A New Relation Between Lerch's Φ and the Hurwitz Zeta

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Abstract

A new relation between the Lerch's transcendent, Φ, and the Hurwitz zeta, $\zeta(k, b)$, at the positive integers is introduced. It is derived simply by inverting the relation presented in the precursor paper with one of two approaches (its generating function or the binomial theorem). This enables one to go from Lerch as a function of Hurwitz zetas (of different orders), to Hurwitz as a function of Lerches. A special case of this new functional equation is a relation between the Riemann's zeta function and the polylogarithm.

1 Introduction

It's possible to invert the formulae that relate the Lerch transcendent function to the Hurwitz zeta from paper [2], using either their generating functions (plus their k-th derivatives) or simply the binomial theorem, and show those formulae from a different perspective. When that is done, the integrals that appear in the formula of the partial cases have finite limits.

The fact these formulas are so simple, relatively speaking, has to do with the fact they are not valid for all k (and that one is being expressed through the other). A formula for the Lerch Φ valid for all complex k can be created using two simple principles (the analytic continuation of the Bernoulli numbers through the zeta function, and a simple transformation, again through the binomial theorem), but the final result involves an integral of a transcendental function, as will be shown in a subsequent paper.

2 The partial Lerch Φ formula

In [2], we created the below formula for the partial Lerch Φ sums, which holds always as long as there are no singularities and k is an integer greater than zero:

$$\sum_{j=0}^{n}\frac{e^{m(j+b)}}{(j+b)^k}=\frac{e^{m\,b}}{2b^k}+\frac{e^{m(n+b)}}{2(n+b)^k}+\frac{1}{2b^k}\sum_{j=0}^{k-1}\frac{(mb)^j}{j!}-\frac{1}{2(n+b)^k}\sum_{j=0}^{k-1}\frac{(m(n+b))^j}{j!}$$
$$+\sum_{j=1}^{k}\frac{m^{k-j}}{(k-j)!}\sum_{q=0}^{n}\frac{1}{(q+b)^j}+\frac{m^k}{2(k-1)!}\int_0^1(1-u)^{k-1}\left(e^{m(n+b)u}-e^{m\,b\,u}\right)\coth\frac{mu}{2}\,du \quad (1)$$

Roughly speaking, this equation presents a formula for the partial Lerch Φ as a function of the various partial Hurwitz zeta functions. Note it's not straightforward to take the limit of (1) as n approaches infinity, as the harmonic progression (of order 1) referenced by the formula explodes out to infinity. This implies that the integral on the right-hand side also explodes out to infinity, but together they cancel out.

If $b = 0$, we have an interesting particular case, obtained by taking the limit as b tends to 0, as explained in [2] ($H_j(n)$ are the so-called harmonic numbers):

$$\sum_{j=1}^{n} \frac{e^{mj}}{j^k} = \frac{e^{mn}}{2n^k} - \frac{1}{2n^k} \sum_{j=0}^{k} \frac{(mn)^j}{j!} + \sum_{j=1}^{k} \frac{m^{k-j}}{(k-j)!} H_j(n) + \frac{m^k}{2(k-1)!} \int_0^1 (1-u)^{k-1} \left(e^{mnu} - 1\right) \coth \frac{mu}{2} \, du \quad (2)$$

It's possible to invert formula (1) and obtain the partial Hurwitz zeta function as a function of the various partial Lerch functions.

The idea is to develop the generating function of both the left and right-hand sides, as doing so we obtain the below function (which comes from the sum of the various partial Hurwitz zeta functions on the right-hand side):

$$e^{m\,x} \sum_{k=1}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(j+b)^k}$$

Therefore, taking the k-th derivatives of the generating function at $x = 0$, we're able to produce a formula for the partial Hurwitz function. Without showing the simple but time-consuming calculations, one arrives at:

$$(-m)^{-k} \sum_{j=1}^{n} \frac{1}{(j+b)^k} = \frac{(-m)^{-k}}{2(n+b)^k} - \frac{e^{m\,b}}{2} \sum_{j=1}^{k} \frac{(-m\,b)^{-j}}{(k-j)!} - \frac{e^{m(n+b)}}{2} \sum_{j=1}^{k} \frac{(-m(n+b))^{-j}}{(k-j)!} - \frac{1}{2} \sum_{j=0}^{k} \frac{(-m\,b)^{-j}}{(k-j)!} \sum_{q=0}^{j} \frac{(m\,b)^q}{q!} + e^{m\,b} \sum_{j=1}^{k} \frac{(-m)^{-j}}{(k-j)!} \sum_{q=0}^{n} \frac{e^{m\,q}}{(q+b)^j} + \frac{1}{2(k-1)!} \int_0^1 u^{k-1} \left(e^{m(n+b)u} - e^{m\,b\,u}\right) \coth \frac{mu}{2} \, du$$

Besides, using the below identity (discovered through the comparison of different methods used to derive this formula, as explained later on in this text),

$$\sum_{j=0}^{k} \frac{(-m\,b)^{-j}}{(k-j)!} \sum_{q=0}^{j} \frac{(m\,b)^q}{q!} = \frac{1}{(-m\,b)^k},$$

one finally obtains the neat final expression below:

$$\sum_{j=0}^{n}\frac{1}{(j+b)^k}=\frac{1}{2b^k}+\frac{1}{2(n+b)^k}-\frac{(-m)^k e^{m\,b}}{2}\sum_{j=1}^{k}\frac{(-m\,b)^{-j}}{(k-j)!}-\frac{(-m)^k e^{m(n+b)}}{2}\sum_{j=1}^{k}\frac{(-m(n+b))^{-j}}{(k-j)!}$$
$$+\,e^{m\,b}\sum_{j=1}^{k}\frac{(-m)^{k-j}}{(k-j)!}\sum_{q=0}^{n}\frac{e^{m\,q}}{(q+b)^j}+\frac{(-m)^k}{2(k-1)!}\int_0^1 u^{k-1}\left(e^{m(n+b)u}-e^{m\,b\,u}\right)\coth\frac{mu}{2}\,du \quad (3)$$

Now, unlike the previous case, it's straightforward to take the limit of (3) as n approaches infinity, as no terms explode out to infinity, unless m is a multiple of $2\pi\boldsymbol{i}$.

2.1 The partial zeta formula

A particular case is obtained taking the limit of (3) as b approaches zero:

$$\sum_{j=1}^{n}\frac{1}{j^k}=\frac{1}{2n^k}-\frac{(-m)^k}{2k!}-\frac{(-m)^k e^{m\,n}}{2}\sum_{j=1}^{k}\frac{(-m\,n)^{-j}}{(k-j)!}$$
$$+\sum_{j=1}^{k}\frac{(-m)^{k-j}}{(k-j)!}\sum_{q=1}^{n}\frac{e^{m\,q}}{q^j}+\frac{(-m)^k}{2(k-1)!}\int_0^1 u^{k-1}\left(e^{m\,n\,u}-1\right)\coth\frac{mu}{2}\,du \quad (4)$$

To find out this limit more easily it's necessary to go back to the form prior to the transformation.

3 The full Lerch Φ formula

Because one ends up with an integral with u^k instead of $(1-u)^k$, we are led to deduce that we'd be able to obtain the same result using the binomial theorem (for example, applied to $((1-u)-1)^k$).

The below is the Lerch Φ formula from reference [2]:

$$e^{m\,b}\,\Phi(e^m,k,b)=\frac{1}{2b^k}\left(e^{m\,b}-\sum_{j=0}^{k-2}\frac{(mb)^j}{j!}\right)+\sum_{j=2}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j,b)$$
$$+\frac{\pi\,m^k}{2(k-1)!}\cot\pi b-\frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{2\pi}\right)$$
$$-\frac{m^k}{2(k-1)!}\int_0^1(1-u)^{k-1}e^{m\,b\,u}\coth\frac{mu}{2}+\frac{2\pi}{m}\left(-1+\frac{\sin 2\pi bu}{\sin 2\pi b}\right)\cot\pi u\,du, \quad (5)$$

And to recap, the limit of this expression when b approaches zero yields a similar relation for the polylogarithm:

$$\mathrm{Li}_k(e^m) = -\frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{2\pi}\right) + \sum_{\substack{j=0\\j\neq 1}}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j) - \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\coth\frac{mu}{2} - \frac{2\pi}{m}(1-u)\cot\pi u\,du \quad (6)$$

Below, the k in equation (5) was replaced with j so this idea can be demonstrated more clearly. The first noticeable thing is that it's necessary to divide both sides of the equation by $m^j/(j-1)!$ for the aforementioned reasoning to work (which causes some terms to become independent of j and subsequently vanish). Below, the parts that sum up to zero have been removed and possible simplifications have already been made on the right-hand side:

$$e^{m\,b}\sum_{j=1}^{k}\frac{(-1)^{k-j}}{(j-1)!(k-j)!}\frac{(j-1)!}{m^j}\Phi(e^m,j,b) =$$

$$\frac{1}{2}\sum_{j=1}^{k}\frac{(-1)^{k-j}}{(k-j)!}\left(\frac{1}{(m\,b)^j}\left(e^{m\,b} - \sum_{q=0}^{j-2}\frac{(mb)^q}{q!}\right) + 2\sum_{q=2}^{j}\frac{m^{-q}}{(j-q)!}\zeta(q,b) - \int_0^1 \frac{(1-u)^{j-1}}{(j-1)!}e^{m\,b\,u}\coth\frac{mu}{2}\,du\right)$$

After all the calculations are done, we obtain the following:

$$\zeta(k,b) = -\frac{(-m)^k}{2}\sum_{j=1}^{k}\frac{(-m\,b)^{-j}}{(k-j)!}\left(e^{m\,b} - \sum_{q=0}^{j-2}\frac{(mb)^q}{q!}\right) + e^{m\,b}\sum_{j=1}^{k}\frac{(-m)^{k-j}}{(k-j)!}\Phi(e^m,j,b) - \frac{(-m)^k}{2(k-1)!}\int_0^1 u^{k-1}e^{m\,b\,u}\coth\frac{mu}{2}\,du$$

This formula can be further simplified with the surprising identity (discovered through comparison of the results from the binomial and the generating function approaches):

$$(-m)^k\sum_{j=1}^{k}\frac{(-m\,b)^{-j}}{(k-j)!}\sum_{q=0}^{j-2}\frac{(mb)^q}{q!} = \frac{1}{b^k},$$

which finally yields the final relation that we were looking for (which should hold always, except when m has real part greater than or equal to zero and absolute imaginary part greater than 2π, that is, $\Re(m) >= 0$ and $|\Im(m)| > 2\pi$):

$$\zeta(k,b) = \frac{1}{2b^k} - \frac{(-m)^k e^{m\,b}}{2}\sum_{j=1}^{k}\frac{(-m\,b)^{-j}}{(k-j)!} + e^{m\,b}\sum_{j=1}^{k}\frac{(-m)^{k-j}}{(k-j)!}\Phi(e^m,j,b) - \frac{(-m)^k}{2(k-1)!}\int_0^1 u^{k-1}e^{m\,b\,u}\coth\frac{mu}{2}\,du \quad (7)$$

Applying the same reasoning to the particular case, equation (4), leads to:

$$\zeta(k) = -\frac{(-m)^k}{2k!} + \sum_{j=1}^{k} \frac{(-m)^{k-j}}{(k-j)!}\mathrm{Li}_j(e^m) - \frac{(-m)^k}{2(k-1)!}\int_0^1 u^{k-1}\coth\frac{mu}{2}\,du \tag{8}$$

A great upside of (7) is the fact that it now holds for integer or half-integer b.

3.1 Removing singularities

When $m = \pm 2\pi\boldsymbol{i}$, the formulae (7) and (8) don't work, since $\Phi(1,1,b)$ and $\mathrm{Li}_1(1)$ are infinity. It's still possible to fix these formulae by replacing $\Phi(1,1,b)$ and $\mathrm{Li}_1(1)$ with their borderline expressions, from (5) and (6), respectively (that is, by considering that they hold for $k = 1$ as well).

For example, when b is zero, if $k > 1$, one has:

$$\zeta(k) = -\frac{(-2\pi\boldsymbol{i})^k}{2k!} + \frac{(-2\pi\boldsymbol{i})^k}{4(k-1)!} + \sum_{j=2}^{k}\frac{(-2\pi\boldsymbol{i})^{k-j}\zeta(j)}{(k-j)!}$$

4 The limits of the integrals

From the comparison of formulae (3) and (7), we can deduce that the following limit holds for $k \geq 2$:

$$\lim_{n\to\infty}\int_0^1 u^{k-1}\left(e^{m(n+b)u} - e^{m\,b\,u}\right)\coth\frac{mu}{2}\,du = -\int_0^1 u^{k-1}e^{m\,b\,u}\coth\frac{mu}{2}\,du$$

For $k = 1$ the limit is not finite, but its asymptotic behavior can be inferred from (3).

Even though it's possible to figure out the limit of the similar integral from equation (1) by means of the binomial theorem and relation (7), we already know the result. It appears in formula (5) and holds if $k \geq 1$:

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j+b} + \int_0^1 (1-u)^{k-1}\left(e^{m(n+b)u} - e^{m\,b\,u}\right)\coth\frac{mu}{2}\,du = -\frac{1}{2b} + \frac{\pi}{2}\cot\pi b - \log\left(-\frac{m}{2\pi}\right)\\ -\frac{m}{2}\int_0^1 (1-u)^{k-1}e^{m\,b\,u}\coth\frac{mu}{2} + \frac{2\pi}{m}\left(-1 + \frac{\sin 2\pi bu}{\sin 2\pi b}\right)\cot\pi u\,du\end{aligned} \tag{9}$$

Here it's possible to make a transformation which makes clearer the process by which this limit was calculated. Noting the generating function of the zeta function at the positive integers from [1],

$$\sum_{j=2}^{\infty}(-b)^{j-1}\zeta(j) = -\frac{1}{2b} + \frac{\pi}{2}\cot\pi b - \pi\int_0^1\left(\frac{\sin 2\pi bu}{\sin 2\pi b} - u\right)\cot\pi u\,du,$$

it's possible to rearrange the formula in a way such that,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j+b}+\int_0^1 (1-u)^{k-1}\left(e^{m(n+b)u}-e^{m\,b\,u}\right)\coth\frac{mu}{2}\,du =$$
$$\sum_{j=2}^{\infty}(-b)^{j-1}\zeta(j)-\log\left(-\frac{m}{2\pi}\right)+\int_0^1 -\frac{m}{2}(1-u)^{k-1}e^{m\,b\,u}\coth\frac{mu}{2}+\pi(1-u)\cot\pi u\,du,$$

and therefore formula (9) provides an analytic continuation for both the limit of the integral and the generating function of the zeta at the positive integers.

This alternative formula that features a series comprised of the zeta function at the integers is no coincidence, as the harmonic progression can be expressed as the sum over j of the geometric series, as shown below:

$$\sum_{j=1}^{n}\frac{1}{j+b}=\sum_{j=1}^{n}\sum_{q=0}^{\infty}\frac{(-b)^q}{j^{q+1}}\sim H(n)+\sum_{j=2}^{\infty}(-b)^{j-1}\zeta(j)$$

References

[1] Risomar Sousa, Jose *Generalized Harmonic Numbers, eprint arXiv:1810.07877,* 2018.

[2] Risomar Sousa, Jose *Lerch's Φ and the Polylogarithm at the Positive Integers, eprint arXiv:2006.08406,* 2020.