Absolute Convergence of the Series of Fourier–Haar Coefficients

Boris Golubov
Moscow Institute of Physical Technologies
Institute Lane 9, Dolgoproudny, Moscow region, 141700, Russia
Email: golubov@mail.mipt.ru

Sergey Volosivets
Saratov State University
Astrakhanskaya Str. 83, Saratov, 410012, Russia
Email: VolosivetsSS@mail.ru

***Abstract*—We give some sharp statements on absolute convergence of the series of Fourier-Haar coefficients in terms of L_p- and p-variation best approximations by Haar polynomials.**

INTRODUCTION.

The Haar orthonormal system $\{\chi_n\}_{n=1}^{\infty}$ had been constructed in 1909 (see [1]). By this system A. Haar gave positive answer on the question of D. Hilbert: is there an orthogonal system such that Fourier series with respect to this system of any continuous function converges uniformly to that function?

Let us recall the definition of Haar system. We set $\chi_1(x) = 1$ on $[0, 1]$. After that we introduce the open dyadic intervals $I_i^k = \left(2^{-k}(i-1), 2^{-k}i\right)$, $i = 1, ..., 2^k$, $k = 0, 1, ...,$ and represent the natural number $n \geq 2$ in the form $n = 2^k + i$, $i = 1, ..., 2^k$, $k = 0, 1, ...$ Then we set $\chi_n(x) = 2^{k/2}$ for $x \in I_{2i-1}^{k+1}$, $\chi_n(x) = -2^{k/2}$ for $x \in I_{2i}^{k+1}$ and $\chi_n(x) = 0$ for $x \in [0, 1]\backslash\overline{I_i^k}$, where $\overline{I_i^k}$ is closure of the interval I_i^k. If the Haar function $\chi_n(x)$ has a jump in some poimt $x \in (0, 1)$, then $\chi_n(x) = [\chi_n(x-0) + \chi_n(x+0)]/2$. In the end points of interval $[0, 1]$ we set $\chi_n(0) = \chi_n(0+0)$ and $\chi_n(1) = \chi_n(1-0)$. The Haar functions $\chi_n(x)$ are step functions.

The principal information on Fourier-Haar series may be found in the book [2].

For a function $f \in L_p[0, 1]$, $1 \leq p < \infty$, we introduce the integral modulus of continuity

$$\omega\left(\delta, f\right)_p = \sup_{0\leq h\leq\delta}\left(\int_0^{1-h}|f(t+h) - f(t)|^p\,dt\right)^{1/p}, \quad (1)$$

$0 \leq \delta \leq 1$, and the Fourier-Haar coefficients

$$\hat{f}(n) = \int_0^1 f(x)\chi_n(x)dx, \qquad n \in \mathbf{N}.$$

Z. Ciesielski and J. Musielak [3] proved the following

Theorem A. *Let $\beta > 0$, $\gamma \geq 0$, $p = \max(\beta, 1)$, $f \in L_p[0, 1)$, and $\sum\limits_{n=1}^{\infty} n^{\gamma-\beta/2}\omega^\beta\left(1/n, f\right)_p < \infty$. Then the series $\sum\limits_{n=1}^{\infty} n^\gamma\left|\hat{f}(n)\right|^\beta$ converges.*

Let us observe that in the paper [3] the authors introduced a slightly different definition of the integral modulus of continuity in the space $L^p[0, 1]$, $1 \leq p < \infty$. They extended the function f to the real axis by setting $f(x) = 0$ for $x \notin [0, 1]$, and evaluated the integral in the right-hand side of (1) over the interval $[0, 1]$. But if we analyze the proof of Theorem 2 from [3], we see that the statement of Theorem A is valid.

Let us define the Wiener's class $V_p[0, 1]$, $1 \leq p < \infty$, of functions of bounded p^{th}-power variation on the interval $[0, 1]$ (see [4]). We set $f \in V_p[0, 1]$, if

$$V(f)_p = \sup_\tau\left\{\sum_{i=1}^n|f(x_i) - f(x_{i-1})|^p\right\}^{1/p} < \infty,$$

where $\tau = \{0 = x_0 < x_1 < ... < x_n = 1\}$ is arbitrary partition of the interval $[0, 1]$. Let us note that the inclusion $Lip(1/p) \subset V_p[0, 1]$ holds for $1 \leq p < \infty$.

P. L. Ulyanov [5] proved the following theorem.

Theorem B. *For the function $f \in V_1[0, 1]$ the series*

$$\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^\beta \quad \text{or} \quad \sum_{n=1}^{\infty} n^{\gamma-1/2}\left|\hat{f}(n)\right| \qquad (2)$$

converge, if $\beta > 2/3$ or $\gamma < 1$ respectively. But this statement does not true for $\beta = 2/3$ or $\gamma = 1$ respectively.

The first author (see [6]) generalized Theorem B to functions $f \in V_p[0, 1]$, $1 \leq p < \infty$.

Theorem C. *For the function $f \in V_p[0, 1]$, $1 \leq p < \infty$, the series (2) converge, if $\beta > 2p/(p + 2)$ or $\gamma < 1/p$. But this statement is not true for $\beta = 2p/(p + 2)$ or $\gamma = 1/p$ respectively.*

In the paper [7] a two-dimensional analog of this theorem was proved.

In our paper we give some sharp generalizations of Theorems A and C. We use the weight sequences belonging to the classes $A(\alpha)$, $\alpha \geq 1$. These classes were introduced by L. Gogoladze and R. Meskhia [8].

MAIN RESULTS.

We shall say that the positive sequence $\gamma = \{\gamma_k\}_{k=1}^{\infty}$ belongs to the class $A(\alpha)$, $\alpha \geq 1$, if there is a constant $C > 0$ such that

$$\left(\sum_{k=2^n+1}^{2^{n+1}}\gamma_k^\alpha\right)^{1/\alpha} \leq C2^{n(1-\alpha)/\alpha}\Gamma_n,$$

where

$$\Gamma_n = \sum_{k=2^{n-1}+1}^{2^n} \gamma_k, \quad n \in \mathbf{N}.$$

For $n = 0$ we assume that the above inequality holds for $\Gamma_0 = \gamma_1$.

This definition is a partial case of one introduced by L. Gogoladze and R. Meskhia [8]. It is easy to prove that $A(\alpha_1) \subset A(\alpha_2)$ for $\alpha_1 > \alpha_2 \geq 1$.

Let us recall that for a function $f \in L_p[0,1)$, $1 \leq p < \infty$, the norm is defined by the equality $\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$. Below we shall use the best approximation $E_n(f)_p = \inf_{\{a_k\}} \|f - t_n\|_p$ of the function $f \in L_p[0,1)$ by Haar polynomials $t_n(x) = \sum_{k=1}^{n} a_k \chi_k(x)$ of order n.

Theorem 1. *Let $f \in L_p[0,1)$, $1 \leq p < \infty$, and*

$$\sum_{k=1}^{\infty} \gamma_k \left[k^{-1/2} E_k(f)_p\right]^\beta < \infty, \tag{3}$$

where $0 < \beta < p$, $\gamma \in A(p/(p-\beta))$. Then the series

$$\sum_{n=1}^{\infty} n^\gamma \left|\hat{f}(n)\right|^\beta \tag{4}$$

converges.

From the Theorem 1 and the inequality $E_n(f)_p \leq 2^{1+1/p}\omega(n^{-1}, f)_p$, $1 \leq p < \infty$, $n \in \mathbf{N}$, (see [5]) it follows

Theorem 2. *The assertion of the Theorem 1 is also valid, if instead of the condition (3) we assume $\sum_{k=1}^{\infty} \gamma_k \left[k^{-1/2}\omega\left(k^{-1}, f\right)_p\right]^\beta < \infty$.*

For the function $f \in V_p[0,1]$, $1 \leq p < \infty$, we set $\omega_{1-1/p}(\delta, f) = \sup_{\lambda(\tau) \leq \delta} \left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p\right\}^{1/p}$, where $\tau = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ is a partition of interval $[0,1]$ and $\lambda(\tau) = \max_{1 \leq i \leq n}(x_i - x_{i-1})$. This notation was introduced in [9]. It is known the inequality $\omega(\delta, f)_p \leq \delta^{1/p}\omega_{1-1/p}(\delta, f)$ for the function $f \in V_p[0,1]$, $1 < p < \infty$ (see [6], Lemma 2 and [9], Theorem 2.5). Therefore from the Theorem 1 it follows

Corollary 1. *If $f \in V_p[0,1]$, $1 < p < \infty$, and*

$$\sum_{k=1}^{\infty} \gamma_k \left[k^{-1/2-1/p}\omega_{1-1/p}(1/k, f)\right]^\beta < \infty, \tag{5}$$

where $0 < \beta < p$, $\gamma \in A(p/(p-\beta))$, then the series (4) converges.

For the function $f \in V_p[0,1]$, $1 \leq p < \infty$, we define the norm $\|f\|_{V_p} = \max(V_p(f), \|f\|_\infty)$, where $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$. Let us define the best approximation $E_n(f)_{V_p} = \inf_{\{a_k\}} \|f - t_n\|_{V_p}$ of the function $f \in V_p[0,1]$, $1 \leq p < \infty$, by Haar polynomials $t_n = \sum_{k=1}^{n} a_k \chi_k(x)$ of order n. It is easy to prove the inequality $E_n(f)_p \leq C_p n^{-1/p} E_n(f)_{V_p}$. Therefore from the Theorem 1 it follows

Corollary 2. *The assertion of the Corollary 1 is valid, if instead of the condition (5) we assume*

$$\sum_{k=1}^{\infty} \gamma_k \left[k^{-1/2-1/p} E_k(f)_{V_p}\right]^\beta < \infty.$$

The following two theorems show that under some conditions the statement of Theorem 1 is sharp.

Theorem 3. *Let $1 \leq p < \infty$, $0 < \beta < p$, $\gamma \in A(p/(p-\beta))$, and let be given some decreasing and tending to zero sequence $\varepsilon = \{\varepsilon_i\}_{i=1}^\infty$ satisfying Bary condition*

$$\sum_{i=k}^{\infty} \varepsilon_i / i = O(\varepsilon_k), \quad k \in \mathbf{N}, \tag{6}$$

and $\sum_{k=1}^{\infty} \gamma_k \left(k^{-1/2}\varepsilon_k\right)^\beta = \infty$. Then there exists a function $f \in L_p[0,1]$ such that $E_n(f)_p \leq \varepsilon_n$, $n \in \mathbf{N}$, and the series (4) diverges.

Theorem 4. *Let $0 < \beta \leq 1$ for $1 < p < \infty$ and $0 < \beta < 1$ for $p = 1$. Moreover, let be given the sequence $\gamma \in A(p/(p-\beta))$ such that $(1-\alpha)2^{\beta/2}\Gamma_{n+1} \geq \Gamma_n$ for some $\alpha \in (0,1)$ and a decreasing and tending to zero sequence $\varepsilon = \{\varepsilon_i\}_{i=1}^\infty$ such that $\sum_{k=1}^{\infty} \gamma_k \left(k^{-1/2}\varepsilon_k\right)^\beta = \infty$. Then there exists a function $f \in L_p[0,1)$ such that $E_n(f)_p \leq \varepsilon_n$, $n \in \mathbf{N}$, and the series (4) diverges.*

The following theorem shows that under some conditions the statement of the Corollary 2 is sharp.

Theorem 5. *Let $1 < p < \infty$, $0 < \beta < p$, $\gamma \in A(p/(p-\beta))$, and let be given some decreasing and tending to zero sequence $\varepsilon = \{\varepsilon_i\}_{i=1}^\infty$ satisfying Bary condition (6) and such that $\sum_{k=1}^{\infty} \gamma_k \left(k^{-1/2-1/p}\varepsilon_k\right)^\beta = \infty$. Then there exists a function $f \in V_p[0,1]$ such that $E_n(f)_{V_p} \leq \varepsilon_n$, $n \in \mathbf{N}$, and the series (4) diverges.*

Remark. Theorem 2 and Corollaries 1 and 2 have two-dimensional analogs which will appear elsewhere.

Acknowledgment

The work of the first author is supported by the Russian Foundation for Basic Research under Grant No 11-01-00321. The work of the second author is supported by the Russian Foundation for Basic Research under Grant No 13-01-00238.

References

[1] A.Haar, Theorie der orthogonalen Functionensysteme, Math. Ann., **69** (1910), 331-371.

[2] B.S. Kashin, A.A. Sahakyan, Orthogonal series (Russian), Moscow, Nauka, 1984, 496p.

[3] Z. Ciesielski, J. Musielak, On absolute convergence of Haar series, Colloquium Math., **7**:1 (1959), 61-65.

[4] N. Wiener, The quadratic variation of a function and its Fourier coefficients, J. Math. and Phys., **3** (1924), 72-94.

[5] P.L. Ul'yanov, On the series with respect to Haar system (Russian), Mat. Sb., **63**: 3 (1964), 356-391.
[6] B.I. Golubov, On the Fourier series of continuous functions with respect to the Haar system (Russian), Izv. Akad. Nauk SSSR, Ser. mat., **28**:6 (1964), 1271-1296.
[7] B.I. Golubov, Absolute convergence of double series of Fourier-Haar coefficients for functions of bounded p-Variation (Russian), Izv. Vyssh. Ucheb. Zaved. Matematika, **6** (2012), pp. 3-13. Engl.transl.: Russian Math. (Iz. VUZ), **56**:6 (2012), 1-10.
[8] L. Gogoladze, R. Meskhia, On the absolute convergence of trigonometric Fourier series, Proc. Razmadze Math. Institute, **141** (2006), 29-40.
[9] A.P. Terekhin, Approximation of functions of bounded p-variation (Russian), Izv. Vyssh. Ucheb. Zaved. Matematika, **2** (1965), 171-187.