The Hurwitz Zeta Function at the Positive Integers

Jose Risomar Sousa

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Abstract

We address the problem of finding out the values of the Hurwitz zeta function at the positive integers k, $\zeta(k,b)$, by working out their real and imaginary parts separately and then combining them. A few different formulae for the Hurwitz zeta function are known from the literature, but they are very general and usually hold for $\Re(k) > 1$. The advantage of formulae that only hold at the positive integers is the fact that they are usually simpler and easier to work with. We also obtain an analytic continuation for the generating function of $\zeta(k,b)$ as $\sum_{k\geq 2} x^k(\zeta(k,b) - 1/b^k)$, valid for complex x and b, where term $1/b^k$ was subtracted for convenience.

Summary

1 Introduction

The Hurwitz zeta function, $\zeta(k,b)$, is one of the many generalizations that were thought out for the Riemann zeta function. Its importance lies on its relation to Riemann's zeta function and consequently to the Riemann hypothesis itself, to some degree.

In this article we create a formula for $\zeta(k,b)$ that holds at the positive integers k. The advantage of formulae that only hold at the positive integers is the fact we expect them to be simpler and easier to work with. It's an obvious statement if, for example, we think about the closed-forms of the zeta function at the positive integers greater than 1 and its general integral, valid for $\Re(k) > 1$.

Findings from previous papers I wrote on generalized harmonic numbers, $H_k(n)$[2], and generalized harmonic progressions, $HP_k(n)$[3,4], make it extremely easy to figure out the limits of both when n goes to infinity, which matters because $\zeta(k, b)$ is the limit of $HP_k(n)$ when n tends to infinity.

Hence, for the problem at hand we can rely on the formula for $HP_k(n)$ from [4], since it allows for non-integer b.

First, let's recall the formula, which holds for any complex b, except for a zero measure subset of $\mathbb{C}$ (that is, $\boldsymbol{i}b \in \mathbb{Z}$). For simplicity, let's assume that $a = 1$ and that b is real, so we know which part of the formula is real and which part is a pure complex. Then, $HP_k(n)$ is given by:

$$\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j + b)^k} = -\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n + b)^k}$$
$$+ (2\pi)^k e^{-2\pi b} \int_0^1 \left(\frac{(1-u)^{k-1}}{(k-1)!} + \sum_{j=1}^{k} \frac{\mathrm{Li}_{-j+1}(e^{-2\pi b})(1-u)^{k-j}}{(j-1)!(k-j)!} \right) e^{\pi u(\boldsymbol{i}n+2b)} \sin \pi n u \cot \pi u \, du,$$

where $\mathrm{Li}_{-j+1}(e^{-2\pi b})$ is the polylogarithm of order $-j+1$. The polylogarithm, $\mathrm{Li}_k(z)$, is the analytic continuation of a Dirichlet series given by:

$$\mathrm{Li}_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k}$$

The polynomial in u that goes in the integrand is generated by the below function:

$$f(x) = -\frac{2\pi x \, e^{2\pi x(1-u)}}{e^{2\pi x} - e^{2\pi b}} \Rightarrow \frac{f^{(k)}(0)}{k!} = (2\pi)^k e^{-2\pi b} \left(\frac{(1-u)^{k-1}}{(k-1)!} + \sum_{j=1}^{k} \frac{\mathrm{Li}_{-j+1}(e^{-2\pi b})(1-u)^{k-j}}{(j-1)!(k-j)!} \right)$$

Assuming b is real, this integral can be transformed using Euler's formula for the exponential of a complex argument, and then replacing $\cos \pi n u \sin \pi n u$ and $\sin^2 \pi n u$ with equivalent expressions. Let's also introduce the Kronecker delta (δ_{ij}) in the formula to make it shorter:

$$\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j + b)^k} = -\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n + b)^k}$$
$$+\frac{(2\pi)^k}{2} \int_0^1 \sum_{j=1}^{k} \frac{\left(\delta_{1j} + \mathrm{Li}_{-j+1}(e^{-2\pi b})\right) u^{k-j}}{(j-1)!(k-j)!} e^{-2\pi b u} \left(\sin 2\pi n(1-u) + \boldsymbol{i}(1 - \cos 2\pi n(1-u))\right) \cot \pi(1-u) \, du,$$

where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

2 Limit of $HP_k(n)$ for real b

In this section we obtain the real and imaginary parts of the limit of $HP_k(n)$ as n approaches infinity separately, assuming b is real. At the end, we obtain $\zeta(k, -\boldsymbol{i}b)$ by means of the relation:

$$\zeta(k, -\boldsymbol{i}b) = \boldsymbol{i}^k \sum_{j=0}^{\infty} \frac{1}{(\boldsymbol{i}j + b)^k} \tag{1}$$

This solution provides another proof that $HP(n)$ diverges, though that is not the focus.

2.1 The real part

For the real part, when n is large, we have:

$$\Re\left(\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k}\right) \sim -\frac{1}{2b^k} + \frac{(2\pi)^k}{2} \int_0^1 \sum_{j=1}^{k} \frac{\left(\delta_{1j} + \mathrm{Li}_{-j+1}(e^{-2\pi b})\right) u^{k-j}}{(j-1)!(k-j)!} e^{-2\pi b u} \sin 2\pi n(1-u) \cot \pi(1-u)\, du$$

Let's recall a result that appeared in reference [2], whose proof depends on formulae that feature in Abramowitz and Stegun[1]:

Theorem 1 $\displaystyle \lim_{n\to\infty} \int_0^1 u^k \sin 2\pi n(1-u) \cot \pi(1-u)\, du = \begin{cases} 1, & \text{if } k = 0 \\ \frac{1}{2}, & \text{if integer } k > 0 \end{cases}$

If we look at the real part of the formula only, we can split the sum inside the integral into a polynomial in u and a constant, since the limits for each are different. And since $e^{-2\pi b u}$ is a polynomial in u itself, when they are multiplied together, as below, in only one of the four possible cases the limit as n approaches infinity is 1 (all others being 1/2):

$$\lim_{n\to\infty} \int_0^1 \left(\frac{\delta_{1k} + \mathrm{Li}_{-k+1}(e^{-2\pi b})}{(k-1)!} + \sum_{j=1}^{k-1} \frac{\left(\delta_{1j} + \mathrm{Li}_{-j+1}(e^{-2\pi b})\right) u^{k-j}}{(j-1)!(k-j)!} \right) \left(1 + \sum_{v=1}^{\infty} \frac{(-2\pi b)^v u^v}{v!}\right) \sin 2\pi n(1-u) \cot \pi(1-u)\, du$$

This leads us to the below limit:

$$\frac{\delta_{1k} + \mathrm{Li}_{-k+1}(e^{-2\pi b})}{(k-1)!} \left(1 + \frac{-1 + e^{-2\pi b}}{2}\right) + \frac{e^{-2\pi b}}{2} \sum_{j=1}^{k-1} \frac{\left(\delta_{1j} + \mathrm{Li}_{-j+1}(e^{-2\pi b})\right) u^{k-j}}{(j-1)!(k-j)!}$$

Therefore, as a consequence of theorem 1, after we do all the necessary algebra we conclude that if b is real then:

$$\Re\left(\sum_{j=0}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \frac{1}{2b^k} + \frac{(2\pi)^k \left(\delta_{1k} + \mathrm{Li}_{-k+1}(e^{-2\pi b})\right)}{4(k-1)!} + \frac{(2\pi)^k e^{-2\pi b}}{4} \sum_{j=1}^{k} \frac{\delta_{1j} + \mathrm{Li}_{-j+1}(e^{-2\pi b})}{(j-1)!(k-j)!}$$

Note j starts at 0 to coincide with the Hurwitz zeta function, so we need to add $1/b^k$.

The real part of this infinite sum is finite even when $k = 1$, unlike the imaginary part. Next, we try to apply the same reasoning for the imaginary part.

2.2 The imaginary part

For the imaginary part, when n is large, we have:

$$\Im\left(\sum_{j=1}^{n}\frac{1}{(\boldsymbol{i}j+b)^k}\right)\sim\frac{(2\pi)^k}{2}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi b})\right)u^{k-j}}{(j-1)!(k-j)!}e^{-2\pi bu}(1-\cos 2\pi n(1-u))\cot\pi(1-u)\,du$$

So, to fully understand what happens in the case of the imaginary part, we need to go back to one of the results from [2]. It provides us with an equivalence between certain integrals and generalized harmonic numbers. More specifically, we've seen that:

$$\int_0^1(1-u)^{2k+1}(1-\cos 2\pi nu)\cot\pi u\,du=\frac{2(-1)^k(2k+1)!}{(2\pi)^{2k+1}}\left(\sum_{j=0}^{k}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!}H_{2j+1}(n)-\frac{1}{2n^{2k+1}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+1)!}\right)$$

$$\int_0^1(1-u)^{2k+2}(1-\cos 2\pi nu)\cot\pi u\,du=\frac{2(-1)^k(2k+2)!}{(2\pi)^{2k+1}}\left(\sum_{j=0}^{k}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}H_{2j+1}(n)-\frac{1}{2n^{2k+1}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!}\right)$$

Let's consider their limit as n approaches infinity. If n is sufficiently large:

$$\int_0^1(1-u)^{2k+1}(1-\cos 2\pi nu)\cot\pi u\,du\sim\frac{\gamma+\log n}{\pi}+\frac{2(-1)^k(2k+1)!}{(2\pi)^{2k+1}}\sum_{j=1}^{k}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!}\zeta(2j+1)$$

$$\int_0^1(1-u)^{2k+2}(1-\cos 2\pi nu)\cot\pi u\,du\sim\frac{\gamma+\log n}{\pi}+\frac{2(-1)^k(2k+2)!}{(2\pi)^{2k+1}}\sum_{j=1}^{k}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}\zeta(2j+1)$$

Using findings from [2], we can obtain the following analytic continuation for these approximations[1], which holds for any complex k such that $\Re(k)>0$:

$$\int_0^1(1-u)^k(1-\cos 2\pi nu)\cot\pi u\,du\sim\frac{\gamma+\log n}{\pi}-\int_0^1(u^k-u)\cot\pi u\,du$$

Now we can see that, unlike the integrals that appear in the formula of the real part, these new ones all explode out to infinity as n goes to infinity.

Hence, like in theorem 7 from [2], a linear combination of these integrals, say $p(u)=\sum_k a_k u^k$,

$$\int_0^1 p(u)(1-\cos 2\pi n(1-u))\cot\pi(1-u)\,du,$$

will only converge if $p(1)=0$. But that is not a problem, since fortunately,

$$\int_0^1(1-\cos 2\pi n(1-u))\cot\pi(1-u)\,du=0,\ \forall\text{ integer } n,$$

[1]It stems from the expressions for $C^m_{2k+1}(n)$ and $S^m_{2k}(n)$ and for their limits, C^m_{2k+1} and S^m_{2k}.

which means that, without altering the result whatsoever for integer n, we can change the formula at the beginning of this section into:

$$\frac{(2\pi)^k}{2}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi b})\right)\left(u^{k-j}e^{-2\pi bu}-e^{-2\pi b}\right)}{(j-1)!(k-j)!}(1-\cos 2\pi n(1-u))\cot\pi(1-u)\,du,$$

and therefore, since the infinities now cancel out, we can conclude that:

$$\Im\left(\sum_{j=0}^{\infty}\frac{1}{(\boldsymbol{i}j+b)^k}\right)=-\frac{(2\pi)^k}{2}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi b})\right)\left(u^{k-j}e^{-2\pi bu}-e^{-2\pi b}\right)}{(j-1)!(k-j)!}\cot\pi u\,du$$

The imaginary part doesn't converge when $k=1$ since the above integral doesn't converge.

3 Hurwitz zeta function

The relation (1) implies that when b is a real or purely imaginary number, the formulae we created allow us to separate the real and imaginary parts of $\zeta(k,-\boldsymbol{i}b)$.

For some lucky coincidence, when the two parts are combined the formula holds even when b is not real, so let's make a simple transformation to obtain the proper Hurwitz zeta function. For integer $k\geq 2$:

$$\zeta(k,b)=\frac{1}{2b^k}+\frac{(2\pi\boldsymbol{i})^k\left(\delta_{1k}+\mathrm{Li}_{-k+1}(e^{-2\pi\boldsymbol{i}b})\right)}{4(k-1)!}+\frac{(2\pi\boldsymbol{i})^k e^{-2\pi\boldsymbol{i}b}}{4}\sum_{j=1}^{k}\frac{\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi\boldsymbol{i}b})}{(j-1)!(k-j)!}$$
$$-\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi\boldsymbol{i}b})\right)\left(u^{k-j}e^{-2\pi\boldsymbol{i}b\,u}-e^{-2\pi\boldsymbol{i}b}\right)}{(j-1)!(k-j)!}\cot\pi u\,du$$

3.1 A special case

I didn't find the below closed-form in the literature (meaning, the software Mathematica), though I can't be sure if it's really unknown (it's very probably not):

$$\zeta\left(2,\frac{5}{4}\right)=-16+\pi^2+8G,\text{ where } G \text{ is Catalan's constant.}$$

4 The $\zeta(k,b)$ generating function

Now, assuming again that b is real, we derive a generating function for $HP_k(n)$ and determine its limit when n goes to infinity:

$$\sum_{k=1}^{\infty}x^k\sum_{j=1}^{n}\frac{1}{(\boldsymbol{i}j+b)^k}=\sum_{k=1}^{\infty}-\frac{x^k}{2b^k}+\frac{x^k}{2(\boldsymbol{i}n+b)^k}+$$
$$\frac{1}{2}\sum_{k=1}^{\infty}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi b})\right)(2\pi x)^j}{(j-1)!}\frac{(2\pi x(1-u))^{k-j}}{(k-j)!}e^{-2\pi b(1-u)}\left(\sin 2\pi nu+\boldsymbol{i}(1-\cos 2\pi nu)\right)\cot\pi u\,du$$

The sum within the integral is the sum of the product of the general terms of two power series whose generating functions we know:

$$p(x) = \frac{-2\pi x}{(e^{2\pi x} - e^{2\pi b})} \cdot e^{2\pi x(1-u)}$$

Therefore, the integrand is the power series of the product of these two functions, and we can rewrite the formula as:

$$\sum_{k=1}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = \sum_{k=1}^{\infty} x^k \left(-\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k}\right) - \frac{\pi x}{e^{2\pi x} - e^{2\pi b}} \int_0^1 e^{2\pi x(1-u)} e^{2\pi b u} \left(\sin 2\pi n u + \boldsymbol{i}(1-\cos 2\pi n u)\right) \cot \pi u \, du$$

Since we want to take the limit of that expression as n goes to infinity, we need to subtract the harmonic progression of order 1, which diverges (our method will show us once more why this is the case, later down the line). Thus we have:

$$\sum_{k=2}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = \sum_{k=2}^{\infty} x^k \left(-\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k}\right) - \pi x \int_0^1 \left(\frac{1}{e^{2\pi b} - 1} + \frac{e^{2\pi x(1-u)}}{e^{2\pi x} - e^{2\pi b}}\right) e^{2\pi b u} \left(\sin 2\pi n u + \boldsymbol{i}(1-\cos 2\pi n u)\right) \cot \pi u \, du$$

4.1 The real part

Let's tackle the limit of the real part first. By following the same thought process that has been laid out in section (2.1), we conclude that the real part is given by:

$$\Re\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \frac{x^2}{2b(x-b)} + \frac{\pi x(e^{2\pi x} - 1)}{(e^{-2\pi b} - 1)(e^{2\pi x} - e^{2\pi b})}$$

4.2 The imaginary part

When I first wrote this demonstration, I still hadn't had the insight that I used to derive the limit of the imaginary part from section (2.2), so the following rationale may be unnecessarily convoluted.

When it comes to the imaginary part, we can obviously discard the two terms outside of the integral, as one is real and the other one goes to zero as n goes to infinity, leaving us with:

$$\Im\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \lim_{n\to\infty} -\pi x \int_0^1 \left(\frac{1}{e^{2\pi b} - 1} + \frac{e^{2\pi x(1-u)}}{e^{2\pi x} - e^{2\pi b}}\right) e^{2\pi b u} (1-\cos 2\pi n u) \cot \pi u \, du$$

Here we realize that in order to figure out the above limits, we need to solve the below problem. To solve the two limits at once, let's use $c = c(x)$ as the coefficient of u:

$$\lim_{n\to\infty} \int_0^1 e^{-c\,u}(1-\cos 2\pi n u)\cot \pi u\, du$$

Just to be sure we're not missing anything, let's make our initial formula more in sync with the asymptotic formulae from (2.2) by changing u for $1-u$:

$$\Im\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \lim_{n\to\infty} -\pi x e^{2\pi b}\int_0^1 \left(\frac{e^{-2\pi b u}}{e^{2\pi b}-1} + \frac{e^{2\pi(x-b)u}}{e^{2\pi x}-e^{2\pi b}}\right)(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du$$

Also, let's introduce a variable y in the integral we need to evaluate, to help us with the power series manipulations that we need to perform. In the end we just need to remember to set y to 1 (and c to $2\pi b$ or $2\pi(x-b)$):

$$\int_0^1 e^{-cy\,u}(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du =$$
$$\int_0^1 \left(1+\sum_{k=0}^{\infty} \frac{(-cy)^{2k+1}}{(2k+1)!}u^{2k+1} + \frac{(-cy)^{2k+2}}{(2k+2)!}u^{2k+2}\right)(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du$$

Note we can ignore the constant 1, since that integral is 0 for all integer n. After we replace each $u^k(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du$ with their asymptotic formulae from section (2.2), part of the above expression reduces to:

$$\frac{\gamma+\log n}{\pi}\sum_{k=0}^{\infty} \frac{-c^{2k+1}}{(2k+1)!} + \frac{c^{2k+2}}{(2k+2)!} = \frac{\gamma+\log n}{\pi}(-\sinh c - 1 + \cosh c)$$

This part explodes out to infinity, and it's due to $HP(n)$, as mentioned in the introduction, which we subtracted from the generating function. Therefore, we expect these infinities to cancel out when we add up the terms that have $c = 2\pi b$ and $c = -2\pi(x-b)$ in our initial formula.

Now, let's see the part that doesn't diverge:

$$\frac{1}{\pi}\sum_{k=0}^{\infty} -(cy)^{2k+1}\sum_{j=1}^{k} \frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1) + (cy)^{2k+2}\sum_{j=1}^{k} \frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1)$$

It's not a very simple sum, and it's not very easy to know how to proceed from here. But, let's recall the integral representation we derived for $\zeta(2j+1)$ in [2], only here it's been slightly modified:

$$\zeta(2j+1) = -\frac{(-1)^j(2\pi)^{2j+1}}{2}\int_0^1 \sum_{p=0}^{j} \frac{B_{2p}\,(2-2^{2p})\,u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot \pi u\, du$$

Let's start with the first part of the previous sum:

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\left(-\frac{(-1)^j(2\pi)^{2j+1}}{2}\int_0^1\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot\pi u\,du\right)\Rightarrow$$

$$\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{1}{(2k+1-2j)!}\int_0^1\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot\pi u\,du=$$
$$\frac{1}{cy}\int_0^1\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(cy)^{2k+1-2j}}{(2k+1-2j)!}(cy)^{2j+1}\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}-\frac{(cy)^{2k+2}u}{(2k+1)!}\right)\cot\pi u\,du$$

Again, the above power series is the product of two somewhat familiar functions. One of them is obviously $\sinh cy$, and the other one (which appeared in [2] in non-hyperbolic form) is:

$$\sum_{j=0}^{\infty}(cy)^{2j+1}\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}=cy\operatorname{csch}cy\sinh cuy$$

Therefore, the final conclusion is:

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1)=\sinh cy\int_0^1(\operatorname{csch}cy\sinh cuy-u)\cot\pi u\,du$$

And since the second part follows an analogous thought process, its development is omitted, but the end result is below:

$$\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1)=(1-\cosh cy)\int_0^1(\operatorname{csch}cy\sinh cuy-u)\cot\pi u\,du$$

Now, we can sum it all up by making $y=1$:

$$\frac{1}{\pi}\sum_{k=0}^{\infty}-c^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1)+c^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1)=$$
$$-\left(e^{-c}-1\right)\int_0^1(\operatorname{csch}c\sinh cu-u)\cot\pi u\,du$$

To wrap it up, we just need to evaluate the initial expression with the above identity, and when we do so we find that:

$$\Im\left(\sum_{k=2}^{\infty}x^k\sum_{j=1}^{\infty}\frac{1}{(\boldsymbol{i}j+b)^k}\right)=\pi x\int_0^1(\operatorname{csch}2\pi(x-b)\sinh 2\pi(x-b)u-\sinh 2\pi b\operatorname{csch}2\pi bu)\cot\pi u\,du$$

4.3 Conclusion

If we don't assume that b is real then, provided that $\boldsymbol{i}b \notin \mathbb{Z}$ (and provided that the right-hand side doesn't contain singularities, such as $b = x$), we can simply state that, for $b, x \in \mathbb{C}$, when the limit exists it's given by:

$$\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k} = \frac{x^2}{2b(x-b)} + \frac{\pi x(e^{2\pi x}-1)}{(e^{-2\pi b}-1)(e^{2\pi x}-e^{2\pi b})} + \boldsymbol{i}\pi x \int_0^1 (\operatorname{csch} 2\pi(x-b) \sinh 2\pi(x-b)u - \operatorname{csch} 2\pi b \sinh 2\pi b u) \cot \pi u \, du$$

However, when the left-hand side diverges, the expression on the right can still converge, if it doesn't have singularities, meaning that it's an analytic continuation of the left-hand side.

The above can be turned into a better looking equation without non-real numbers, which in principle holds if $b \neq 0$, $b \neq x$ and $2b$ and $2(x-b)$ are not integers:

$$f(x) = \sum_{k=2}^{\infty} x^k \left(\zeta(k,b) - \frac{1}{b^k} \right) = \frac{x^2}{2b(x-b)} - \frac{\pi x \sin \pi x}{2 \sin \pi b} \csc \pi(x-b) - \pi x \int_0^1 \left(\frac{\sin 2\pi(x-b)u}{\sin 2\pi(x-b)} - \frac{\sin 2\pi b u}{\sin 2\pi b} \right) \cot \pi u \, du$$

Notice that $x = 2b$ causes the integral to vanish, which hints at a possible set of solutions for the zeros of this equation, though we don't pursue it.

To know what the formula looks like when b is a positive integer, we can rely on the below identity:

$$f(x) = \sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(j+b)^k} = \sum_{k=2}^{\infty} x^k (\zeta(k) - H_k(b))$$

In general, when b is an integer, we can obtain $f(x)$ by using the generating functions we created for $H_k(n)$ and $\zeta(k)$ in [2], which, after all the necessary calculations are performed, leads us to:

$$f(x) = \begin{cases} \frac{x^2}{2b(x-b)} - \frac{\pi x \sin \pi x}{2 \sin \pi b} \csc \pi(x-b) - \pi x \int_0^1 \left(\frac{\sin 2\pi(x-b)u}{\sin 2\pi(x-b)} - \frac{\sin 2\pi b u}{\sin 2\pi b} \right) \cot \pi u \, du & , \text{if } 2b \notin \mathbb{Z} \\ \frac{1}{2} - \frac{\pi x}{2} \cot \pi x - \pi x \int_0^1 \left(\frac{\sin 2\pi x u}{\sin 2\pi x} - u \right) \cot \pi u \, du & , \text{if } b = 0 \\ \frac{x^2}{2b(x-b)} - \frac{\pi x}{2} \cot \pi x - \pi x \int_0^1 \left(\frac{\sin 2\pi(x-b)u}{\sin 2\pi x} - u \cos 2\pi b u \right) \cot \pi u \, du & , \text{if } b \in \mathbb{Z}_+ \\ 1 + \frac{x^2}{2b(x-b)} - \frac{\pi x}{2} \cot \pi x - \pi x \int_0^1 \left(\frac{\sin 2\pi(x-b)u}{\sin 2\pi x} - u \cos 2\pi b u \right) \cot \pi u \, du & , \text{if } b \in \mathbb{Z}_-, \end{cases}$$

where we're skipping over singularities in the case of negative integer b, and leaving only the half-integers b unaccounted for.

Note some of these functions have a removable singularity at 0, so they are analytic at 0. So, for $k \geq 2$, we can also obtain $\zeta(k, b)$ as:

$$\zeta(k, b) = \frac{1}{b^k} + \frac{f^{(k)}(0)}{k!}$$

References

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