Generalized Harmonic Numbers

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Abstract

This paper presents new formulae for the harmonic numbers of order k, $H_k(n)$, and for the partial sums of two Fourier series associated with them, denoted here by $C_k^m(n)$ and $S_k^m(n)$. I believe this new formula for $H_k(n)$ is an improvement over the digamma function, ψ, because it's simpler and it stems from Faulhaber's formula, which provides a closed-form for the sum of powers of the first n positive integers. We demonstrate how to create an exact power series for the harmonic numbers, a new integral representation for $\zeta(2k+1)$ and a new generating function for $\zeta(2k+1)$, among many other original results. The approaches and formulae discussed here are entirely different from solutions available in the literature.

Summary

1 Introduction

Although formulae for the harmonic numbers have been known for some time, they're not simple or very useful. For example, a formula due to Euler expresses $H(n)$ as $\int_0^1 (1 + x + \cdots + x^{n-1})\, dx$ for integer n, but it's frequently dismissed by scholars, who prefer the approximation $H(n) \sim \log(n) + \gamma$.

In this paper we figure how to obtain a more natural and elegant formula for the generalized harmonic numbers:

$$H_k(n) = \sum_{j=1}^{n} \frac{1}{j^k}$$

This new formula has the advantage of being easier to work with. For example, it can be used to obtain the sum of $H(n)/n^2$ over the positive integers relatively easy.

We also show how to obtain the partial sums of two Fourier series, denoted here by $S_k^m(n)$ and $C_k^m(n)$, which cover some notable particular cases, such as the alternating harmonic numbers, $C_k^2(n)$, and the odd alternating harmonic numbers, $S_k^4(n)$ ($S_2^4(n)$ converges to Catalan's constant). These two functions are given below (for all integer $k \geq 1$ and complex m):

$$C_k^m(n) = \sum_{j=1}^{n} \frac{1}{j^k} \cos \frac{2\pi j}{m}, \text{ and } S_k^m(n) = \sum_{j=1}^{n} \frac{1}{j^k} \sin \frac{2\pi j}{m}$$

We create general formulae for $C_k^m(n)$ and $S_k^m(n)$ and find out their limits as n approaches infinity as a function of Riemann's zeta function. (After looking up previous results in the literature, I found that the limits of $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$ are not new, they are a function of the so-called Bernoulli polynomials,[1] though the limits of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$ are possibly new.)

So, to begin, let's recall Faulhaber's formula for the sum of the i-th powers of the first n positive integers:

$$\sum_{k=1}^{n} k^i = \sum_{j=0}^{i} \frac{(-1)^j i! B_j n^{i+1-j}}{(i+1-j)!j!}$$

where B_j are the Bernoulli numbers.[2]

Since odd Bernoulli numbers are always 0, except for B_1, we can simplify the above formula for even and odd powers as follows:

$$\sum_{k=1}^{n} k^{2i} = \frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{2i+1-2j}}{(2j)!(2i+1-2j)!} \tag{1}$$

$$\sum_{k=1}^{n} k^{2i+1} = \frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{2i+2-2j}}{(2j)!(2i+2-2j)!} \tag{2}$$

2 Indicator Function $\mathbb{1}_{k|n}$

One key component of the method used to solve the generalized harmonic numbers is the indicator function $\mathbb{1}_{k|n}$, defined as 1 if k divides n, and 0 otherwise. This function and its analog (that will appear in the next section) play a key role in the solution that is presented here:

$$\mathbb{1}_{k|n} = \frac{1}{k} \sum_{j=1}^{k} \cos \frac{2\pi n j}{k}$$

A closed-form for $\mathbb{1}_{k|n}$ can be obtained by means of the so-called Lagrange's trigonometric identities:

$$\mathbb{1}_{k|n} = \frac{1}{2k} \frac{\sin\left(2\pi n + \frac{\pi n}{k}\right)}{\sin \frac{\pi n}{k}} - \frac{1}{2k} = \frac{1}{2k} \sin 2\pi n \cot \frac{\pi n}{k} + \frac{\cos 2\pi n - 1}{2k} \tag{3}$$

We can also create a power series for $\mathbb{1}_{k|n}$ by expanding the cosine with Taylor series:

$$\mathbb{1}_{k|n} = \frac{1}{k}\sum_{j=1}^{k}\cos\frac{2\pi nj}{k} = 1 + \frac{1}{k}\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{2\pi n}{k}\right)^{2i}\sum_{j=1}^{k} j^{2i}$$

Now, by replacing the sum of j^{2i} over j with Faulhaber's formula, (1), we get:

$$\mathbb{1}_{k|n} = 1 + \frac{1}{k}\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{2\pi n}{k}\right)^{2i}\left(\frac{k^{2i}}{2} + \sum_{j=0}^{i}\frac{(2i)!B_{2j}k^{2i+1-2j}}{(2j)!(2i+1-2j)!}\right) \Rightarrow$$

$$\mathbb{1}_{k|n} = \frac{\cos 2\pi n - 1}{2k} + \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2j)!(2i+1-2j)!} \tag{4}$$

From (3) and (4), after re-scaling n to $n/2$, we conclude that:

$$\sum_{i=0}^{\infty}(-1)^i(\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2i+1-2j)!(2j)!} = \frac{1}{2k}\cot\frac{\pi n}{2k}\sin\pi n \tag{5}$$

2.1 The Analog of $\mathbb{1}_{k|n}$

Now, just as we created a power series for $\mathbb{1}_{k|n}$, we have to create one for its analog, which is the sum:

$$\frac{1}{k}\sum_{j=1}^{k}\sin\frac{2\pi nj}{k}$$

Again, we can find a closed-form for the above sum using Lagrange's trigonometric identities:

$$\frac{1}{k}\sum_{j=1}^{k}\sin\frac{2\pi nj}{k} = -\frac{1}{2k}\frac{\cos\left(2\pi n + \frac{\pi n}{k}\right)}{\sin\frac{\pi n}{k}} + \frac{1}{2k}\cot\frac{\pi n}{k} = \frac{\sin 2\pi n}{2k} + \frac{1}{k}\cot\frac{\pi n}{k}\sin^2\pi n \tag{6}$$

As previously, we can obtain a power series for the above by expanding the sine with Taylor series and making use of (2):

$$\frac{1}{k}\sum_{j=1}^{k}\sin\frac{2\pi nj}{k} = \frac{\sin 2\pi n}{2k} + \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i+1}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2i+2-2j)!(2j)!} \tag{7}$$

From (6) and (7), after re-scaling n to $n/2$, it follows that:

$$\sum_{i=0}^{\infty}(-1)^i(\pi n)^{2i+1}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2i+2-2j)!(2j)!} = \frac{1}{k}\cot\frac{\pi n}{2k}\left(\sin\frac{\pi n}{2}\right)^2 \tag{8}$$

3 Generalized Harmonic Numbers

3.1 Formula Rationale

The rationale to build a formula for $H_k(n)$ is to use the Taylor series expansion of $\sin \pi k$, and exploit the fact that it's 0 for all integer k. We refer to the below as initial equation (note the k in the sum is not the same k used as subscript on $H_k(n)$):

$$\sin \pi k = 0 \Rightarrow \pi k = \sum_{i=1}^{\infty} \frac{-(-1)^i (\pi k)^{2i+1}}{(2i+1)!} \tag{9}$$

If we divide both sides of (9) by πk^2 we end up with a power series for $1/k$ about 0 that only holds for integer k (after all $1/k$ is not analytic at 0).

Besides, on the right-hand side of the resulting equation, the exponents of k are positive integers, allowing us to apply Faulhaber's formula mentioned in the introduction. By doing so we end up with a convoluted power series that fortunately can be transformed into an integral by means of the closed-form we derived for $\mathbb{1}_{k|n}$ (or its analog) using Lagrange's identities. That is a high level summary of the reasoning.

To not make this paper long, we only give two fully detailed demonstrations based on the initial equation $\sin \pi k = 0$, and jump straight to the final formulae in a few other cases, before we state a general formula. We also briefly show how the outcomes change with the choice of different initial equations.

3.2 Harmonic Number

We start by dividing both sides of (9) by πk^2:

$$\frac{1}{k} = \sum_{i=1}^{\infty} \frac{-(-1)^i \pi^{2i} k^{2i-1}}{(2i+1)!} = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} k^{2i+1}}{(2i+3)!} \tag{10}$$

Below we take the sum of (10) over k and use equation (2), thus extending the domain of $H_1(n)$ ($H(n)$ for short) to the real numbers, in an analytic continuation:

$$H(n) = \sum_{k=1}^{n} \frac{1}{k} = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2}}{(2i+3)!} \sum_{k=1}^{n} k^{2i+1} = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2}}{(2i+3)!} \left(\frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{2i+2-2j}}{(2i+2-2j)!(2j)!} \right)$$

$$H(n) = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i+1}}{2(2i+3)!} + \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i+2}}{(2i+3)!} \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{-2j}}{(2i+2-2j)!(2j)!}$$

The 1st sum is straightforward:

$$\sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i+1}}{2(2i+3)!} = -\frac{1}{2\pi n^2} \sum_{i=1}^{\infty} \frac{(-1)^i (\pi n)^{2i+1}}{(2i+1)!} = \frac{1}{2\pi n^2} (\pi n - \sin \pi n)$$

The 2nd sum is an exact power series for $H(n) - 1/(2n)$ and can be rewritten as:

$$\sum_{i=0}^{\infty}\left(-\frac{1}{2i+3}+\frac{1}{2i+2}\right)(-1)^i\pi^{2i+2}n^{2i+2}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} \tag{11}$$

The above sums are tricky, but they can be obtained from (8), one of the formulae derived previously. In order to do that, let's replace (n,k) by (x,n) and define a function $f(x,n)$ such that:

$$f(x,n) = \sum_{i=0}^{\infty}(-1)^i(\pi x)^{2i+1}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} = \frac{1}{n}\cot\frac{\pi x}{2n}\left(\sin\frac{\pi x}{2}\right)^2 \tag{12}$$

To build each piece of (11) we start from the above $f(x,n)$.

For the 1st sum, we multiply $f(x,n)$ by $-\pi\cdot x/n$ and integrate with respect to x as below:

$$-\frac{1}{n}\sum_{i=0}^{\infty}(-1)^i\pi^{2i+2}\left(\int_0^n x^{2i+2}\,dx\right)\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} = -\frac{\pi}{n}\int_0^n xf(x,n)\,dx$$

$$-\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+2}}{2i+3}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} = -\frac{\pi}{n}\int_0^n x\frac{1}{n}\cot\frac{\pi x}{2n}\left(\sin\frac{\pi x}{2}\right)^2 dx$$

For the 2nd sum, we multiply $f(x,n)$ by π and integrate with respect to x as below:

$$\sum_{i=0}^{\infty}(-1)^i\pi^{2i+2}\left(\int_0^n x^{2i+1}\right)\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!}\,dx = \pi\int_0^n f(x,n)\,dx$$

$$\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+2}}{2i+2}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} = \pi\int_0^n \frac{1}{n}\cot\frac{\pi x}{2n}\left(\sin\frac{\pi x}{2}\right)^2 dx$$

Now, by summing up the two resulting integrals, we get the below equivalence (which holds for all real n, not just integers):

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+2}}{(2i+3)!}\sum_{j=0}^{i}\frac{(2i+1)!B_{2j}n^{-2j}}{(2i+2-2j)!(2j)!} &= \int_0^n\frac{\pi(n-x)}{n^2}\cot\frac{\pi x}{2n}\left(\sin\frac{\pi x}{2}\right)^2 dx\\ &= \pi\int_0^1 u\cot\frac{\pi(1-u)}{2}\left(\sin\frac{\pi n(1-u)}{2}\right)^2 du\end{aligned}$$

where we've used the transformation $u = 1 - x/n$.

Now, by adding up the simple part (disregarding $\sin \pi n$ and changing u for $1-u$), we finally arrive at a formula for $H(n)$:

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2n} + \frac{\pi}{2} \int_0^1 (1-u)\,(1-\cos \pi n u) \cot \frac{\pi u}{2}\, du \tag{13}$$

This formula has a certain resemblance to Faulhaber's formula, especially the term $1/(2n)$ outside of the integral. If we compare this formula with the below, due to Euler,[3] based on the digamma function, it seems to me that the former is more natural and tractable than the latter. The two functions approach one another very quickly as n gets large.

$$\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1-x^n}{1-x}\, dx = \gamma + \psi(n+1), \text{ where } \gamma \text{ is the Euler-Mascheroni constant.}$$

3.3 Harmonic Number of Order 2

We divide both sides of (10) by k:

$$\frac{1}{k^2} = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} k^{2i}}{(2i+3)!}$$

We sum the above over k, using equation (1) this time, noting that because equation (1) doesn't work exactly for $i=0$, we need to make a little correction by adding up $-1/2$:

$$H_2(n) = \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2}}{(2i+3)!} \sum_{k=1}^{n} k^{2i} = -\frac{1}{2}\frac{\pi^2}{3!} + \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2}}{(2i+3)!} \left(\frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{2i+1-2j}}{(2i+1-2j)!(2j)!} \right)$$

$$H_2(n) = -\frac{1}{2}\frac{\pi^2}{3!} + \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i}}{(2i+3)!} + \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i+1}}{(2i+3)!} \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{-2j}}{(2i+1-2j)!(2j)!}$$

The 1st sum, again, is straightforward:

$$-\frac{1}{2}\frac{\pi^2}{3!} + \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+2} n^{2i}}{(2i+3)!} = -\frac{1}{2\pi n^3} \sum_{i=2}^{\infty} \frac{(-1)^i (\pi n)^{2i+1}}{(2i+1)!} = \frac{1}{2\pi n^3} \left(\pi n - \frac{(\pi n)^3}{3!} - \sin \pi n \right)$$

The 2nd sum can be rewritten as:

$$\sum_{i=0}^{\infty} \left(\frac{1}{2(2i+3)} - \frac{1}{2i+2} + \frac{1}{2(2i+1)} \right) (-1)^i \pi^{2i+2} n^{2i+1} \sum_{j=0}^{i} \frac{B_{2j} n^{-2j}}{(2i+1-2j)!(2j)!}$$

The above sums are tricky, but they can be derived from (5), another one of the formulae derived previously. In order to do that, let's replace (n, k) by (x, n) and define a function $g(x, n)$ such that:

$$g(x,n) = \sum_{i=0}^{\infty} (-1)^i (\pi x)^{2i} \sum_{j=0}^{i} \frac{B_{2j} n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{1}{2n} \cot \frac{\pi x}{2n} \sin \pi x \tag{14}$$

To build each piece of (14) we start from $g(x,n)$.

For the 1st sum, we multiply both sides of $g(x,n)$ by $\pi^2/2 \cdot x^2/n^2$ and integrate with respect to x as below:

$$\frac{1}{2n^2}\sum_{i=0}^{\infty}(-1)^i\pi^{2i+2}\left(\int_0^n x^{2i+2}\,dx\right)\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{\pi^2}{2n^2}\int_0^n x^2 g(x,n)\,dx$$

$$\frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+1}}{2i+3}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{\pi^2}{2n^2}\int_0^n x^2\frac{1}{2n}\cot\frac{\pi x}{2n}\sin\pi x\,dx$$

For the 2nd sum, we multiply both sides of $g(x,n)$ by $-\pi^2 \cdot x/n$ and integrate with respect to x as below:

$$-\frac{1}{n}\sum_{i=0}^{\infty}(-1)^i\pi^{2i+2}\left(\int_0^n x^{2i+1}\,dx\right)\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{-\pi^2}{n}\int_0^n xg(x,n)\,dx$$

$$-\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+1}}{2i+2}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{-\pi^2}{n}\int_0^n x\frac{1}{2n}\cot\frac{\pi x}{2n}\sin\pi x\,dx$$

For the 3rd sum, we multiply both sides of $g(x,n)$ by $\pi^2/2$ and integrate with respect to x as below:

$$\frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\pi^{2i+2}\left(\int_0^n x^{2i}\,dx\right)\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{\pi^2}{2}\int_0^n g(x,n)\,dx$$

$$\frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+1}}{2i+1}\sum_{j=0}^{i}\frac{B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} = \frac{\pi^2}{2}\int_0^n \frac{1}{2n}\cot\frac{\pi x}{2n}\sin\pi x\,dx$$

Let's summarize the convoluted part by summing up the three resulting integrals:

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+2}n^{2i+1}}{(2i+3)!}\sum_{j=0}^{i}\frac{(2i)!B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!} &= \int_0^n\frac{\pi^2(n-x)^2}{4n^3}\cot\frac{\pi x}{2n}\sin\pi x\,dx\\ &= \frac{\pi^2}{4}\int_0^1 u^2\sin\pi n(1-u)\cot\frac{\pi(1-u)}{2}\,du\end{aligned}$$

where we've made a change of variables, $u = 1 - x/n$.

Now, by summing up the two parts, we get a formula for $H_2(n)$:

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{1}{2n^2} - \frac{\pi^2}{12} + \frac{\pi^2}{4}\int_0^1 u^2\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du,$$

where the identity $\cot \pi(1-u)/2 = \tan \pi u/2$ was used.

In section (3.7) we find out the general polynomial, $p_{2k}(u)$, that goes under the integral sign, and it's convenient to move the constant $-\pi^2/12$ under the integral sign (this also makes the $H_2(n)$ formula look more similar to $H(n)$).

First, we note that for all positive integer n:

$\int_0^1 \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = 1$, which stems from the below equation:

$$H_0^*(n) = \sum_{k=1}^{n} 1 = n = n + \frac{1}{2} - \frac{1}{2}\int_0^1 \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = n + H_0(n)$$

From the above we conclude that $H_0(n) = 0$ for all positive integer n (this is not a usual definition, but it will make sense when we reach section (3.7)). Therefore our modified formula is:

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{2n^2} + \pi^2 \int_0^1 \left(-\frac{1}{12} + \frac{u^2}{4}\right) \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du \tag{15}$$

It can be proved that for all integer $k \geq 0$:

$$\lim_{n\to\infty} \int_0^1 u^{2k} \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = 1 \Rightarrow \lim_{n\to\infty} H_2(n) = \pi^2 \left(-\frac{1}{12} + \frac{1}{4}\right) = \frac{\pi^2}{6} = \zeta(2)$$

The above limits are justified by Theorem 1, section (6.1.1), and Theorem 3, section (8.1). The latter assumes that the closed-form of $\zeta(2k)$ is known, as the limits of the above integrals stem from the limits of $H_{2k}(n)$ and vice-versa.

3.4 Harmonic Number of Order 3

We divide both sides of (10) by k^2 and simplify:

$$H_3(n) = \frac{\pi^2}{3!} H_1(n) - \frac{1}{2}\sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+4} n^{2i+1}}{(2i+5)!} - \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+4} n^{2i+2}}{(2i+5)!} \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{-2j}}{(2i+2-2j)!(2j)!},$$

which gives us the below recurrence:

$$H_3(n) = \frac{\pi^2}{3!} H_1(n) + \frac{1}{2\pi n^4}\left(\pi n - \frac{(\pi n)^3}{3!} - \sin \pi n\right) - \frac{\pi^3}{12}\int_0^1 u^3 \left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, dx$$

Performing all the necessary calculations, we get:

$$\sum_{k=1}^{n} \frac{1}{k^3} = \frac{1}{2n^3} + \frac{\pi^3}{12}\int_0^1 \left(u - u^3\right)\left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, du \tag{16}$$

Besides, due to the below identity, whose proof is given in section (8.2.1):

$\frac{\pi^3}{12}\int_0^1 \left(u-u^3\right)\tan\frac{\pi u}{2}\,du = \zeta(3)$, the previous equation can be rewritten as:

$$\sum_{k=1}^{n}\frac{1}{k^3} = \frac{1}{2n^3} + \zeta(3) - \frac{\pi^3}{12}\int_0^1 \left(u-u^3\right)\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du$$

And since the limit of $H_3(n)$ when n tends to infinity is $\zeta(3)$, it means that:

$$\lim_{n\to\infty}\int_0^1 \left(u-u^3\right)\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du = 0$$

3.5 Harmonic Number of Order 4

We divide both sides of (10) by k^3 and simplify:

$$H_4(n) = \frac{\pi^2}{3!}H_2(n) + \frac{1}{2}\frac{\pi^4}{5!} - \frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+4}n^{2i}}{(2i+5)!} - \sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+4}n^{2i+1}}{(2i+5)!}\sum_{j=0}^{i}\frac{(2i)!B_{2j}n^{-2j}}{(2i+1-2j)!(2j)!},$$

which brings us to the below recurrence:

$$H_4(n) = \frac{\pi^2}{3!}H_2(n) + \frac{1}{2\pi n^5}\left(\pi n - \frac{(\pi n)^3}{3!} + \frac{(\pi n)^5}{5!} - \sin\pi n\right) - \frac{\pi^4}{48}\int_0^1 u^4\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du$$

Now, performing the calculations, we get the formula for $H_4(n)$:

$$\sum_{k=1}^{n}\frac{1}{k^4} = \frac{1}{2n^4} - \frac{7\pi^4}{720} + \pi^4\int_0^1 \left(\frac{u^2}{24} - \frac{u^4}{48}\right)\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du$$

Moving the constant under the integral sign, as we did for $H_2(n)$:

$$\sum_{k=1}^{n}\frac{1}{k^4} = \frac{1}{2n^4} + \pi^4\int_0^1 \left(-\frac{7}{720} + \frac{u^2}{24} - \frac{u^4}{48}\right)\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du \quad (17)$$

Since each limit is 1, as mentioned in section (3.3), we conclude that the limit of $H_4(n)$ is:

$$\zeta(4) = \pi^4\left(-\frac{7}{720} + \frac{1}{24} - \frac{1}{48}\right) = \frac{\pi^4}{90}$$

3.6 General Formula

As we've seen, lots of patterns emerge when we create formulae for $H_1(n)$, $H_2(n)$, and so on. We now assume that these patterns always repeat and see if we can figure out what the general rule is for each k.

Because in each case the term that goes outside of the integral sign is easy to deduce ($1/(2n^{2k})$ or $1/(2n^{2k+1})$), we can focus on the polynomials in u that go under the integral sign, $p_{2k}(u)$ and $p_{2k+1}(u)$, and see if we can find out their generating function, $g(x)$.

Note that since for each recursive equation the coefficients π^{2k} (or π^{2k+1}) cancel out, we can ignore them for simplification purposes.

3.7 Harmonic Numbers of Order $2k$

Let $f(u,n)$ be the below function (not to be confused with the same function from previous sections):

$$f(u,n) = \sin \pi n(1-u) \tan \frac{\pi u}{2}$$

As we've seen in previous sections, calculating the $2k$-th harmonic number involves a recurrence with prior ones:

$$H_{2k}(n) = \frac{1}{2n^{2k}} \sum_{j=0}^{k} \frac{(-1)^j (\pi n)^{2j}}{(2j+1)!} - \sum_{j=0}^{k-1} \frac{(-1)^{k-j} \pi^{2k-2j}}{(2k+1-2j)!} H_{2j}(n) - \frac{(-1)^k \pi^{2k}}{2(2k)!} \int_0^1 u^{2k} f(u,n)\, du$$

That is, the harmonic numbers of even orders obey the below recursive equations:

$$\begin{cases} H_0(n) = \frac{1}{2} - \frac{1}{2} \int_0^1 f(u,n)\, du \\ H_2(n) = \frac{\pi^2}{3!} H_0(n) + \frac{1}{2\pi n^3} \left(\pi n - \frac{\pi^3 n^3}{3!} \right) + \frac{\pi^2}{2} \int_0^1 \frac{u^2}{2!} f(u,n)\, du \\ H_4(n) = \frac{\pi^2}{3!} H_2(n) - \frac{\pi^4}{5!} H_0(n) + \frac{1}{2\pi n^5} \left(\pi n - \frac{\pi^3 n^3}{3!} + \frac{\pi^5 n^5}{5!} \right) - \frac{\pi^4}{2} \int_0^1 \frac{u^4}{4!} f(u,n)\, du \\ H_6(n) = \frac{\pi^2}{3!} H_4(n) - \frac{\pi^4}{5!} H_2(n) + \frac{\pi^6}{7!} H_0(n) + \frac{1}{2\pi n^7} \left(\pi n - \frac{\pi^3 n^3}{3!} + \frac{\pi^5 n^5}{5!} - \frac{\pi^7 n^7}{7!} \right) + \frac{\pi^6}{2} \int_0^1 \frac{u^6}{6!} f(u,n)\, du \\ \vdots \end{cases}$$

Let's try to solve that recurrence, we have:

$$\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right) \left(p_0 + p_2 x^2 + p_4 x^4 + p_6 x^6 + \cdots \right) = -1 + \frac{u^2}{2!} x^2 - \frac{u^4}{4!} x^4 - \frac{u^6}{6!} x^6 + \cdots$$

But the product on the left-hand side gives us:

$$p_0 + \left(p_2 - \frac{1}{3!} p_0\right) x^2 + \left(p_4 - \frac{1}{3!} p_2 + \frac{1}{5!} p_0\right) x^4 + \left(p_6 - \frac{1}{3!} p_4 + \frac{1}{5!} p_2 - \frac{1}{7!} p_0\right) x^6 + \cdots,$$

where the coefficient of each x^{2k} is the recurrence that produces the polynomials p_{2k} that we're interested in. The generating function for $p_{2k}(u)$ is therefore given by:

$$g(x) = -\frac{x\cos xu}{\sin x} = -1 + \left(-\frac{1}{6} + \frac{u^2}{2}\right)x^2 + \left(-\frac{7}{360} + \frac{u^2}{12} - \frac{u^4}{24}\right)x^4 + \left(-\frac{31}{15120} + \frac{7u^2}{720} - \frac{u^4}{144} + \frac{u^6}{720}\right)x^6 + \cdots$$

To obtain the power series of the function $g(x)$, we need to obtain the power series of each of its components individually:[1]

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty} \frac{(-1)^i B_{2i}(2-2^{2i})}{(2i)!}x^{2i}, \text{ and } -\cos xu = -\sum_{i=0}^{\infty} \frac{(-1)^i u^{2i}}{(2i)!}x^{2i}$$

Therefore, the $2k$-th term of the power series of $g(x)$ is $p_{2k}(u)$, which is given by the below expression:

$$p_{2k}(u) = \sum_{j=0}^{k} \frac{(-1)^j B_{2j}\left(2-2^{2j}\right)}{(2j)!} \cdot \frac{(-1)^{k-j+1}u^{2k-2j}}{(2k-2j)!}$$

Now, putting it all together, we get that for all integer $k \geq 1$:

$$H_{2k}(n) = \frac{1}{2n^{2k}} + \frac{\pi^{2k}}{2}\int_0^1 p_{2k}(u) f(u,n)\, du \Rightarrow$$

$$H_{2k}(n) = \frac{1}{2n^{2k}} - \frac{(-1)^k \pi^{2k}}{2}\int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)u^{2k-2j}}{(2j)!(2k-2j)!} \sin \pi n(1-u) \tan\frac{\pi u}{2}\, du$$

Note that this formula applies to $H_0(n)$ as well, but remember that per this definition $H_0(n)$ is such that $H_0(n) = 0$ for all positive integer n.

We can rewrite $H_{2k}(n)$ by means of Bernoulli polynomials, which are given by:[1]

$$B_k(u) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} u^j$$

In doing so we get an expression that resembles, but is not exactly, an Euler polynomial:[1]

$$H_{2k}(n) = \frac{1}{2n^{2k}} - \frac{(-1)^k \pi^{2k}}{(2k)!}\int_0^1 \left(B_{2k}(u) - 2^{2k-1}B_{2k}\left(\frac{u}{2}\right)\right) f(u,n)\, du$$

3.7.1 Generating Function of $H_{2k}(n)$

A generating function for $H_{2k}(n)$ can be obtained by means of the generating function $g(x)$, that we previously found for $p_{2k}(u)$, as follows:

$$\sum_{k=0}^{\infty} H_{2k}(n)x^{2k} = \frac{n^2}{2(n^2-x^2)} - \frac{\pi x}{2\sin \pi x}\int_0^1 \cos \pi x u \sin \pi n(1-u) \tan\frac{\pi u}{2}\, du$$

Note the convergence radius of the power series on the left-hand side is the open interval $(-1, 1)$, but the domain of the function on the right-hand side is $\mathbb{R}\backslash\mathbb{Z}$. This generating function is probably an analytic continuation of the power series to the left.

As an example, if $n = 2$ the above function is $\dfrac{5x^2}{4}+\dfrac{17x^4}{16}+\dfrac{65x^6}{64}+\dfrac{257x^8}{256}+\cdots$

Notice it doesn't have the independent term, which is a result of $H_0(n) = 0$ for all positive integer n.

3.7.2 Limit of the Generating Function of $H_{2k}(n)$

The limit of the generating function we just found as n goes to infinity is:

$$h(x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} H_{2k}(n)x^{2k} = \lim_{n\to\infty} \frac{n^2}{2(n^2 - x^2)} - \frac{\pi x}{2\sin \pi x} \int_0^1 \cos \pi xu \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du \Rightarrow$$

$$h(x) = \sum_{k=1}^{\infty} \zeta(2k)x^{2k} = \frac{1}{2} - \frac{\pi x \cos \pi x}{2\sin \pi x}$$

Note $h(x)$ also doesn't have the independent term, due to $H_0(n) = 0$.

Proof The proof of the above is simple:

$$\lim_{n\to\infty} \int_0^1 \cos \pi xu \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = \lim_{n\to\infty} \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k(\pi xu)^{2k}}{(2k)!} \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du \Rightarrow$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(\pi x)^{2k}}{(2k)!} \int_0^1 u^{2k} \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = \cos \pi x,$$

as the limits of the above integrals are always 1, per the two different proofs that are provided in sections (6.1.1), Theorem 1, and (8.1), Theorem 3. ■

3.8 Harmonic Numbers of Order $2k + 1$

Let $f(u, n)$ be the below function:

$$f(u, n) = (1 - \cos \pi n(1-u)) \tan \frac{\pi u}{2}$$

Calculating the odd harmonic numbers involves a recurrence with prior ones:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j(\pi n)^{2j}}{(2j+1)!} - \sum_{j=0}^{k-1} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k+1-2j)!} H_{2j+1}(n) + \frac{(-1)^k\pi^{2k+1}}{2(2k+1)!} \int_0^1 u^{2k+1} f(u, n)\, du$$

The reasoning employed to figure out the generating function of $p_{2k+1}(u)$ is entirely analogous to what we've done previously, and $g(x)$ is given by:

$$g(x) = \frac{x \sin xu}{\sin x} = ux + \left(\frac{u}{6} - \frac{u^3}{6}\right) x^3 + \left(\frac{7u}{360} - \frac{u^3}{36} + \frac{u^5}{120}\right) x^5 + \left(\frac{31u}{15120} - \frac{7u^3}{2160} + \frac{u^5}{720} - \frac{u^7}{5040}\right) x^7 + \cdots$$

The $(2k+1)$-th term of the power series of $g(x)$ is therefore:

$$p_{2k+1}(u) = (-1)^k \sum_{j=0}^{k} \frac{B_{2j}\left(2 - 2^{2j}\right) u^{2k+1-2j}}{(2j)!(2k+1-2j)!}$$

Now, putting it all together, we get that for all integer $k \geq 0$:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} + \frac{\pi^{2k+1}}{2} \int_0^1 p_{2k+1}(u) f(u,n)\, du \Rightarrow$$

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} + \frac{(-1)^k \pi^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2 - 2^{2j}\right) u^{2k+1-2j}}{(2j)!(2k+1-2j)!} \left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, du$$

Because of Theorem 8, section (8.2.1), we can also rewrite $H_{2k+1}(n)$ as:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} + \zeta(2k+1) - \frac{(-1)^k \pi^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2 - 2^{2j}\right) u^{2k+1-2j}}{(2j)!(2k+1-2j)!} \cos \pi n(1-u) \tan \frac{\pi u}{2}\, du$$

Finally, with the aforementioned Bernoulli polynomials, we can also rewrite $H_{2k+1}(n)$ as:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} + \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} \int_0^1 \left(B_{2k+1}(u) - 2^{2k} B_{2k+1}\left(\frac{u}{2}\right)\right) f(u,n)\, du$$

3.8.1 Generating Function of $H_{2k+1}(n)$

A generating function for $H_{2k+1}(n)$ can be obtained from the function $g(x)$ that we found for $p_{2k+1}(u)$ previously, as follows:

$$\sum_{k=0}^{\infty} H_{2k+1}(n) x^{2k+1} = \frac{nx}{2(n^2 - x^2)} + \frac{\pi x}{2 \sin \pi x} \int_0^1 \sin \pi xu \left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, du$$

For example, if $n = 2$, the above function becomes $\frac{3x}{2} + \frac{9x^3}{8} + \frac{33x^5}{32} + \frac{129x^7}{128} + \cdots$

3.8.2 Limit of the Generating Function of $H_{2k+1}(n)$

Before we can take the limit of this generating function as n approaches infinity, we need to exclude term $H_1(n)x$, since $H(n)$ is unbounded. Hence, using the expression for $H(n)$ from section (3.2), the limit of the generating function as n increases is:

$$\sum_{k=1}^{\infty} H_{2k+1}(n) x^{2k+1} = \frac{nx}{2(n^2 - x^2)} - \frac{x}{2n} + \frac{\pi x}{2} \int_0^1 \left(\frac{\sin \pi xu}{\sin \pi x} - u\right) \left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, du \Rightarrow$$

$$h(x) = \sum_{k=1}^{\infty} \zeta(2k+1) x^{2k+1} = \frac{\pi x}{2} \int_0^1 \left(\frac{\sin \pi x u}{\sin \pi x} - u \right) \tan \frac{\pi u}{2} \, du$$

Proof To prove that the generating function converges to the above limit, we need to show that the below integral goes to 0 as n approaches infinity. But,

$$\int_0^1 (\sin \pi x u - u \sin \pi x) \cos \pi n(1-u) \tan \frac{\pi u}{2} \, du = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k (\pi x)^{2k+1} \left(u^{2k+1} - u\right)}{(2k+1)!} \cos \pi n(1-u) \tan \frac{\pi u}{2} \, du$$

it follows that $\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!} \int_0^1 \left(u^{2k+1} - u\right) \cos \pi n(1-u) \tan \frac{\pi u}{2} \, du = 0,$

as the limits of the integrals are 0, per Corollary 1 of section (6.2.1). ■

The above representation of the generating function of $\zeta(2k+1)$ is different from the one found in the literature, which employs the digamma function, though they must be equivalent:

$$h(x) = \sum_{k=1}^{\infty} \zeta(2k+1) x^{2k+1} = -x\gamma - \frac{x}{2}(\psi(1+x) + \psi(1-x))$$

3.9 Initial Equation $\sin 2\pi k = 0$

In this section we set the initial equation to $\sin 2\pi k = 0$. To avoid redundancy, we omit the step by step demonstrations and present only the final formulae.

Using this initial equation, we get slightly different formulae for $H(n)$ and $H_2(n)$:

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2n} + \pi \int_0^1 (1-u)\,(1 - \cos 2\pi n u) \cot \pi u \, du$$

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{2n^2} - \frac{\pi^2}{3} - \pi^2 \int_0^1 u^2 \sin 2\pi n(1-u) \cot \pi u \, du$$

3.9.1 General Formula

We conclude that not much really changes in the system of recurrence equations, except for the introduction of a coefficient 2 on π. Therefore, the polynomial solution is the same as before, only the multiplier of the integral and the integrand change.

3.9.2 Harmonic Numbers of Order $2k$

The recurrence equation changes slightly:

$$H_{2k}(n) = \frac{1}{2n^{2k}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi n)^{2j}}{(2j+1)!} - \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (2\pi)^{2k-2j}}{(2k+1-2j)!} H_{2j}(n) + \frac{(-1)^k (2\pi)^{2k}}{2(2k)!} \int_0^1 u^{2k} \sin 2\pi n(1-u) \cot \pi u \, du$$

For all integer $k \geq 0$:

$$H_{2k}(n) = \frac{1}{2n^{2k}} + \frac{(-1)^k (2\pi)^{2k}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2 - 2^{2j}\right) u^{2k-2j}}{(2j)!(2k-2j)!} \sin 2\pi n(1-u) \cot \pi u \, du,$$

3.9.3 Harmonic Numbers of Order $2k+1$

The recurrence equation also changes slightly:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+1)!} - \sum_{j=0}^{k-1}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!}H_{2j+1}(n) - \frac{(-1)^k(2\pi)^{2k+1}}{2(2k+1)!}\int_0^1 u^{2k+1}\left(1-\cos 2\pi n(1-u)\right)\cot \pi u\, du$$

For all integer $k \geq 0$:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} - \frac{(-1)^k(2\pi)^{2k+1}}{2}\int_0^1 \sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\left(1-\cos 2\pi n(1-u)\right)\cot \pi u\, du$$

Here the integral sign has changed due to $\cot \pi(1-u) = -\cot \pi u$.

3.10 Initial Equation $\cos 2\pi k = 1$

When we switch to cosine-based harmonic numbers, the degree of the polynomials $p_k(u)$ go up by one. Below are two examples for $H(n)$ and $H_2(n)$:

$$\sum_{k=1}^{n}\frac{1}{k} = \frac{1}{2n} - \pi\int_0^1 u^2\left(1-\cos 2\pi n(1-u)\right)\cot \pi u\, du$$

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{1}{2n^2} - \frac{\pi^2}{6} - \frac{2\pi^2}{3}\int_0^1 u^3 \sin 2\pi n(1-u)\cot \pi u\, du$$

3.10.1 General Formula

Using this initial equation, we add more entropy to the formula. Here we only show a detailed demonstration for the odd case, and the even case is just stated.

3.10.2 Harmonic Numbers of Order $2k$

$H_{2k}(n)$ is given by the below recurrence equation:

$$H_{2k}(n) = \frac{1}{n^{2k}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!} - 2\sum_{j=0}^{k-1}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}H_{2j}(n) + \frac{(-1)^k(2\pi)^{2k}}{(2k+1)!}\int_0^1 u^{2k+1}\sin 2\pi n(1-u)\cot \pi u\, du$$

The polynomial $p_{2k}(u)$ can be obtained using a similar approach to $p_{2k+1}(u)$ (see the next section for a detailed demonstration), which results in the formula below:

$$H_{2k}(n) = \frac{1}{2n^{2k}} + \frac{(-1)^k\pi^{2k}}{2}\int_0^1\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{B_{2j}B_{2i-2j}\left(2-2^{2j}\right)\left(2-2^{2i-2j}\right)(2u)^{2k+1-2i}}{(2j)!(2i-2j)!(2k+1-2i)!}\sin 2\pi n(1-u)\cot \pi u\, du$$

3.10.3 Harmonic Numbers of Order $2k+1$

Let $f(u,n)$ be the below function:

$$f(u,n) = (1-\cos 2\pi n(1-u))\cot \pi u$$

$H_{2k+1}(n)$ is given by the below recurrence equation:

$$H_{2k+1}(n) = \frac{1}{n^{2k+1}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!} - 2\sum_{j=0}^{k-1}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}H_{2j+1}(n) - \frac{(-1)^k(2\pi)^{2k+1}}{(2k+2)!}\int_0^1 u^{2k+2}f(u,n)\,du$$

That is, the harmonic numbers of odd orders obey the below recursive equations (notice we're ignoring $\cos 2\pi n - 1$):

$$\begin{cases} H_1(n) = \frac{1}{2n} - \frac{2\pi}{2!}\int_0^1 u^2 f(u,n)\,du \\ H_3(n) = 2\frac{(2\pi)^2}{4!}H_1(n) + \frac{1}{n^3}\left(\frac{1}{2!} - \frac{(2\pi n)^2}{4!}\right) + \frac{(2\pi)^3}{4!}\int_0^1 u^4 f(u,n)\,du \\ H_5(n) = 2\left(\frac{(2\pi)^2}{4!}H_3(n) - \frac{(2\pi)^4}{6!}H_1(n)\right) + \frac{1}{n^5}\left(\frac{1}{2!} - \frac{(2\pi n)^2}{4!} + \frac{(2\pi n)^4}{6!}\right) - \frac{(2\pi)^5}{6!}\int_0^1 u^6 f(u,n)\,du \\ H_7(n) = 2\left(\frac{(2\pi)^2}{4!}H_5(n) - \frac{(2\pi)^4}{6!}H_3(n) + \frac{(2\pi)^6}{8!}H_1(n)\right) + \frac{1}{n^7}\left(\frac{1}{2!} - \frac{(2\pi n)^4}{4!} + \frac{(2\pi n)^4}{6!} - \frac{(2\pi n)^6}{8!}\right) + \frac{(2\pi)^7}{8!}\int_0^1 u^8 f(u,n)\,du \\ \vdots \end{cases},$$

Let $p(x)$ be the generating function that we're interested in. We have:

$$p(x) - p(x)\frac{1}{2x^2}\left(\cos 2x - 1 + \frac{(2x)^2}{2}\right) = -\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+1}u^{2k+2}}{(2k+2)!}x^{2k+1} = \frac{1}{2x}(\cos 2ux - 1) \Rightarrow$$

$$p_1 x + \left(-\frac{p_1}{3} + p_3\right)x^3 + \left(\frac{2p_1}{45} - \frac{p_3}{3} + p_5\right)x^5 + \left(-\frac{p_1}{315} + \frac{2p_3}{45} - \frac{p_5}{3} + p_7\right)x^7 + \cdots =$$
$$-u^2x + \frac{u^4}{3}x^3 - \frac{2u^6}{45}x^5 + \frac{u^8}{315}x^7 + \cdots$$

The generating function for $p_{2k+1}(u)$ is therefore given by:

$$g(x) = \left(\frac{x}{\sin x}\right)^2\frac{\cos 2ux - 1}{2x} = -u^2x + \left(-\frac{u^2}{3} + \frac{u^4}{3}\right)x^3 + \left(-\frac{u^2}{15} + \frac{u^4}{9} - \frac{2u^6}{45}\right)x^5 + \left(-\frac{2u^2}{189} + \frac{u^4}{45} - \frac{2u^6}{135} + \frac{u^8}{315}\right)x^7 + \cdots$$

where we've used the transformation: $-x\dfrac{\cos 2ux - 1}{\cos 2x - 1 + \frac{(2x)^2}{2} - 2x^2} = -x\dfrac{\cos 2xu - 1}{\cos 2x - 1} = x\dfrac{\cos 2ux - 1}{2\sin^2 x}$

To obtain the power series of $g(x)$, we need to obtain the power series of each function individually:

$$\left(\frac{x}{\sin x}\right)^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i B_{2j}B_{2i-2j}\left(2-2^{2j}\right)\left(2-2^{2i-2j}\right)}{(2j)!(2i-2j)!}x^{2i}, \text{ and } \frac{\cos 2xu - 1}{x} = \sum_{i=1}^{\infty}\frac{(-1)^i(2u)^{2i}}{(2i)!}x^{2i-1}$$

Therefore, the $(2k+1)$-th term of the power series of $g(x)$ is given by:

$$p_{2k+1}(u) = \frac{1}{2}\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{(-1)^i B_{2j}B_{2i-2j}\left(2-2^{2j}\right)\left(2-2^{2i-2j}\right)}{(2j)!(2i-2j)!}\cdot\frac{(-1)^{k+1-i}(2u)^{2k+2-2i}}{(2k+2-2i)!},$$

which goes into the final formula:

$$H_{2k+1}(n) = \frac{1}{2n^{2k+1}} + \pi^{2k+1}\int_0^1 p_{2k+1}(u)\left(1-\cos 2\pi n(1-u)\right)\cot \pi u\,du$$

4 Alternating Harmonic Numbers: $C_k^2(n)$

Setting the initial equation to $\cos\pi k = (-1)^k$ drastically changes the picture. It no longer enables us to calculate $H_k(n)$, but the alternating harmonic numbers instead, $C_k^2(n)$.

Below are a couple of examples of formulae for the alternating harmonic numbers:

$$C_1^2(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k} = H_1(n) + \frac{1}{2n}\left(-1 + \cos\pi n\right) - \frac{\pi}{2}\int_0^1 (1 - \cos\pi n(1-u))\tan\frac{\pi u}{2}\,du$$

$$C_2^2(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k^2} = H_2(n) + \frac{1}{2n^2}\left(-1 + \frac{(\pi n)^2}{2!} + \cos\pi n\right) - \frac{\pi^2}{2}\int_0^1 u\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du$$

4.1 General Formula: $C_k^2(n)$

The recurrence equations for the generalized alternating harmonic numbers are:

$$C_{2k}^2(n) = \sum_{j=1}^{n} \frac{(-1)^j}{j^{2k}} = \frac{1}{2n^{2k}}\left(\cos\pi n - \sum_{j=0}^{k} \frac{(-1)^j(\pi n)^{2j}}{(2j)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k-2j)!}H_{2j}(n)$$
$$+ \frac{(-1)^k\pi^{2k}}{2(2k-1)!}\int_0^1 u^{2k-1}\sin\pi n(1-u)\tan\frac{\pi u}{2}\,du$$

$$C_{2k+1}^2(n) = \sum_{j=1}^{n} \frac{(-1)^j}{j^{2k+1}} = \frac{1}{2n^{2k+1}}\left(\cos\pi n - \sum_{j=0}^{k} \frac{(-1)^j(\pi n)^{2j}}{(2j)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k-2j)!}H_{2j+1}(n)$$
$$- \frac{(-1)^k\pi^{2k+1}}{2(2k)!}\int_0^1 u^{2k}\left(1 - \cos\pi n(1-u)\right)\tan\frac{\pi u}{2}\,du$$

5 Odd Alternating Harmonic Numbers: $S_k^4(n)$

If we set the initial equation to $\sin\pi k/2$, we are able to obtain formulae for the odd alternating harmonic numbers, $S_k^4(n)$.

Two examples of formulae for the odd alternating harmonic numbers are below:

$$S_1^4(n) = \sum_{k=1}^{n} \frac{1}{k} sin\frac{\pi k}{2} = \frac{1}{2n}\left(-\frac{\pi n}{2} + \sin\frac{\pi n}{2}\right) + \frac{\pi}{4}\int_0^1 \sin\frac{\pi n(1-u)}{2}\left(\sec\frac{\pi u}{2} + \tan\frac{\pi u}{2}\right)du$$

$$S_2^4(n) = \sum_{k=1}^{n} \frac{1}{k^2}\sin\frac{\pi k}{2} = \frac{\pi}{2}H_1(n) + \frac{1}{2n^2}\left(-\frac{\pi n}{2} + \sin\frac{\pi n}{2}\right) - \frac{\pi^2}{8}\int_0^1 u\left(1 - \cos\frac{\pi n(1-u)}{2}\right)\left(\sec\frac{\pi u}{2} + \tan\frac{\pi u}{2}\right)du$$

5.1 General Formula: $S_k^4(n)$

The recurrence equations for the generalized odd alternating harmonic numbers are:

$$S_{2k}^4(n) = \sum_{j=1}^{n} \frac{1}{j^{2k}} \sin\frac{\pi j}{2} = \frac{1}{2n^{2k}} \left(\sin\frac{\pi n}{2} - \sum_{j=0}^{k-1} \frac{(-1)^j (\frac{\pi n}{2})^{2j+1}}{(2j+1)!} \right) - \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (\frac{\pi}{2})^{2k-1-2j}}{(2k-1-2j)!} H_{2j+1}(n)$$
$$+ \frac{(-1)^k (\frac{\pi}{2})^{2k}}{2(2k-1)!} \int_0^1 u^{2k-1} \left(1 - \cos\frac{\pi n(1-u)}{2}\right) \left(\sec\frac{\pi u}{2} + \tan\frac{\pi u}{2}\right) du$$

$$S_{2k+1}^4(n) = \sum_{j=1}^{n} \frac{1}{j^{2k+1}} \sin\frac{\pi j}{2} = \frac{1}{2n^{2k+1}} \left(\sin\frac{\pi n}{2} - \sum_{j=0}^{k} \frac{(-1)^j (\frac{\pi n}{2})^{2j+1}}{(2j+1)!} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j} (\frac{\pi}{2})^{2k+1-2j}}{(2k+1-2j)!} H_{2j}(n)$$
$$+ \frac{(-1)^k (\frac{\pi}{2})^{2k+1}}{2(2k)!} \int_0^1 u^{2k} \sin\frac{\pi n(1-u)}{2} \left(\sec\frac{\pi u}{2} + \tan\frac{\pi u}{2}\right) du$$

6 General Formula: $C_k^m(n)$ and $S_k^m(n)$

There's a striking similarity between the formulae derived from initial equation $\cos \pi k = 1$ and the ones derived with $\sin \pi k/2$. Based on this similarity, we are able to generalize the pattern.

6.1 $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$

We've grouped these two under the same section because they share an integral and they both have $H_{2j}(n)$ in their recursions.

For all complex m, $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$ are given by:

$$C_{2k}^m(n) = \sum_{j=1}^{n} \frac{1}{j^{2k}} \cos\frac{2\pi j}{m} = \frac{1}{2n^{2k}} \left(\cos\frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j (\frac{2\pi n}{m})^{2j}}{(2j)!} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j} (\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} H_{2j}(n)$$
$$+ \frac{(-1)^k (\frac{2\pi}{m})^{2k}}{2(2k-1)!} \int_0^1 (1-u)^{2k-1} \sin\frac{2\pi n u}{m} \cot\frac{\pi u}{m}\, du, \ \forall \text{ integer } k \geq 1$$

$$S_{2k+1}^m(n) = \sum_{j=1}^{n} \frac{1}{j^{2k+1}} \sin\frac{2\pi j}{m} = \frac{1}{2n^{2k+1}} \left(\sin\frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j (\frac{2\pi n}{m})^{2j+1}}{(2j+1)!} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j} (\frac{2\pi}{m})^{2k+1-2j}}{(2k+1-2j)!} H_{2j}(n)$$
$$+ \frac{(-1)^k (\frac{2\pi}{m})^{2k+1}}{2(2k)!} \int_0^1 (1-u)^{2k} \sin\frac{2\pi n u}{m} \cot\frac{\pi u}{m}\, du, \ \forall \text{ integer } k \geq 0$$

Notice that in order for these equations to hold, we need to have $H_0(n) = 0$ for all positive integer n, as mentioned before.

6.1.1 Limits of $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$

At infinity, $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$ become Fourier series, denoted here by C_{2k}^m and S_{2k+1}^m, whose closed-forms are given by Bernoulli polynomials, per Abramowitz and Stegun:[1]

$$\sum_{j=1}^{\infty} \frac{1}{j^{2k}} \cos 2\pi x j = \frac{-(-1)^k (2\pi)^{2k}}{2(2k)!} B_{2k}(x) \text{ and } \sum_{j=1}^{\infty} \frac{1}{j^{2k+1}} \sin 2\pi x j = \frac{(-1)^k (2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(x)$$

The above result implies the following theorem, which holds for all integer $k \geq 0$ and real $m \geq 1$:

Theorem 1 $\lim_{n\to\infty} \int_0^1 (1-u)^k \sin \frac{2\pi n u}{m} \cot \frac{\pi u}{m}\, du = \begin{cases} 1, & \text{if } k=0 \text{ and } m=1 \\ \frac{m}{2}, & \text{otherwise} \end{cases}$

Therefore, with the exception of $S_1^1 = 0$, the limits of $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$, for real $m \geq 1$, are given by:

$$C_{2k}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \cos \frac{2\pi j}{m} = \sum_{j=0}^{k} \frac{(-1)^{k-j} \left(\frac{2\pi}{m}\right)^{2k-2j}}{(2k-2j)!} \zeta(2j) + \frac{(-1)^k m}{4(2k-1)!} \left(\frac{2\pi}{m}\right)^{2k} \quad (\forall\ k \geq 1)$$

$$S_{2k+1}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k+1}} \sin \frac{2\pi j}{m} = \sum_{j=0}^{k} \frac{(-1)^{k-j} \left(\frac{2\pi}{m}\right)^{2k+1-2j}}{(2k+1-2j)!} \zeta(2j) + \frac{(-1)^k m}{4(2k)!} \left(\frac{2\pi}{m}\right)^{2k+1} \quad (\forall\ k \geq 0)$$

These are just rewrites of the expressions for C_{2k}^m and S_{2k+1}^m from reference [1], with $x = 1/m$.

6.2 $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$

For all complex m, $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$ are given by:

$$C_{2k+1}^m(n) = \sum_{j=1}^{n} \frac{1}{j^{2k+1}} \cos \frac{2\pi j}{m} = \frac{1}{2n^{2k+1}} \left(\cos \frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j (\frac{2\pi n}{m})^{2j}}{(2j)!} \right) + \sum_{j=0}^{k} \frac{(-1)^{k-j} (\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} H_{2j+1}(n)$$
$$- \frac{(-1)^k (\frac{2\pi}{m})^{2k+1}}{2(2k)!} \int_0^1 (1-u)^{2k} \left(1 - \cos \frac{2\pi n u}{m}\right) \cot \frac{\pi u}{m}\, du, \ \forall \text{ integer } k \geq 0$$

$$S_{2k}^m(n) = \sum_{j=1}^{n} \frac{1}{j^{2k}} \sin \frac{2\pi j}{m} = \frac{1}{2n^{2k}} \left(\sin \frac{2\pi n}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j (\frac{2\pi n}{m})^{2j+1}}{(2j+1)!} \right) - \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (\frac{2\pi}{m})^{2k-1-2j}}{(2k-1-2j)!} H_{2j+1}(n)$$
$$+ \frac{(-1)^k (\frac{2\pi}{m})^{2k}}{2(2k-1)!} \int_0^1 (1-u)^{2k-1} \left(1 - \cos \frac{2\pi n u}{m}\right) \cot \frac{\pi u}{m}\, du, \ \forall \text{ integer } k \geq 1$$

6.2.1 Limits of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$

Before taking the limit of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$ as n tends to infinity, we need to remove $H(n)$ from the second sum (since it explodes to infinity), and recombine it with the integral.

In order to do that, we need to use one of the three formulae we created for $H(n)$ in sections (3.2), (3.9) and (3.10). Since the last two are almost identical, let's consider only the first two:

$$H(n) - \frac{1}{2n} = \frac{\pi}{2}\int_0^1 (1-u)\,(1-\cos\pi nu)\cot\frac{\pi u}{2}\,du = \pi\int_0^1 (1-u)\,(1-\cos 2\pi nu)\cot\pi u\,du$$

By using either one of these formulae, we can carve out two integrals, one that doesn't depend on n and one that does, as in the below example:

$$C_{2k+1}^m(n) = \frac{1}{2n^{2k+1}}\left(\cos\frac{2\pi n}{m} - \sum_{j=0}^{k}\frac{(-1)^j(\frac{2\pi n}{m})^{2j}}{(2j)!}\right) + \sum_{j=1}^{k}\frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!}H_{2j+1}(n) - \frac{(-1)^k(\frac{2\pi}{m})^{2k+1}}{2(2k)!}$$
$$\left(-\frac{m}{2\pi n} + \int_0^1 (1-u)^{2k}\cot\frac{\pi u}{m} - m(1-u)\cot\pi u\,du - \int_0^1 (1-u)^{2k}\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - m(1-u)\cos 2\pi nu\cot\pi u\,du\right)$$

Therefore, to know the limit of $C_{2k+1}^m(n)$, we need to know the limit of the integral to the right as n grows. This limit is given in the following theorem, which holds for all integer $k \geq 0$ and real $m \geq 1$ (except $k = 0$ and $m = 1$), and for which we don't provide a proof:

Theorem 2 $\displaystyle\lim_{n\to\infty}\int_0^1 (1-u)^k\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - m(1-u)\cos 2\pi nu\cot\pi u\,du = \frac{m\log(m)}{\pi}$

This limit apparently doesn't exist in the literature. Theorem 2 allows us to deduce the following corollary:

Corollary 1 $\displaystyle\lim_{n\to\infty}\int_0^1 \left(u^k - u\right)\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du = 0 \;\forall$ integer $k \geq 0$

Proof 1 This result stems from Theorem 2 and the fact we can write $C_{2k+1}^m(n)$ or $S_{2k}^m(n)$ using different formulae for $H(n)$, which leads to an equation:

$$\int_0^1 (1-u)^k\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - \frac{m}{2}(1-u)\cos\pi nu\cot\frac{\pi u}{2}\,du - \int_0^1 (1-u)^k\cot\frac{\pi u}{m} - \frac{m}{2}(1-u)\cot\frac{\pi u}{2}\,du$$
$$= \int_0^1 (1-u)^k\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - m(1-u)\cos 2\pi nu\cot\pi u\,du - \int_0^1 (1-u)^k\cot\frac{\pi u}{m} - m(1-u)\cot\pi u\,du \Rightarrow$$

$$\int_0^1 (1-u)^k\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - \frac{m}{2}(1-u)\cos\pi nu\cot\frac{\pi u}{2}\,du = \int_0^1 (1-u)^k\cos\frac{2\pi nu}{m}\cot\frac{\pi u}{m} - m(1-u)\cos 2\pi nu\cot\pi u\,du$$
$$+ \int_0^1 m(1-u)\cot\pi u - \frac{m}{2}(1-u)\cot\frac{\pi u}{2}\,du$$

Now, by making $m = 2$ and using Theorem 2, it follows from the above relation that:

$$\lim_{n\to\infty}\int_0^1 \left(u^k - u\right)\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du = \frac{2\log(2)}{\pi} + \int_0^1 2(1-u)\cot\pi u - (1-u)\cot\frac{\pi u}{2}\,du = 0. \;\blacksquare$$

Now that we have Theorem 2, we can figure out the limits of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$ which, except for $C_1^1 = \infty$, are given by:

$$C_{2k+1}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k+1}} \cos \frac{2\pi j}{m} = \sum_{j=1}^{k} \frac{(-1)^{k-j} \left(\frac{2\pi}{m}\right)^{2k-2j}}{(2k-2j)!} \zeta(2j+1) + \frac{(-1)^k \log(m) \left(\frac{2\pi}{m}\right)^{2k}}{(2k)!}$$
$$- \frac{(-1)^k (\frac{2\pi}{m})^{2k+1}}{2(2k)!} \int_0^1 (1-u)^{2k} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u \, du$$

$$S_{2k}^m = \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \sin \frac{2\pi j}{m} = -\sum_{j=1}^{k-1} \frac{(-1)^{k-j} \left(\frac{2\pi}{m}\right)^{2k-1-2j}}{(2k-1-2j)!} \zeta(2j+1) - \frac{(-1)^k \log(m) \left(\frac{2\pi}{m}\right)^{2k-1}}{(2k-1)!}$$
$$+ \frac{(-1)^k (\frac{2\pi}{m})^{2k}}{2(2k-1)!} \int_0^1 (1-u)^{2k-1} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u \, du$$

Note that $H(n)$ diverges because $\int_0^1 \cot \pi u - (1-u) \cot \pi u \, du$ diverges.

7 Example: Infinite Sum of $H(n)/n^2$

In this section, we derive expressions for sums of the type $H_k(n)/n^r$, over the positive integers n, with k odd and r even, and vice-versa. We will not try to get the result for k and r both even or odd, because these cases lead to integrals that are very hard to evaluate.

Hence, let's start with an example. We want to obtain the sum of $H(n)/n^2$ over the positive integers using the formula for $H(n)$ from section (3.9):

$$H(n) = \frac{1}{2n} + \pi \int_0^1 u \left(1 - \cos 2\pi n(1-u)\right) \cot \pi(1-u) \, du \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{H(n)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{2n} + \pi \int_0^1 (1-u) \left(1 - \cos 2\pi n u\right) \cot \pi u \, du \right) =$$
$$\frac{1}{2}\zeta(3) + \pi \int_0^1 (1-u) \left(\zeta(2) - \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 2\pi n u \right) \cot \pi u \, du$$

The Fourier series can be simplified using the results from section (6.1.1), giving us:

$$\sum_{n=1}^{\infty} \frac{H(n)}{n^2} = \frac{1}{2}\zeta(3) + \pi^3 \int_0^1 u(1-u)^2 \cot \pi u \, du = 2\zeta(3)$$

7.1 General Formula: Sum of $H_{2k}(n)/n^{2r+1}$

Here we use the formula for $H_{2k}(n)$ from section (3.9.2), with a slight transformation only valid for integer n:

$$\sum_{n=1}^{\infty} \frac{H_{2k}(n)}{n^{2r+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \left(\frac{1}{2n^{2k}} - \frac{(-1)^k (2\pi)^{2k}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j} \left(2 - 2^{2j}\right) u^{2k-2j}}{(2j)!(2k-2j)!} \sin 2\pi n u \cot \pi u \, du \right)$$

$$\sum_{n=1}^{\infty}\frac{H_{2k}(n)}{n^{2r+1}}=\frac{\zeta(2k+2r+1)}{2}-\frac{(-1)^k(2\pi)^{2k}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k-2j}}{(2j)!(2k-2j)!}\left(\sum_{n=1}^{\infty}\frac{\sin 2\pi nu}{n^{2r+1}}\right)\cot\pi u\,du$$

Now, the closed-form of the Fourier series above, after a change of variables, $u=1/m$, is given by:

$$\sum_{n=1}^{\infty}\frac{\sin 2\pi nu}{n^{2r+1}}=\sum_{i=0}^{r}\frac{(-1)^{r-i}\zeta(2i)(2\pi u)^{2r+1-2i}}{(2r+1-2i)!}+\frac{(-1)^r(2\pi)^{2r+1}u^{2r}}{4(2r)!}=-\frac{(-1)^r(2\pi)^{2r+1}}{2(2r+1)!}B_{2r+1}(u)$$

Therefore, we can express the sum as function of Bernoulli polynomials, and it's finite for all integer $r\geq 1$:

$$\sum_{n=1}^{\infty}\frac{H_{2k}(n)}{n^{2r+1}}=\frac{\zeta(2k+2r+1)}{2}+\frac{(-1)^{k+r}(2\pi)^{2k+2r+1}}{2(2k)!(2r+1)!}\int_0^1\left(B_{2k}(u)-2^{2k-1}B_{2k}\left(\frac{u}{2}\right)\right)B_{2r+1}(u)\cot\pi u\,du$$

If $k=0$ the sum is always zero (since $H_0(n)=0$), which enables us to deduce another integral representation for $\zeta(2r+1)$, which happens to coincide with the one in Abramowitz and Stegun:[1]

$$\zeta(2r+1)=-\frac{(-1)^r(2\pi)^{2r+1}}{2(2r+1)!}\int_0^1 B_{2r+1}(u)\cot\pi u\,du$$

7.2 General Formula: Sum of $H_{2k+1}(n)/n^{2r}$

Here we use the formula for $H_{2k+1}(n)$ from section (3.9.3), though we could've used the two others as well (notice we made a transformation only valid for integer n):

$$\sum_{n=1}^{\infty}\frac{H_{2k+1}(n)}{n^{2r}}=\sum_{n=1}^{\infty}\frac{1}{n^{2r}}\left(\frac{1}{2n^{2k+1}}-\frac{(-1)^k(2\pi)^{2k+1}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}(1-\cos 2\pi nu)\cot\pi u\,du\right)$$

$$\sum_{n=1}^{\infty}\frac{H_{2k+1}(n)}{n^{2r}}=\frac{\zeta(2k+1+2r)}{2}-\frac{(-1)^k(2\pi)^{2k+1}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\left(\sum_{n=1}^{\infty}\frac{1-\cos 2\pi nu}{n^{2r}}\right)\cot\pi u\,du$$

Now, the closed-form of the Fourier series featured in the above equation is given in section (6.1.1), and it can also be expressed as Bernoulli polynomials. That is, after a change of variables, $u=1/m$, we get:

$$\sum_{n=1}^{\infty}\frac{\cos 2\pi nu}{n^{2r}}=\sum_{i=0}^{r}\frac{(-1)^{r-i}\zeta(2i)(2\pi u)^{2r-2i}}{(2r-2i)!}+\frac{(-1)^r(2\pi)^{2r}u^{2r-1}}{4(2r-1)!}=-\frac{(-1)^r(2\pi)^{2r}}{2(2r)!}B_{2r}(u)$$

In a way, the closed-form from section (6.1.1) is more general than the Bernoulli polynomial. For instance, when the denominator here is n^{2r+1} instead of n^{2r}, we can use its analogous form from section (6.2.1), which is no longer a Bernoulli polynomial.

That said, for integer $k\geq 0$ and $r\geq 1$, we can write:

$$\sum_{n=1}^{\infty}\frac{H_{2k+1}(n)}{n^{2r}}=\frac{\zeta(2k+1+2r)}{2}$$
$$-\frac{(-1)^k(2\pi)^{2k+1}}{(2k+1)!}\int_0^1\left(B_{2k+1}(u)-2^{2k}B_{2k+1}\left(\frac{u}{2}\right)\right)\left(\zeta(2r)+\frac{(-1)^r(2\pi)^{2r}}{2(2r)!}B_{2r}(u)\right)\cot\pi u\,du$$

Or, although we lose the validity of the formula for $k = 0$, we can rewrite the sum as:

$$\sum_{n=1}^{\infty} \frac{H_{2k+1}(n)}{n^{2r}} = \frac{\zeta(2k+1+2r)}{2} + \zeta(2k+1)\zeta(2r) - \frac{(-1)^{k+r}(2\pi)^{2k+1+2r}}{2(2k+1)!(2r)!} \int_0^1 \left(B_{2k+1}(u) - 2^{2k}B_{2k+1}\left(\frac{u}{2}\right)\right) B_{2r}(u) \cot \pi u \, du$$

8 Limits of the Integrals

8.1 Limits of the Integrals on the $H_{2k}(n)$ Recursions

In this section we present proofs for the limits of the integrals that appear on the recursions of $H_{2k}(n)$ from sections (3.7), (3.9.2) and (3.10.2). This approach requires prior knowledge of the closed-forms of $\zeta(2k)$, as mentioned in sections (3.3) and (3.5).

Looking back at the set of recurrence equations from the aforementioned sections, it's evident that we can express each integral as a function of $H_{2j}(n)$:

$$\int_0^1 u^{2k} \sin \pi n(1-u) \tan \frac{\pi u}{2} \, du = \frac{-2(-1)^k(2k)!}{\pi^{2k}} \left(\sum_{j=1}^{k} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k+1-2j)!} H_{2j}(n) - \frac{1}{2n^{2k}} \sum_{j=0}^{k} \frac{(-1)^j(\pi n)^{2j}}{(2j+1)!} \right)$$

$$\int_0^1 u^{2k} \sin 2\pi n(1-u) \cot \pi u \, du = \frac{2(-1)^k(2k)!}{(2\pi)^{2k}} \left(\sum_{j=1}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} H_{2j}(n) - \frac{1}{2n^{2k}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi n)^{2j}}{(2j+1)!} \right)$$

$$\int_0^1 u^{2k+1} \sin 2\pi n(1-u) \cot \pi u \, du = \frac{2(-1)^k(2k+1)!}{(2\pi)^{2k}} \left(\sum_{j=1}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!} H_{2j}(n) - \frac{1}{2n^{2k}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!} \right)$$

We can deduce the limits of these integrals based on the closed-forms of $\zeta(2k)$. Conversely, knowing these limits allows us to deduce the closed-forms of $\zeta(2k)$.

Theorem 3 $\lim_{n\to\infty} \int_0^1 u^{2k} \sin \pi n(1-u) \tan \frac{\pi u}{2} \, du = 1 \ \forall$ integer $k \geq 0$

Proof 3 This integral appears with initial equation $\sin \pi k = 0$ and per Theorem 1, section (6.1.1), its limit is $m/2 = 1$, which we shall confirm now. By taking the limit of the integral as n approaches infinity, we have:

$$\lim_{n\to\infty} \int_0^1 u^{2k} \sin \pi n(1-u) \tan \frac{\pi u}{2} \, du = \frac{-2(-1)^k(2k)!}{\pi^{2k}} \left(\sum_{j=1}^{k} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k-2j+1)!} \zeta(2j) - \frac{(-1)^k \pi^{2k}}{2(2k+1)!} \right)$$

$$= \frac{-2(-1)^k(2k)!}{\pi^{2k}} \sum_{j=0}^{k} \frac{(-1)^{k-j}\pi^{2k-2j}}{(2k-2j+1)!} \zeta(2j) = (2k)! \sum_{j=0}^{k} \frac{2^{2j}B_{2j}}{(2k-2j+1)!(2j)!}$$

(Note that $H_0(n) = 0$, but $\zeta(0) = -1/2$.) Now, to complete the proof, let's show that the above sum equals 1 for all integer $k \geq 0$. For that, let $g(x)$ be the product of the two below functions:

$$x \coth x = x\,\frac{e^x + e^{-x}}{e^x - e^{-x}} = \sum_{j=0}^{\infty} \frac{2^{2j} B_{2j}}{(2j)!} x^{2j}, \text{ and} \sinh x = \frac{e^x - e^{-x}}{2} = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} x^{2j+1} \Rightarrow$$

$$g(x) = x \coth x\, \sinh x = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{2^{2j} B_{2j}}{(2j)!} x^{2j} \cdot \frac{1}{(2k-2j+1)!} x^{2k-2j+1} \Rightarrow$$

$$g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{2^{2j} B_{2j}}{(2k-2j+1)!(2j)!} \right) x^{2k+1} = x\,\frac{e^x + e^{-x}}{2} = x \cosh x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k+1},$$

which implies the theorem. ■

Theorem 4 $\lim_{n\to\infty} \int_0^1 u^k \sin 2\pi n(1-u) \cot \pi u \, du = \begin{cases} -1, & \text{if } k = 0 \\ -\frac{1}{2}, & \text{if integer } k \geq 1 \end{cases}$

Proof 4 This integral appears with initial equations $\sin 2\pi k = 0$ and $\cos 2\pi k = 1$. Here we only prove the case $\sin 2\pi k = 0$, though case $\cos 2\pi k = 1$ should follow a similar reasoning and be straightforward.

By taking the limit of the integral as n goes to infinity, we have:

$$\lim_{n\to\infty} \int_0^1 u^{2k} \sin 2\pi n(1-u) \cot \pi u \, du = \frac{2(-1)^k (2k)!}{(2\pi)^{2k}} \left(\sum_{j=1}^{k} \frac{(-1)^{k-j} (2\pi)^{2k-2j}}{(2k-2j+1)!} \zeta(2j) - \frac{(-1)^k (2\pi)^{2k}}{2(2k+1)!} \right)$$

$$= \frac{2(-1)^k (2k)!}{(2\pi)^{2k}} \sum_{j=0}^{k} \frac{(-1)^{k-j} (2\pi)^{2k-2j}}{(2k-2j+1)!} \zeta(2j) = -(2k)! \sum_{j=0}^{k} \frac{B_{2j}}{(2k-2j+1)!(2j)!}$$

Now, to complete the proof, let's show that the above sum equals -1 if $k = 0$, or $-1/2$ if $k \geq 1$. For that, let $g(x)$ be the product of the two below functions, that also appeared in the proof of Theorem 3, only now the cotangent is re-scaled:

$$\frac{x}{2} \coth \frac{x}{2} = \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!} x^{2j}, \text{ and} \sinh x = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} x^{2j+1}$$

Therefore, we have:

$$g(x) = \frac{x}{2} \coth \frac{x}{2}\, \sinh x = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{B_{2j}}{(2j)!} x^{2j} \cdot \frac{1}{(2k-2j+1)!} x^{2k-2j+1} \Rightarrow$$

$$g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{B_{2j}}{(2k-2j+1)!(2j)!} \right) x^{2k+1} = \frac{x}{2} \left(1 + \frac{e^x + e^{-x}}{2} \right) = x + \sum_{k=1}^{\infty} \frac{1}{2(2k)!} x^{2k+1},$$

which implies the theorem. ■

8.1.1 Limit of $H_{2k}(n)$

The limit of $H_{2k}(n)$ as n approaches infinity is $\zeta(2k)$ for all integer $k \geq 0$, which is proved in the two following theorems:

Theorem 5 $\lim_{n\to\infty} \frac{1}{2n^{2k}} - \frac{(-1)^k \pi^{2k}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right) u^{2k-2j}}{(2j)!(2k-2j)!} \sin \pi n(1-u) \tan \frac{\pi u}{2}\, du = \zeta(2k)$

Proof 5 First we note that due to Theorem 3, the above limit reduces to:

$$-\frac{(-1)^k \pi^{2k}}{2} \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!}$$

Now we can prove that the above expression equals $\zeta(2k)$, using the same approach from the previous section:

$$\sum_{k=0}^{\infty} x^{2k} \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\left(2-2^{2j}\right) B_{2j} x^{2j}}{(2j)!} \cdot \frac{x^{2k-2j}}{(2k-2j)!} = \left(x \coth \frac{x}{2} - x \coth x\right) \cosh x$$

$$= x \coth x = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k} \Rightarrow -\frac{(-1)^k \pi^{2k}}{2} \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!} = -\frac{(-1)^k (2\pi)^{2k} B_{2k}}{2(2k)!} = \zeta(2k) \blacksquare$$

Theorem 6 $\lim_{n\to\infty} \frac{1}{2n^{2k}} + \frac{(-1)^k (2\pi)^{2k}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right) u^{2k-2j}}{(2j)!(2k-2j)!} \sin 2\pi n(1-u) \cot \pi u\, du = \zeta(2k)$

Proof 6 : First we note that due to Theorem 4, the above limit reduces to:

$$-\frac{(-1)^k (2\pi)^{2k}}{2} \left(\frac{B_{2k}(2-2^{2k})}{(2k)!} + \frac{1}{2} \sum_{j=0}^{k-1} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!} \right)$$

Now let's prove that the above expression equals $\zeta(2k)$, using some of the previous results:

$$\frac{B_{2k}(2-2^{2k})}{(2k)!} + \frac{1}{2} \sum_{j=0}^{k-1} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!} = \frac{1}{2} \frac{B_{2k}(2-2^{2k})}{(2k)!} + \frac{1}{2} \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k-2j)!}$$

$$= \frac{1}{2} \frac{B_{2k}(2-2^{2k})}{(2k)!} + \frac{1}{2} \frac{B_{2k} 2^{2k}}{(2k)!} = \frac{B_{2k}}{(2k)!} \Rightarrow -\frac{(-1)^k (2\pi)^{2k}}{2} \frac{B_{2k}}{(2k)!} = \zeta(2k) \blacksquare$$

8.2 Limits of the Integrals on the $H_{2k+1}(n)$ Recursions

We can express each integral as a function of $H_{2j+1}(n)$ for all real n:

$$\int_0^1 u^{2k+1} \left(1 - \cos \pi n(1-u)\right) \tan \frac{\pi u}{2}\, du = \frac{2(-1)^k (2k+1)!}{\pi^{2k+1}} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j} \pi^{2k-2j}}{(2k+1-2j)!} H_{2j+1}(n) - \frac{1}{2n^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j (\pi n)^{2j}}{(2j+1)!} \right)$$

$$\int_0^1 u^{2k+1} \left(1 - \cos 2\pi n(1-u)\right) \cot \pi u\, du = \frac{-2(-1)^k (2k+1)!}{(2\pi)^{2k+1}} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j} (2\pi)^{2k-2j}}{(2k+1-2j)!} H_{2j+1}(n) - \frac{1}{2n^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi n)^{2j}}{(2j+1)!} \right)$$

$$\int_0^1 u^{2k+2}\left(1-\cos 2\pi n(1-u)\right)\cot \pi u\, du = \frac{-2(-1)^k(2k+2)!}{(2\pi)^{2k+1}}\left(\sum_{j=0}^{k}\frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}H_{2j+1}(n) - \frac{1}{2n^{2k+1}}\sum_{j=0}^{k}\frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!}\right)$$

From the above equations, we can infer that each one of the integrals tends to plus or minus infinity as n increases (the reason is that each one contains $H(n)$, which is unbounded).

One consequence of this fact is that the coefficients of $p_{2k+1}(u)$ in the formulae of $H_{2k+1}(n)$ need to sum up to 0 for all $k \geq 1$, in order to cancel out those infinities, the exception being $H(n)$. This statement is translated in the next theorem.

Theorem 7 $p_{2k+1}(1) = \sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k+1-2j)!} = 0, \forall$ integer $k \geq 1$

Proof 7 In section (3.8), we've created a generating function for $p_{2k+1}(u)$, which allows us to deduce the following equivalence (notice the second sum in the double-sum is $p_{2k+1}(u)$):

$$\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!} = \frac{x\sin xu}{\sin x}$$

Now, if $u=1$, we conclude that the above sum equals x, which implies the theorem. ■

Similarly, the below also holds, though the proof is omitted:

$$\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{B_{2j}B_{2i-2j}\left(2-2^{2j}\right)\left(2-2^{2i-2j}\right)2^{2k+2-2i}}{(2j)!(2i-2j)!(2k+2-2i)!} = 0\ \forall\ k \geq 1$$

8.2.1 Limit of $H_{2k+1}(n)$

The values of $\zeta(2k+1)$ are given by the first part of the integral, as explained by the next theorem:

Theorem 8 $\zeta(2k+1) = \frac{(-1)^k\pi^{2k+1}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\tan\frac{\pi u}{2}\,du, \forall$ integer $k \geq 1$

Proof 8 To prove this result, we need to show that:

$$\lim_{n\to\infty}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du = 0$$

Using the result from Theorem 2, section (6.2.1), we know that for large n we can write:

$$\int_0^1 u^{2k+1-2j}\cos\pi n(1-u)\tan\frac{\pi u}{2} + 2u\cos 2\pi n(1-u)\cot\pi u\,du \sim \frac{2\log(2)}{\pi}$$

But per Theorem 7:

$$\frac{2\log(2)}{\pi}\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)}{(2j)!(2k+1-2j)!} = 0 \text{ and } \int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u}{(2j)!(2k+1-2j)!}\cos 2\pi n(1-u)\cot\pi u\,du = 0$$

which implies the theorem. ■

There's a slightly different integral representation for $\zeta(2k+1)$, which stems from the formula derived in section (3.9):

$$\zeta(2k+1) = -\frac{(-1)^k (2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right) u^{2k+1-2j}}{(2j)!(2k+1-2j)!} \cot \pi u \, du$$

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