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Topologized Hamiltonian and Complete Graph

S. Vimala[1*] and K. Priyanka[2]

[1]Department of Mathematics, Mother Teresa Women's University, Kodaikanal, India.
[2]Research Scholar, Mother Teresa Women's University, Kodaikanal, India.

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Topological graph theory deals with embedding the graphs in Surfaces, and the graphs considered as a topological spaces. The concept topology extended to the topologized graph by the S_1 space and the boundary of every vertex and edge. The space is S_1 if every singleton in the topological space is either open or closed. Let G be a graph with n vertices and e edges and a topology defined on graph is called topologized graph if it satisfies the following:

- Every singleton is open or closed and
- For all $x \in X$, $|\partial(x)| \leq 2$, where $\partial(x)$ is the boundary of a point x.

This paper examines some results about the topological approach of the Complete Graph, Path, Circuit, Hamiltonian circuit and Hamiltonian path. And the results were generalized through this work.

Keywords: Path; circuit; Hamiltonian graph; topology.

MSC 2010: 05C38, 05C45.

**Corresponding author: E-mail: tvimss@gmail.com;*
E-mail: priyankaaug97@gmail.com

1 Introduction

Topological approach of the graphs deals with a different type of geometry. It studies the embedding of graphs in surfaces, spatial embedding of graphs, and graphs as topological spaces. It was developed to the other invariants such as skewness, thickness, crossing number, local crossing, coloring and covering ([1,2,3]).

A Hamiltonian path in a graph is a sequence of edges that uses each node precisely once. A Hamiltonian circuit [4], also called Hamiltonian cycle is a cycle in the graph which visits each node exactly once and return to the starting node. Through the comparative study between the Euler and Hamiltonian graphs [5] the idea of Hamiltonian path and Hamiltonian circuits were topologically developed. The concepts of paths and cycles were topologically developed to the finite and infinite graphs with ends and that were generalized of finite results and standard notations [6].

In 2005 Antoine Vella [7] tried to express combinatorial concepts in topological language as a part of investigation the classical topology, topologized graph, prepath, precycle, edge spaces, separation axioms were defined. Given a hypergraph H [8], the classical topology on$V_H \cup E_H$ is the collection of all sets U such that, if U contains a vertex v, then it also contains all hyperedges incident with v. It is interesting to note that all these topologies are either defined on the vertex set, or on the union of vertex set and edge set. All graphs are finite, simple, undirected graphs with no loops, multiple edges and planar. Every topological space considered here are finite. And the topology is defined on the set X which is the union of vertices (V) and edges (E) of the graph G.

2 Preliminaries

2.1 Definition: [9,10]

A topology on a set X is a collection τ of a subset of X with the following properties:

i. Φ and X are in τ.
ii. The union of the element of any subcollection of τ is in τ (arbitrary union).
iii. The intersection of the element of any finite subcollection of τ is in τ.

The set X for which a topology τ has been specified is called a topological space.

2.2 Definition: [7]

A topologized graph is a topological space X such that

- Every singleton is open or closed.
- $\forall x \in X$, $|\partial(x)| \leq 2$, where $\partial(x)$ is the boundary of a point x.

2.3 Definition: [7]

A hyperedge of a topological space is a point which open but not closed. A hyperedge of a topological space is an edge if its boundary consists of at most two points. An edge of a topological space is a loop if it has precisely one boundary point, a proper edge otherwise.

2.4 Definition: [4]

A simple graph in which there exists an edge between every pair of vertices is called a Complete Graph.

2.5 Definition: [4]

A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly one, except of course the starting vertex, at which the walk also terminates. If the removal of any one edge from a Hamiltonian circuit leaves as Hamiltonian Path.

3 Main Results

Theorem: 3.1. If G is a Complete graph K_n, then it is not a topologized graph, for $n \geq 4$.

Proof:

Let G be a complete graph K_n with topology on $V \cup E$. Let (X, τ) be a topological space.

Proof given by the method of mathematical induction,

For n=1, the graph consists one vertex in the graph (i.e.,) $| X | =1$, so the boundary of the vertex is zero. Hence K_1 is a topologized graph.

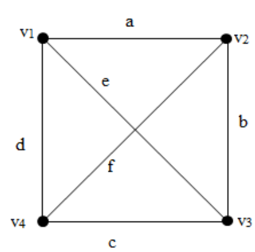
For n=2, $|X|=3$ the graph consists two vertices and one edge. The boundary of the two vertices is one and the boundary of the edge is two. Clearly K_2 is a topologized graph.

For n=3, $|X|=6$ the graph has three vertices and three edges. The boundary of each vertex is two and the boundary of each edge is two. So that K_3 is a topologized graph.

For n=4, $|X|=10$ the graph has four vertices and ten edges. The boundary of each edge is two but the boundary of each vertex is three. Clearly K_4 is not a topologized graph.

Every singleton is open or closed in K_n for $n \geq 4$. Since the singleton defined in the topology is either a vertex or an edge of the graph. But a complete graph with n vertices has degree n-1 for each vertex. Hence the boundary of all the vertices and edges of the graph is greater than 2. Thus for $n \geq 4$, K_n is not topologized graph. Hence the proof.

<u>Example: 3.1.1</u>: Let G be a complete graph K_4. Let $X=\{v_1,v_2,v_3, v_4, a, b, c, d, e, f\}$ be a topological space and a topology defined by $\tau = \{X, \phi,\{ v_1\},\{ v_4\},\{ v_1, v_4, c\},\{ v_1, v_4\}\}$ .Every singleton is open or closed.



K_4 is not a topologized graph

$\partial(v_1)=\{ v_2, v_3 ,v_4\}$; so that we have $| \partial(v_1) | = 3$.. $\partial(v_2)=\{ v_1, v_3 ,v_4\}$; so that we have $| \partial(v_2) | = 3$.
$\partial(v_3)=\{ v_1 ,v_2, v_4\}$; so that we have $| \partial(v_3) | = 3$. $\partial(v_4)=\{ v_1 ,v_2, v_3\}$; so that we have $| \partial(v_4) | = 3$.
$\partial(a)=\{ v_2, v_1\}$; so that we have $| \partial(a) |=2$. $\partial(b)=\{ v_2, v_3\}$; so that we have $| \partial(b) |=2$.
$\partial(c)=\{ v_3, v_4\}$; so that we have $| \partial(c) |=2$. $\partial(d)=\{ v_1, v_4\}$; so that we have $| \partial(d) |=2$.
$\partial(e)=\{ v_1, v_3\}$; so that we have $| \partial(e) |=2$. $\partial(f)=\{ v_2, v_4\}$; so that we have $| \partial(f) |=2$.

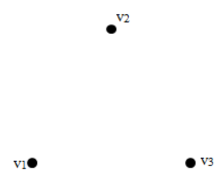
Here for every x ∈ X, | ∂(x) |≤ 2 is not satisfied. Hence, K_4 is not a topologized graph.

Theorem: 3.2. Show that $\overline{K}_n$ is topologized graph.

Proof:

Let G be a graph $\overline{K}_n$ with topology defined on V ∪ E. Let (X, τ) be a topological space. The complement of the complete graph is a null graph, since in complete graph each vertex is adjacent to the other. Hence, boundary of each vertex is zero. And every singleton is closed but not open. Hence, complement of the complete graph is topologized. Hence the proof.

Example: 3.2.1: Let G be a graph $\overline{K}_3$. Let X={ v_1 ,v_2 ,v_3 } be a topological space with topology defined by τ = {X, ϕ,{ v_1},{ v_2},{ v_1, v_2}} and every singleton is closed but not open.



Topologized graph $\overline{K}_3$

∂(v_1) = ϕ; so that we have | ∂(v_1) | = 0.
∂(v_2) = ϕ; so that we have | ∂(v_2) | = 0.
∂(v_3) = ϕ; so that we have | ∂(v_3) | = 0.

Clearly for all x ∈X, | ∂(x) | ≤ 2. Hence it is a topologized graph.

Theorem: 3.3. If G is circuit C_n, then it is a topologized graph, for n≥1.

Proof:

Let C_n be a circuit with topology defined on V ∪ E. Let (X, τ) be a topological space.

Proof given by the method of induction

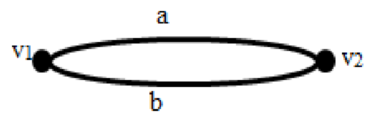
For n=1, |X|=2 graph consisting of a single vertex and a single loop. Clearly is a topologized graph.

For n=2, |X|=4 graph consisting of 2 vertices and 2 edges. Clearly, for all x∈ X, |∂(x) |≤2 and every {x} ∈ X is open or closed, since {x} is closed then any point y ∈ X\ {x} is open, the set E of points which are not closed is open, and its complement V is closed. So, is a topologized graph.

For n=3, |X|=6 graph consisting of 3 vertices and 3 edges. For all x∈ X, |∂(x) |≤2 and every {x} ∈ X is open or closed. Hence C_3 is a topologized graph.

Continuing this way we may assume that C_{n-1} is a topologized graph. Now we have to prove that C_n is a topologized graph. In a circuit degree of every vertex is two, so for all x ∈ X, |∂(x) |≤2 and every singleton is open or closed. So, C_n is a topologized graph. Hence every circuit is a topologized graph.

Example: 3.3.1: Let C_2 be a graph with topology defined on V ∪ E. Let X={ v_1,v_2, a, b} be a topological space with topology defined by τ = { X, ϕ,{ v_1},{v_2},{ v_1, v_2, a},{ v_1, v_2}}. Every singleton is open or closed.



Topologized graph C_2

$\partial(v_1) = \{v_2\}$; so that we have $|\partial(v_1)| = 1$ $\partial(v_2) = \{v_1\}$; so that we have $|\partial(v_2)| = 1$.
$\partial(a) = \{v_2, v_1\}$; so that we have $|\partial(a)| = 2$ $\partial(b) = \{v_2, v_1\}$; so that we have $|\partial(b)| = 2$.

Here for every $x \in X$, $|\partial(x)| \leq 2$. Hence, C_2 is a topologized graph.

Theorem: 3.4. If G is a path P_n , then it is a topologized graph, for $n \geq 1$.

Proof:

Let P_n be any path with topology defined on $V \cup E$. Let (X, τ) be a topological space.

Proof given by the method of induction

For n=1, |X|=1 graph consisting of a single vertex. Clearly is a topologized graph.

For n=2, |X|=3 graph consisting of 2 vertices and 1 edge. Clearly, for all $x \in X$, $|\partial(x)| \leq 2$ and every $\{x\} \in X$ is open or closed since $\{x\}$ is closed then any point $y \in X \setminus \{x\}$ is open, the set E of points which are not closed is open, and its complement V is closed. So, is a topologized graph.

For n=3, |X|=5 graph consisting of 3 vertices and 2 edges. For all $x \in X$, $|\partial(x)| \leq 2$ and every $\{x\} \in X$ is open or closed. Hence P_3 is a topologized graph.

Continuing this way we may assume that P_{n-1} is a topologized graph. Now we have to prove that P_n is a topologized graph. In a path degree of every vertex are two except the initial and terminal vertices which of degree one. So for all $x \in X$, $|\partial(x)| \leq 2$ and every singleton is open or closed. So, P_n is a topologized graph. Hence, every path is a topologized graph. Hence the proof.

<u>Example: 3.4.1</u>: Let P_3 be a graph with topology defined on $V \cup E$. Let $X = \{v_1, v_2, v_3, a, b\}$ be a topological space with topology defined by $\tau = \{X, \phi, \{v_1\}, \{v_2\}, \{v_1, v_2, a\}, \{v_1, v_2\}\}$ and every singleton is open or closed.

Topologized graph P_2

$\partial(v_1) = \{v_2\}$; so that we have $|\partial(v_1)| = 1$ $\partial(v_2) = \{v_1, v_3\}$; so that we have $|\partial(v_2)| = 2$.
$\partial(v_3) = \{v_3\}$; so that we have $|\partial(v_3)| = 1$ $\partial(a) = \{v_2, v_1\}$; so that we have $|\partial(a)| = 2$.
$\partial(b) = \{v_2, v_3\}$; so that we have $|\partial(b)| = 2$.

Here for every $x \in X$, $|\partial(x)| \leq 2$. Hence, P_3 is a topologized graph.

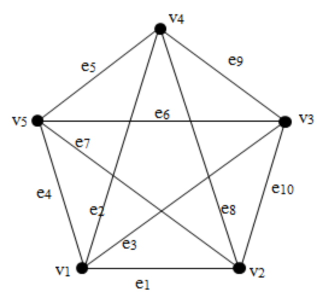
Theorem: 3.5. A graph G is topologized graph if and only if it is a path or a circuit.

Proof: The proof is obvious from the Theorem 3.3 and Theorem 3.4.

Theorem: 3.6. Every Hamiltonian circuit in a complete graph is topologized graph.

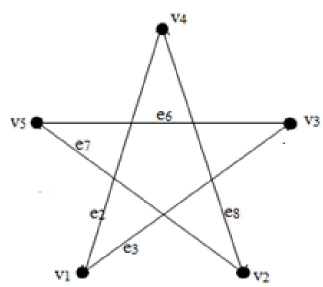
Proof: Let G be any complete graph with topology defined on V ∪ E. Let (X, τ) be a topological space. We know that C_n is a topologized graph. So, clearly every Hamiltonian circuit in a complete graph is a topologized graph. Hence the proof.

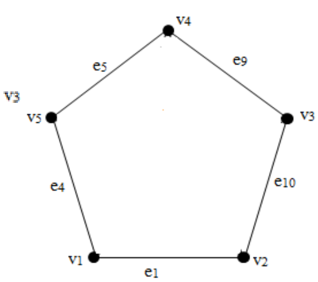
Example: 3.6.1: Let G be a complete graph K_5 with topology defined on V ∪ E. Let X = {v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}} be a topological space.

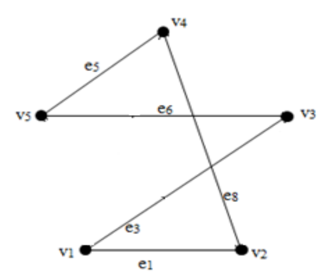


Complete graph K_5

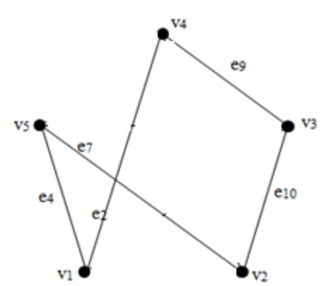
Some Hamiltonian circuits in K_5:

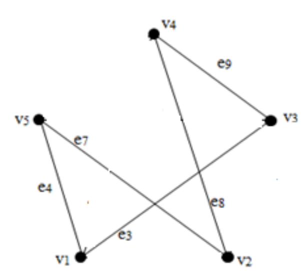






(i) v_1 e_2 v_4 e_8 v_2 e_7 v_5 e_6 v_3 e_3 v_1 (ii) v_1 e_1 v_2 e_{10} v_3 e_9 v_4 e_5 v_3 e_4 v_1 (iii) v_1 e_1 v_2 e_8 v_4 e_5 v_5 e_6 v_3 $e_3 v_1$





(iv) v_5 e_7 v_2 e_{10} v_3 e_9 v_4 e_2 v_1 e_4 v_5 **(v) v_3 e_9 v_4 e_8 v_2 e_7 v_5 e_4 v_1 e_3 v_3**

All the Hamiltonian circuits in this complete graph are C_5. Hence every circuit is topologized graph.

Now, we will show that the Hamiltonian circuit (v) $\mathbf{v_3\ e_9\ v_4\ e_8\ v_2\ e_7\ v_5\ e_4\ v_1\ e_3\ v_3}$ is a topologized graph.

Let H={ $\mathbf{v_1, v_2, v_3, v_4, v_5, e_9, e_8, e_7, e_4, e_3}$} be a topological space with topology defined by

$\tau = \{H, \phi, \{\mathbf{v_1}\}\{\mathbf{v_3}\}\{\mathbf{v_1, v_3, e_9}\}\{\mathbf{v_1, v_3}\}\}$ and every singleton in this topology is open or closed. Clearly the singletons are closed but not open. Here for every $x \in H$, $|\partial(x)| \leq 2$. Hence, it is a topologized graph.

Theorem: 3.7. Every Hamiltonian path in a complete graph is topologized graph.

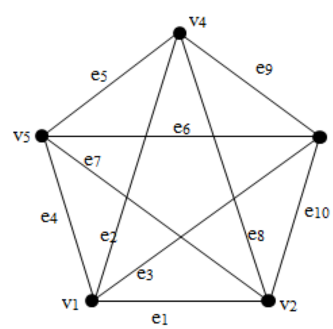
Proof:

Let G be any complete graph with topology defined on $V \cup E$. Let X be a topological space. We know That P_n is a topologized graph. So, clearly every Hamiltonian path in a complete graph is a topologized graph. Hence the proof.

Corollary: 3.7.1

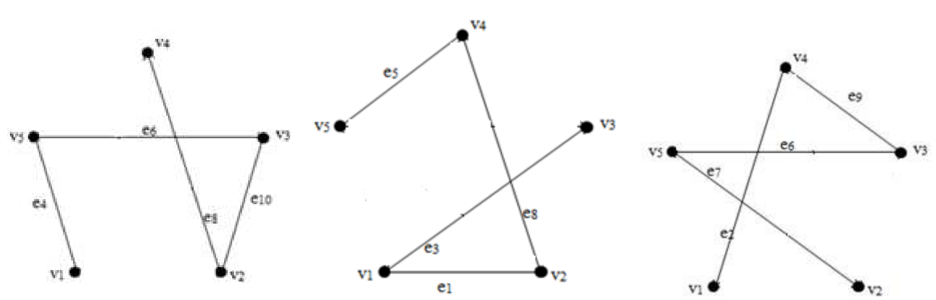
- Every Hamiltonian circuits / path is a topologized graph.
- Every Hamiltonian circuit in a topologized graph G is topologized graph. But the converse is not true. (i.e.) every topologized graph is not a Hamiltonian circuit may be a Hamiltonian path.

Example: 3.7.2: Let G be a complete graph K_5 with topology defined on $V \cup E$. Let X = {$v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$} be a topological space.

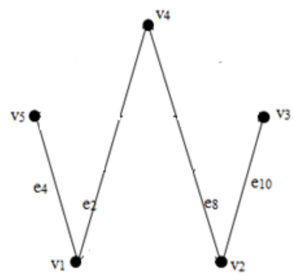


Complete graph K_5

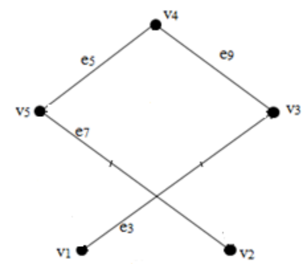
Some Hamiltonian circuits in K_5:



(i) $v_4\ e_8\ v_2\ e_{10}\ v_3\ e_6\ v_5\ e_4\ v_1$ **(ii) $v_3\ e_3\ v_1\ e_1\ v_2\ e_8\ v_4\ e_5\ v_5$** **(iii) $v_2\ e_7\ v_5\ e_6\ v_3\ e_9\ v_4\ e_2\ v_1$**



(iv) v_5 e_4 v_1 e_2 v_4 e_8 v_2 e_{10} v_3



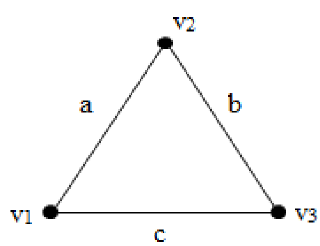
(v) v_1 e_3 v_3 e_9 v_4 e_5 v_5 e_7 v_2

All the Hamiltonian paths in this complete graph are P_5. Hence every path is topologized graph. Now, we will show that the Hamiltonian path (v) $\mathbf{v_4\ e_8\ v_2\ e_{10}\ v_3\ e_6\ v_5\ e_4\ v_1}$ is a topologized graph.

Let H={ $\mathbf{v_1}$,$\mathbf{v_2}$,$\mathbf{v_3}$,$\mathbf{v_4}$,$\mathbf{v_5}$,$\mathbf{e_4}$,$\mathbf{e_6}$,$\mathbf{e_8}$,$\mathbf{e_{10}}$} be a topological space with topology defined by τ = {H, ϕ, {$\mathbf{v_1}$}, { $\mathbf{v_3}$}, { $\mathbf{v_1}$,$\mathbf{v_3}$, $\mathbf{e_8}$}, { $\mathbf{v_1}$,$\mathbf{v_3}$}} and every singleton in this topology is open or closed.

Clearly the singletons are closed but not open. Here for every $x \in H$, $|\ \partial(x)\ | \leq 2$. Hence, it is a topologized graph.

<u>Example: 3.7.3</u>: Let G be complete graph K_3 which is a topologized graph. Let X={v_1,v_2,v_3,a,b,c} be a topological space and a topology defined by τ = {X, ϕ,{a},{b},{ v_1,a},{a, b},{v_1,a,b}}. Every singleton is open or closed.



Hamiltonian circuit of K_3

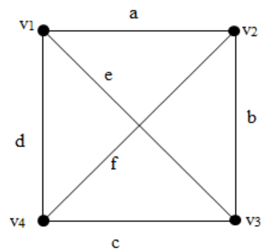
Let v_1 a v_2 bv_3 c v_1 be a Hamiltonian circuit in K_3. Clearly it is a topologized graph.

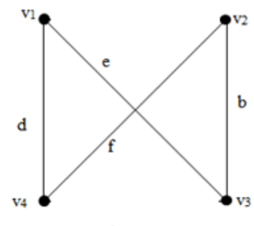
Let H = {v_1,v_2,v_3,a,b,c} be a topological space with the topology defined by

τ = {X, ϕ,{ v_2},{ v_3},{v_3, v_2},{a, b},{v_2, a, b},{ v_3, a, b}{ v_3, v_2, a, b}}.

Here for every $x \in H$, $|\ \partial(x)\ | \leq 2$. Hence it is a topologized graph. Hence a Hamiltonian circuit / path in a topologized graph are topologized.

<u>Example: 3.7.4</u>: Let G be a complete graph K_4. Let X={v_1,v_2,v_3, v_4, a, b, c, d, e, f} be a topological space with a topology defined by τ = {X, ϕ,{ v_1},{ v_4},{ v_1, v_4, c},{ v_1, v_4}} and every singleton is open or closed. But K_4 is not a topologized graph.





Complete graph K_4 **Hamiltonian Circuit of K_4**

Let v_1 e v_3 b v_2 f v_4 d v_1 be a Hamiltonian circuit in K_4. Clearly it is a topologized graph. Let H = {v_1, v_2, v_3, v_4, b, d, e, f} be a topological space with the topology defined by τ = {X, ϕ,{ v_2},{ v_3},{v_3, v_2},{e, b},{v_2, e, b},{ v_3, e, b}, { v_3, v_2, e, b}}.Every singleton is open or closed.

Here for every $x \in H$, $|\partial(x)| \leq 2$ is satisfied. Hence, it is a topologized graph. So that, every Hamiltonian circuit and Hamiltonian path in a non - topologized graph is also topologized.

4 Conclusion

In this paper some new results were found about the topologized graph through the Complete Graph, Path, Circuit, Hamiltonian path and Hamiltonian circuit of a complete graph.

Competing Interests

Authors have declared that no competing interests exist.

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