt{Z_c^2 + V_B\,k_v + 4\,k_v\,v_{f,\,n}}}{k_v}$

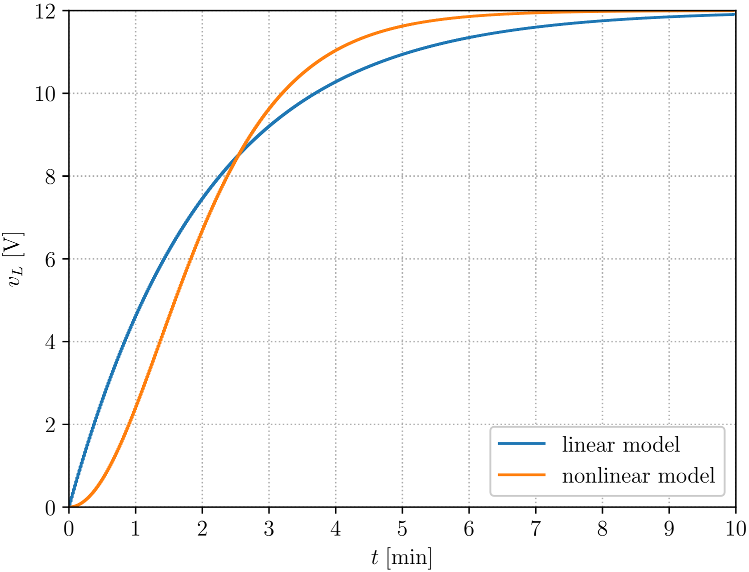
solving the nonlinear equivalent circuit

- equations:
  1. $v_{L,\,n+1} = V_B + 4\,v_{f,\,n} - 2\,Z_c\,i_{L,\,n+1}$
  2. $v_{L,\,n+1} = k_v\,i^2_{L,\,n+1}$
- remember, k_v fited, $k_v = 4.076544679336893\,\frac{\mathrm{V}}{\mathrm{A}^2}$
- let's solve the equations for $v_{L,\,n+1}$ and $i_{L,\,n+1}$
- $v_{L,\,n+1}$ we care for ...
- $i_{L,\,n+1} = \dfrac{-Z_c + \sqrt{Z_c^2 + k_v\,(V_B + 4\,v_{f,\,n})}}{k_v}$
- $v_{L,\,n+1} = \ldots$ choose either 1. or 2. ↑
- $v_{L,\,n+1} = V_B + 4\,v_{f,\,n} - 2\,Z_c\,\dfrac{-Z_c + \sqrt{Z_c^2 + k_v\,(V_B + 4\,v_{f,\,n})}}{k_v}$
- other option: read the measured curve (and iterate)
- some tricks to simplify ↑

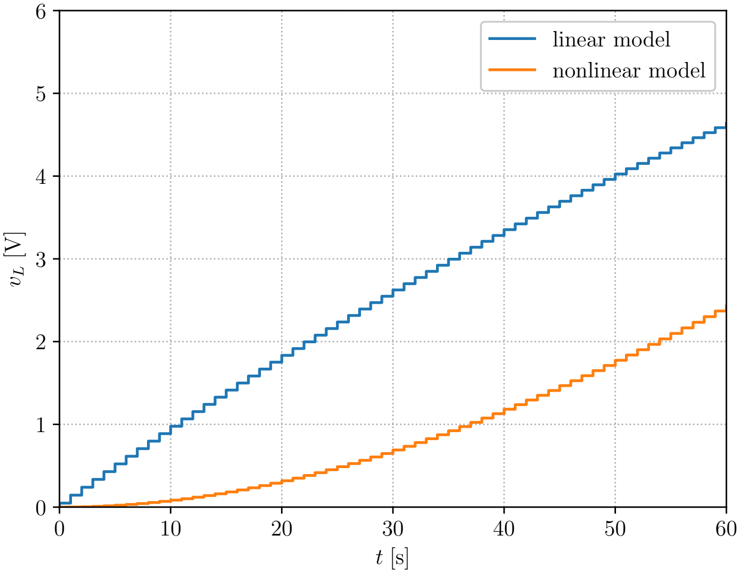
nonlinear effects, comparison, measured $i_L(v_L)$



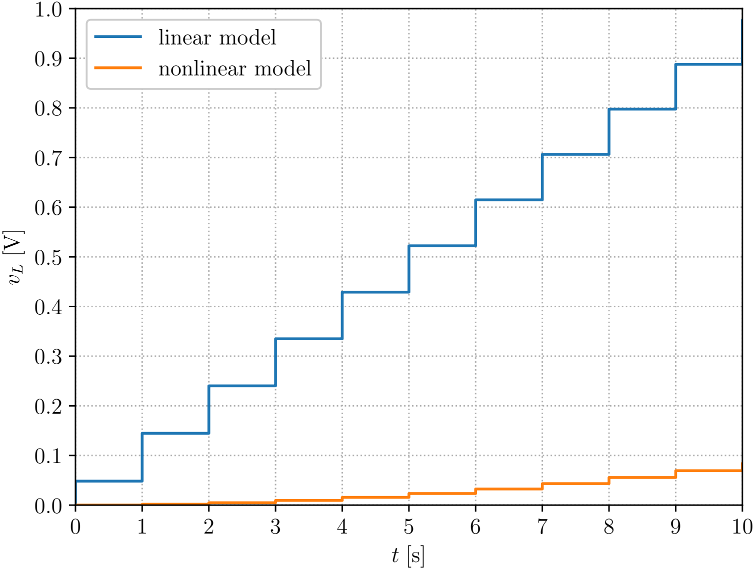
nonlinear effects, comparison, fitted $i_L(v_L)$



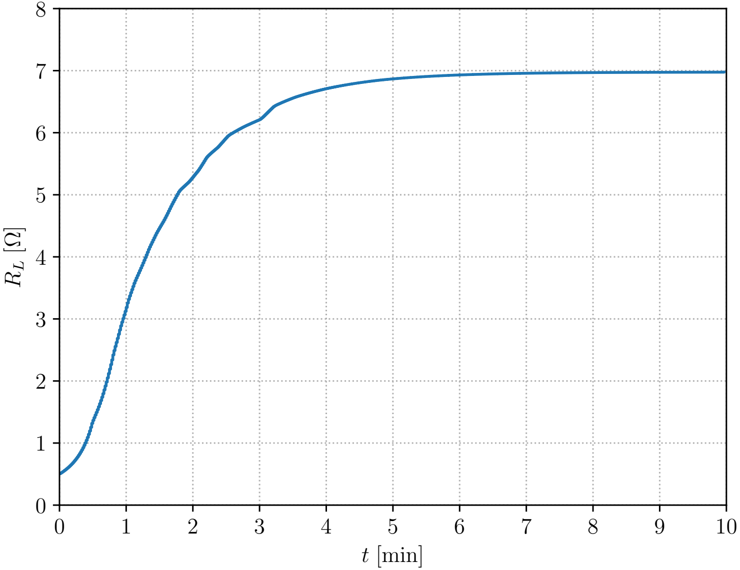
nonlinear effects, comparison, fitted $i_L(v_L)$



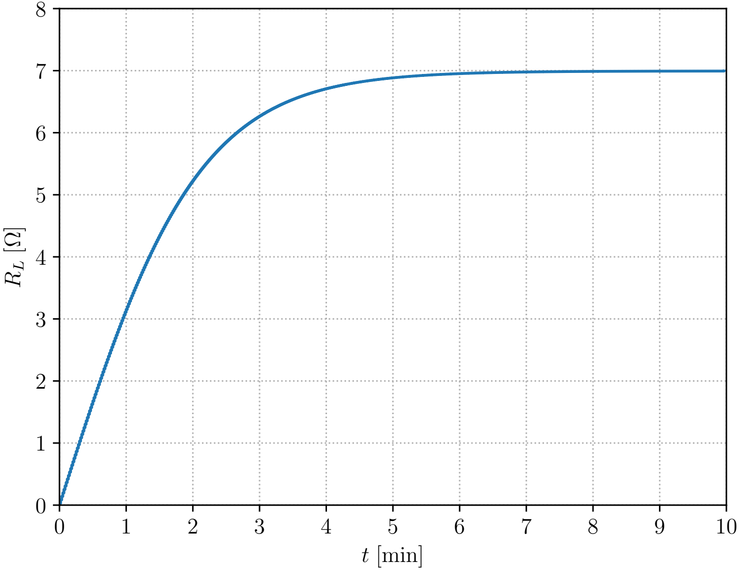
nonlinear effects, comparison, fitted $i_L(v_L)$



heating the light bulb, $R_L(t)$, measured $i_L(v_L)$



heating the light bulb, $R_L(t)$, fitted $i_L(v_L)$



overview of the nonlinear model results

- linear resistor as the light bulb model essentially captured the phenomena ...
- at the beginning, the rise of the light bulb voltage is much slower in comparison to the linear resistor
- this would make us wait for the light even more ...
- however, the current speeds up, and somewhere around $\frac{3}{4}$ of the final voltage the nonlinear model becomes the faster one
- fitted model of the light bulb $i_L(v_L)$ works pretty well, "smoothly"
- analysis much more complicated than for the linear model

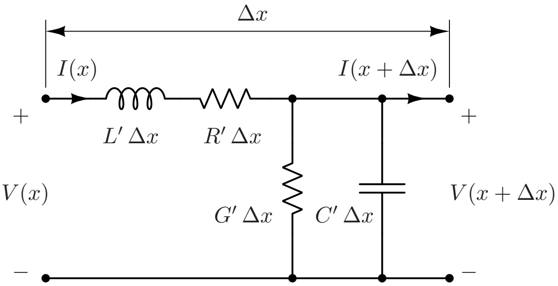
and what if we include losses in the model?

- if I started with this, there would not be any analysis, any slides . . .
- since the light bulb would not produce any light!
- I just wanted to tell you the story about transmission lines
- in the meantime, incandescent light bulbs got involved, and I found them interesting, also
- I find the problem interesting and enlightening . . . regardless it is unlikely to get any light from the bulb . . .
- since only (almost) lossless line would produce any light at the light bulb in our setup

final result

- let us assume a copper wire ...
- $\rho_{Cu} = 1.68\,\mu\Omega\,\mathrm{m}$
- $R' = 22.4\,\frac{\mathrm{m}\Omega}{\mathrm{m}}$ (a problem: "milli" and meter ambiguity, same m for different "objects", for 10^{-3} and for meter)
- total of $R_T = 6.7154\,\mathrm{M}\Omega$, $I_L = 1.79\,\mu\mathrm{A}$, $V_L = 0.89\,\mu\mathrm{V}$, no light according to common sense, but you should double check the fine print for legal interpretation
- assume sinusoidal input for V_B, and perform phasor transform (yes, I know that V_B is assumed DC ...)
- compute the input impedance of the transmission line ...
- reduce to an equivalent circuit, and you'll get the impedance that would never let the bulb shine
- if I started with this, there would never be an opportunity to tell you a nice story about lossless lines ...

transmission line model, phasor transformed



- phasor transform introduced, differentiation over time removed, became $\times j\omega$, in time variable the problem gets reduced from differential to algebraic equations
- in this way, partial differential equations are reduced to ordinary differential equations ...
- which are much easier to solve ...
- frequency response encodes much of the circuit behavior ...

some more tribute to Oliver Heaviside

- steady state response of linear circuits with sine wave generators ...
- phasors, Charles Proteus Steinmetz, another hero!
- easy to extend to steady state response of linear circuits with periodic waveform generators, Fourier analysis and superposition
- easy to extend to (some) aperiodic signals, Fourier transform
- easy to extend to transient response (!), Laplace transform
- ... actually, such analysis of transients started by operational calculus proposed by Oliver Heaviside!
- complex analytic functions encode a lot about themselves ... like continuous functions, but much more!
- you are encouraged to learn complex analysis ...

ordinary differential equations, now

$$V(x+\Delta x) = V(x) - (j\omega\, L'\,\Delta x + R'\,\Delta x)\ I(x)$$
$$I(x+\Delta x) = I(x) - (j\omega\, C'\,\Delta x + G'\,\Delta x)\ V(x+\Delta x)$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = -\,(j\omega\, L' + R')\ I(x)$$
$$\frac{I(x+\Delta x) - I(x)}{\Delta x} = -\,(j\omega\, C' + G')\ V(x+\Delta x)$$

$$\frac{dV(x)}{dx} = -\,(j\omega\, L' + R')\ I(x)$$
$$\frac{dI(x)}{dx} = -\,(j\omega\, C' + G')\ V(x)$$

reduction to one equation, "separation"

$$\frac{d^2V(x)}{dx^2} = (j\omega\, L' + R')\ (j\omega\, C' + G')\ V(x)$$

$$\frac{d^2I(x)}{dx^2} = (j\omega\, L' + R')\ (j\omega\, C' + G')\ I(x)$$

$$\gamma \triangleq \sqrt{(j\omega\, L' + R')\ (j\omega\, C' + G')}$$

$$\frac{d^2V(x)}{dx^2} - \gamma^2\, V(x) = 0$$

$$\frac{d^2I(x)}{dx^2} - \gamma^2\, I(x) = 0$$

linear homogeneous ordinary differential equations!

solution

$$\boxed{V(x) = V_f\, e^{-\gamma\, x} + V_b\, e^{\gamma\, x}}$$

$$\frac{dV(x)}{dx} = -\gamma\, V_f\, e^{-\gamma\, x} + \gamma\, V_b\, e^{\gamma\, x} = -\left(j\omega\, L' + R'\right)\, I(x)$$

$$I(x) = \frac{\gamma}{j\omega\, L' + R'}\, \left(V_f\, e^{-\gamma\, x} - V_b\, e^{\gamma\, x}\right)$$

$$Z_c \triangleq \sqrt{\frac{j\omega\, L' + R'}{j\omega\, C' + G'}}$$

$$\boxed{I(x) = \frac{V_f}{Z_c}\, e^{-\gamma\, x} - \frac{V_b}{Z_c}\, e^{\gamma\, x}}$$

familiar?

two port network

$$V(0) = V_f + V_b$$

$$I(0) = \frac{V_f}{Z_c} - \frac{V_b}{Z_c}$$

$$V(l) = V_f\, e^{-\gamma\, l} + V_b\, e^{\gamma\, l}$$

$$I(l) = \frac{V_f}{Z_c}\, e^{-\gamma\, l} - \frac{V_b}{Z_c}\, e^{\gamma\, l}$$

eliminate V_b and V_f:

$$\boxed{\begin{aligned} V(0) &= V(l)\, \cosh(\gamma\, l) + I(l)\, Z_c\, \sinh(\gamma\, l) \\ I(0) &= V(l)\, \frac{\sinh(\gamma\, l)}{Z_c} + I(l)\, \cosh(\gamma\, l) \end{aligned}}$$

really useful equations, looks nicer into a matrix form ...

two port network, matrix form

I could not resist ...

$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\, l) & Z_c \sinh(\gamma\, l) \\ \frac{1}{Z_c} \sinh(\gamma\, l) & \cosh(\gamma\, l) \end{bmatrix} \begin{bmatrix} V(l) \\ I(l) \end{bmatrix}$$

$$\begin{vmatrix} \cosh(\gamma\, l) & Z_c \sinh(\gamma\, l) \\ \frac{1}{Z_c} \sinh(\gamma\, l) & \cosh(\gamma\, l) \end{vmatrix} = 1 \leftarrow \text{clearly invertible!}$$

transfer functions are not rational functions of $j\omega$ as in linear lumped parameter circuits; the system is more complicated to analyze, transforms are harder to get ...

short at l

plug in $V(l) = 0$:

$$V(0) = I(l)\, Z_c \sinh(\gamma\, l)$$
$$I(0) = I(l)\, \cosh(\gamma\, l)$$

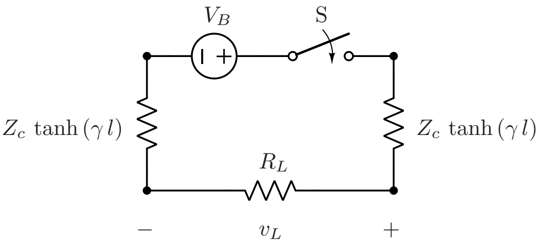
input impedance:

$$Z_{sc} = \frac{V(0)}{I(0)} = Z_c \tanh(\gamma\, l)$$

linear circuit, but not lumped parameter one; impedance is not a rational function of s, $s = j\omega$

$$Z_{sc} = \sqrt{\frac{s\,L' + R'}{s\,C' + G'}} \tanh\left(l\,\sqrt{(s\,C' + G')\,(s\,L' + R')}\right)$$

another equivalent circuit

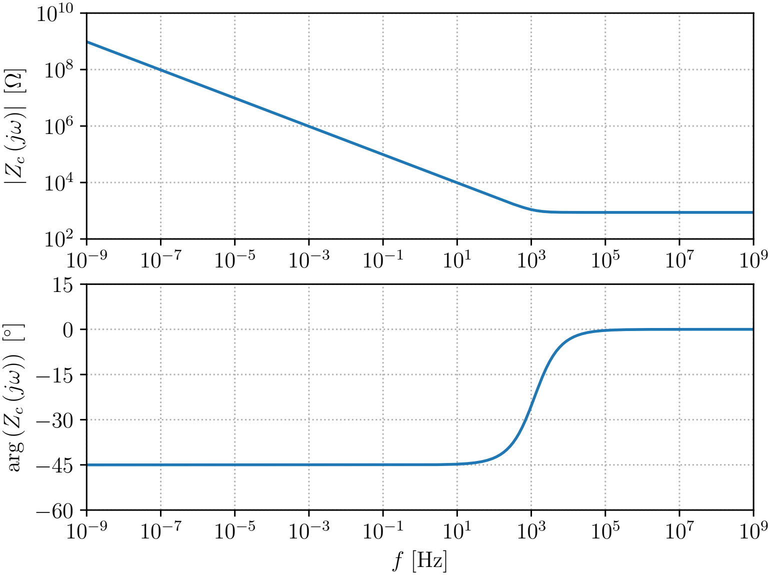


$$V_L = \frac{R_L}{R_L + 2\,Z_c\ \tanh(\gamma\,l)}$$

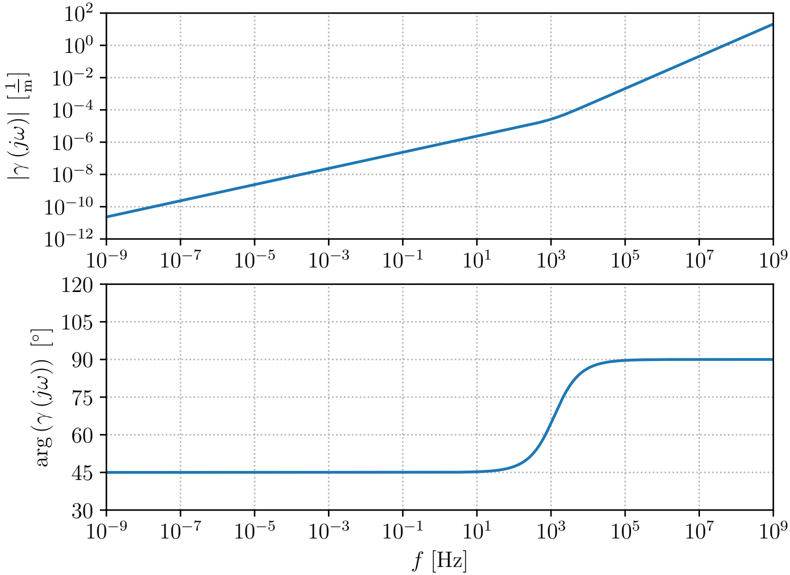
$$L' = 2.91\,\frac{\mu\mathrm{H}}{\mathrm{m}},\ R' = 22.4\,\frac{\mathrm{m}\Omega}{\mathrm{m}},\ C' = 3.82\,\frac{\mathrm{pF}}{\mathrm{m}},\ G' = 0 \rightarrow \gamma(f),\ Z_c(f)$$

$$R_L = 0.5\,\Omega,\ V_B = 12\,\mathrm{V}\ \text{(amplitude)}$$

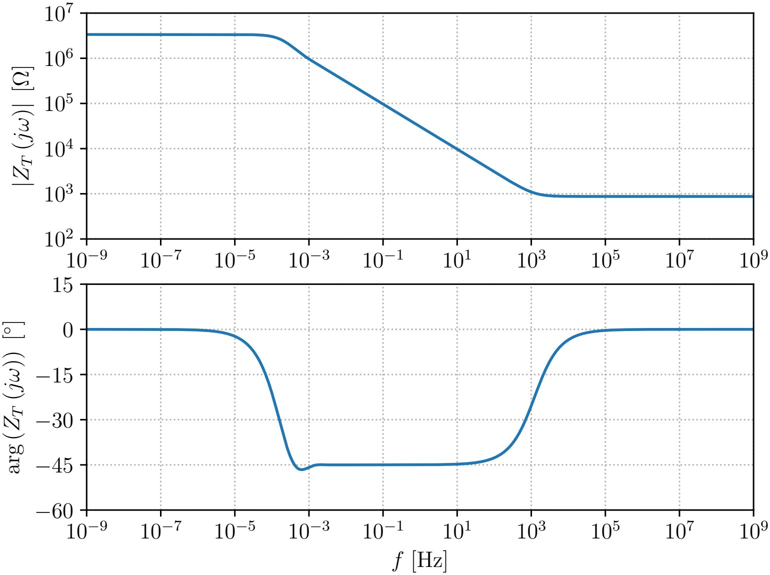
frequency response of Z_c



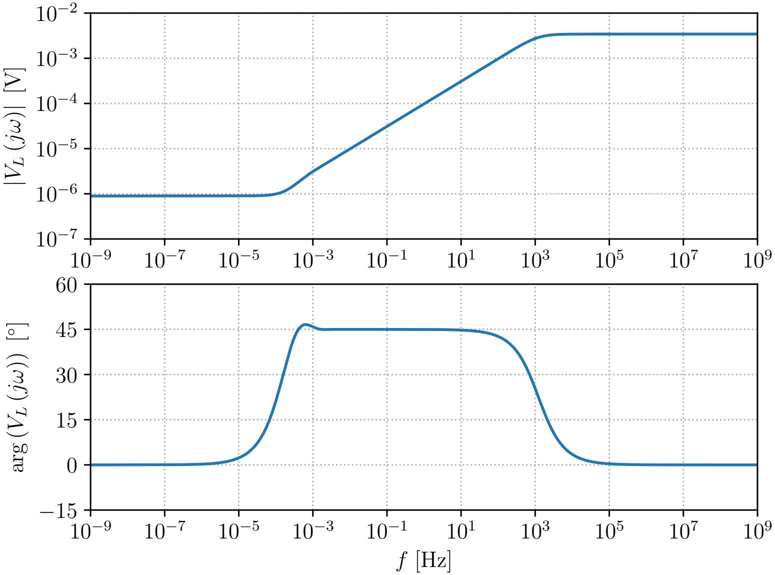
frequency response of γ



frequency response of Z_{sc}



frequency response of V_L



the start and the stop

a trick to get to the Laplace transform: substitute ω with $\frac{s}{j}$

Laplace transform, final value theorem, applied for step response

$V_L(0) = 8.93\,\mu\text{V} \rightarrow v_L(\infty) = 8.93\,\mu\text{V}$

Laplace transform, initial value theorem, applied for step response

$V_L(\infty) = 3.44\,\text{mV} \rightarrow v_L(0^+) = 3.44\,\text{mV}$

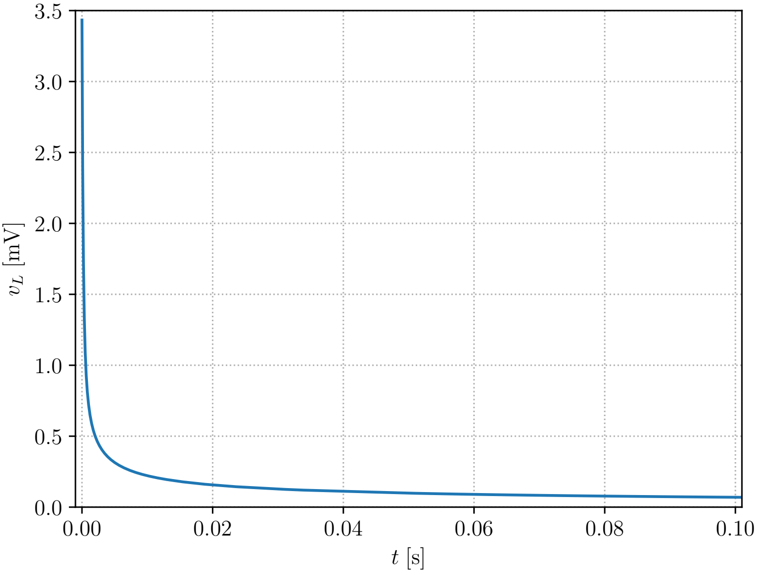
from $3.44\,\text{mV}$ to $8.93\,\mu\text{V}$, not enough voltage to make the light bulb shine! so, when the light bulb would start to shine? well, actually **never**! not $0.5\,\text{s}$, not $1\,\text{s}$, not $2\,\text{s}$, not $\frac{1\,\text{m}}{c}$ ... **never!**

fine print: unless you have excellent deep infrared visual abilities

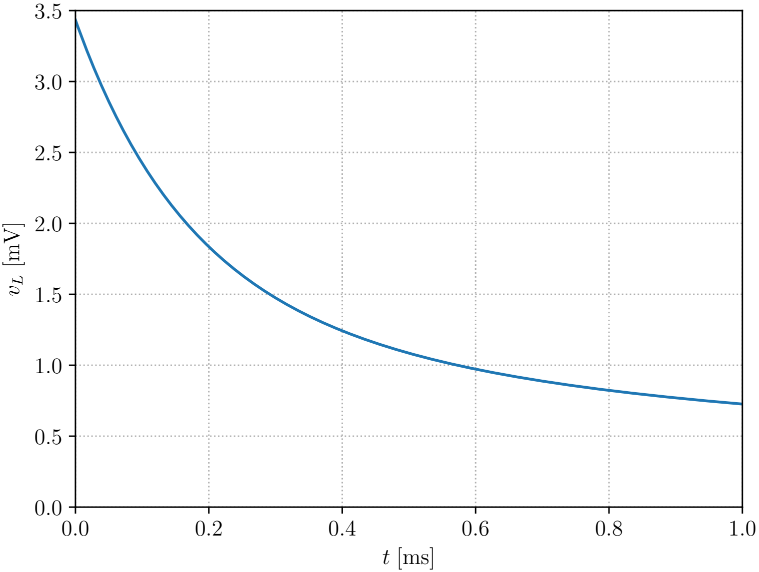
step response from the frequency response ...

- frequency response is the Fourier transform of the system impulse response
- get the impulse response from the frequency response by inverting the Fourier transform
- even analytical approach might work ...
- integrals not worth bothering with ... from almost zero to almost zero ...
- numerical approach, using FFT (DFT) applied here, frequency response sampled ...
- tried and verified on simple examples, to gain confidence in the method and the program that implements it ...
- turned out to be computationally demanding ...
- but it worked!

the diagram which sinks all of our hopes . . .



the diagram which sinks all of our hopes ...



. . . overview of the case with losses

- to handle equations, we had to apply phasor transform to get frequency response . . .
- frequency response? but the source is DC?
- frequency response would tell us enough to compute the transient
- applying the FFT, the step response is obtained . . .
- not an impressive one: the light bulb would never shine
- the losses would consume all of the source voltage . . .
- actually, the transient of the light bulb voltage is very fast, in the order of 100 ms it would change from very small to very very small . . .

conclusions

- this was a long story . . .
- we covered two seemingly simple systems, which turned out to have not so simple behavior:
  1. incandescent light bulb
  2. transmission lines
- the analysis is initiated by a system proposed in a popular science video
- a perfect opportunity to make a tutorial about transmission lines, I already needed to make
- . . . since the question how the transient evolves really is interesting

conclusions: scale

- at first, we put the problem in scale ...
- when creating mathematical models it is important to select relevant phenomena from negligible effects
- this is a quantitative issue we frequently perceive as qualitative, not expressed in numerical terms
- to gain feeling about the problem, quantitative, numerical data, are important, and they are frequently hidden behind symbols
- so, we have to plug in figures from time to time ...
- our problem turned out to be "dimension asymmetric" (actually, symmetry is not the best word here): one spatial dimension is really pronounced in comparison to the other two dimensions; we have planar and a linear effects (almost) decoupled
- which makes the transmission line a suitable model to study the problem!

conclusions: light bulbs

- incandescent light bulbs study followed ...
- used as an indicator, an unfortunate choice ...
- which caused all of the problems, as well as all of the benefits: interest in the topic
- when a light bulb starts to shine turned out to be a complex question by itself!
- an incandescent light bulb requires from 10% to 20% of its rated voltage to produce any visible light ...
- ... after the filament is heated up, which takes time
- light bulbs are DC nonlinear, while AC linear, which is caused by thermal effects ...
- it turned out that at DC current of the light bulb is proportional to the square root of its voltage ...
- as shown to be a good model in two analyzed examples ...
- and useful to provide numerical simulation results

conclusions: methods

- to solve an electromagnetic system, Maxwell's equations are the tool ...
- which is hard to use, since the models are overly complex, and solving the equations is even more complex
- we reviewed Kirchhoff's circuit laws that reduce the electrical system to a sort of "material point", with negligible spatial dimensions
- also, we reviewed transmission line theory, which lays in between Maxwell's equations and the circuit theory
- both the circuit theory and the transmission line theory use voltages and currents; Maxwell's equations use fields
- for our application, a combination of the circuit theory and the transmission line theory should work!

conclusions: lossless transmission lines

- wave equation ...
- and two waves that make the solution ...
- at any given position on the line, the voltage and the current determine which is the wave that travels forward, and which is the wave that travels backward
- step response of a transmission line terminated by a resistor
- reflections and a closed form solution ...
- and the initial step that does not depend on the line termination ...
- numerous examples and experiments to gain intuition ...
- experimental results could be obtained using relatively inexpensive equipment and setup!

conclusions: solving the problem, lossless case

- yes, finally we solved the problem!
- elementary math …
- … after you successfully traveled through not so elementary math and concepts
- everything reduced to successive solving of DC circuits
- there is a closed form solution …
- there **is** some voltage at the beginning, at $t = 0$ in our model …
- but this is minor; the transient is a long one …
- related to the wire thickness …
- and it takes minutes!
- completely different picture than a really fast transient!

conclusions: nonlinear effects

- light bulb is a DC nonlinear element ...
- and that nonlinearity is relevant in our transient
- simplified model, light bulb treated as a nonlinear resistor, thermal inertia neglected, dynamics of the bulb transients assumed fast enough ...
- reasonable, there is 1 s to settle down ...
- numerical simulation ...
- linear resistor essentially captured the phenomena
- but the transient looks different with the nonlinear model
- startup of the bulb slowed down
- but it eventually speeds up and becomes faster than the transient computed using linear resistor model
- for our application, lower starting voltage and about the same long lasting transient are the conclusions

conclusions: losses in the line

- telegrapher's equations for a line with losses …
- phasors applied to transform the problem from partial differential equations to ordinary differential equations
- scope reduced to sinusoidal signals, but that would be quite enough …
- frequency response of the system obtained …
- step response computed from the frequency response using the FFT technique
- initial voltage dies out pretty fast, in order of 100 ms
- no hope to get any light at the light bulb if you include losses in the model …

overall about transmission lines: such a simple system, two wires placed in parallel, and such a complex analysis!

conclusions, about the answer

so, what is correct (if it matters at all)?

a) $0.5\,\mathrm{s}$

b) $1\,\mathrm{s}$

c) $2\,\mathrm{s}$

d) $\frac{1\,\mathrm{m}}{c}$

e) none of the above

I would opt for **e)**; the system response is too complex to be reduced to any single number

however, the important thing is that people discussed electromagnetic theory; the goal is reached

the problem is **great!**

disclaimer

- while preparing these slides, I did not follow the topic online ...
- so many video materials became available ...
- I had no time to review them all, neither I would have ...
- for some reason this problem attracted attention, which is good
- and for some reason it initiated hard feelings, which I do not find good, neither suitable
- overall impression is that **something good happened**, we talked about electromagnetism
- so, no hard feelings, please!

thank you!