NEW DESIGN METHOD OF FIR FILTERS WITH SP2 COEFFICIENTS BASED ON A NEW LINEAR PROGRAMMING RELAXATION WITH TRIANGLE INEQUALITIES

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ABSTRACT

In this paper, we propose new design methods for linear phase FIR filters with signed power-of-two (SP2) coefficients based on a semi-definite programming (SDP) relaxation method. The proposed methods include a linear programming (LP) relaxation and a relaxation by adding triangle inequalities. Although such the design problems are known as one of the NP-hard problems, these methods can solve the design problems in a low computational cost in comparison with a traditional SDP relaxation method. It is shown by several numerical experiments that those method are superior to the simple SDP relaxation method.

1. INTRODUCTION

Recently, many studies on a design method for linear phase FIR filters with discrete coefficients have been published, in which, a numerical representation by a sum of signed power of two(SP2) has been used in several methods [1], [2], [3], [5]. It is a reason that a small number of non-zero digits is often required to include for a representation of the coefficients in a VLSI implementation of the filters.

However, it is difficult to design such the filters since it reduces to an integer programming problem(IP), which is well-known as one of the NP-hard problems [8].

Lu [9] proposed a new design method of filters with SP2 coefficients, in which, the design problem was formulated as a semi-definite programming(SDP) relaxation problem and solved by the interior point method. The SDP relaxation is one of the techniques for approximation of a non-convex programming problem including the IP [10]. Although the SDP relaxation is attractive to be solved in a polynomial time under some constraints qualification [11], it also involves a large computational complexity with a proportion of the designed filter's order [12].

It is well known that SDP problem is a convex programming problem and can be solved in polynomial time under some constraint qualification [11]. In this paper, we propose two new relaxation methods based on the SDP relaxation, i.e., a simple linear programming (LP) relaxation and a relaxation by adding triangle inequalities. It is shown by some numerical experiments that these relaxations are superior to the simple SDP relaxation.

2. DESIGN METHOD OF DIGITAL FILTERS BY USING $\{-1, 1\}$-OPTIMIZATION PROBLEM

In this section, we introduce the design method of digital filters with SP2 coefficients using SDP problem based on [9]. This design method is constructed by two steps: (1) solve the design problem of digital filters with desired frequency characteristics by using continuous variables, (2) formulate the design problem of digital filters with SP2 coefficients as a $\{-1, 1\}$-optimization problem. To consider the structure of the $\{-1, 1\}$-optimization problem obtained, we will convert the $\{-1, 1\}$-optimization problem to a minimum cut problem with negative coefficients which belongs to the class NP-hard.

2.1. Design problem of FIR digital filters with continuous coefficients

We design the FIR filters with SP2 coefficients so as to minimize the square error defined as,

$$e = \int_0^{\pi} W(\omega)[H(e^{j\omega}) - H_d(\omega)]^2 d\omega \tag{1}$$

where $W(\omega) \geq 0$ is a weight function and $H_d(\omega)$ is the desired frequency response function. In the first, we consider the continuous coefficient case. Then the transfer function of the FIR filter is:

$$H_c(z) = \sum_{k=0}^{N-1} h_k z^{-k} \tag{2}$$

where h_k $(k = 0, 1, \ldots, N-1)$ are real numbers. Now, we assume M is the total number of SP2 terms that can be used in $H(z)$ and m_k is the number of SP2 terms used in the k term of the frequency response $H(e^{j\omega})$, i.e., $\sum_{k=0}^{N-1} m_k =$

M. Then we denote

$$H(z) = \sum_{k=0}^{N-1} d_k z^{-k}. \quad (3)$$

The allocation of SP2 terms is determined, for example, by [13].

We assume that the absolute value of each SP2 term $|\{d_k\}|$ is in the interval $[2^0, 2^{-U}]$ where U is a natural number. Then, by (3),

$$d_k = \sum_{i=1}^{m_k} b_i^{(k)} 2^{-q_i^{(k)}}. \quad (4)$$

Here, we have $b_i^{(k)} \in \{-1, 1\}$ and $q_i^{(k)} \leq U, \ (1 \leq i \leq m_k, \ 0 \leq k \leq N-1)$.

For given $\{m_k, \ k = 0, \ldots, N-1\}$ and U, when an optimal continuous solution $H_c(z) = \sum_{k=0}^{N-1} h_k z^{-k}$ is obtained, it is easy to find the maximum SP2 number $\underline{d}_k$ and the minimum SP2 number $\bar{d}_k$ that satisfy $\bar{d}_k \leq h_k \leq \underline{d}_k$ whose $\underline{d}_k$ and $\bar{d}_k$ satisfy (4) for the given m_k.

Let $d_{mk} = (\underline{d}_k + \bar{d}_k)/2$ be the middle point of the interval $[\underline{d}_k, \bar{d}_k]$ and $\delta_k = (\underline{d}_k - \bar{d}_k)/2$ be the half length of the interval. Then, $\underline{d}_k$ and $\bar{d}_k$ are expressed as $d_{mk} + x_k\delta_k (x_k = -1)$ and $d_{mk} + x_k\delta_k \ \ (x_k = 1)$, respectively. Hence, the transfer function $H(z)$ with discrete coefficient function becomes

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=0}^{N-1} d_k^{(s)} e^{-jk\omega} && (5) \\ &= H_m(e^{j\omega}) + \boldsymbol{x}^T[\boldsymbol{c}_\delta(\omega) - j\boldsymbol{s}_\delta(\omega)] && (6) \end{aligned}$$

by using

$$d_k = d_{mk} + x_k\delta_k, \quad (7)$$

and

$$\begin{aligned} & H_m(e^{j\omega}) = \boldsymbol{d}_m^T[\boldsymbol{c}(\omega) - j\boldsymbol{s}(\omega)], \\ & \boldsymbol{d}_m = (d_{m0}, d_{m1}, \ldots, d_{m,N-1})^T, \\ & \boldsymbol{c}(\omega) = (1, \cos\omega, \ldots, \cos(N-1)\omega)^T, \\ & \boldsymbol{s}(\omega) = (0, \sin\omega, \ldots, \sin(N-1)\omega)^T, \\ & \boldsymbol{c}_\delta(\omega) = (\delta_0, \ldots, \delta_{N-1}\cos(N-1)\omega)^T, \\ & \boldsymbol{s}_\delta(\omega) = (0, \ldots, \delta_{N-1}\sin(N-1)\omega)^T, \\ & \boldsymbol{x} = (x_0, x_1, \ldots, x_{N-1})^T, \quad x_i \in \{-1, 1\}. \end{aligned} \quad (8)$$

By (6), we can easily verify that the objective function (1) becomes

$$e = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + 2\boldsymbol{x}^T\boldsymbol{q} + \text{const} \quad (9)$$

where

$$\begin{aligned} & \boldsymbol{Q} = \int_0^\pi W(\omega)[\boldsymbol{c}_\delta(\omega)\boldsymbol{c}_\delta^T(\omega)]d\omega \\ & \boldsymbol{q} = \int_0^\pi W(\omega)[E_r(\omega)\boldsymbol{c}_\delta(\omega) + E_i(\omega)\boldsymbol{s}_\delta(\omega)]d\omega \\ & E_r(\omega) = \boldsymbol{d}_m^T\boldsymbol{c}(\omega) - H_{dr}(\omega), \\ & E_i(\omega) = \boldsymbol{d}_m^T\boldsymbol{s}(\omega) - H_{di}(\omega), \\ & H_d(\omega) = H_{dr}(\omega) - jH_{di}(\omega). \end{aligned} \quad (10)$$

Now, design problem of $H(z)$ with SP2 coefficients for minimizing weighted least square becomes a $\{-1, 1\}$-quadratic integer programming problems [9]:

$$\begin{aligned} \text{min} \quad & \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + 2\boldsymbol{q}^T\boldsymbol{x} \\ \text{sub. to} \quad & \boldsymbol{x} \in \{-1, 1\}^N. \end{aligned} \quad (11)$$

2.2. Derivation to the minimum cut problem

To know the structure of the optimization problem is always important. In the following, we will point out that the problem (11) can be easily converted to the minimum cut problem with negative coefficients. Minimum cut problem with negative coefficients belongs to the class of NP-hard. Hence, this shows (11) is a hard problem to solve.

Let $G = (V, E)$ be a perfect graph with a vertex set $V = \{0, 1, \ldots, N\}$ and an edge set $E = \{ij \mid 0 \leq i < j \leq N\}$. And, a weight $w_e(e \in E)$ of the each edge be

$$\begin{cases} w_{jN} = -q_j \ (j = 0, \ldots, N-1), \\ w_{ij} = -Q_{ij} \ (0 \leq i < j \leq N-1). \end{cases} \quad (12)$$

Then the minimum cut problem of the graph G is:

$$\begin{aligned} \text{min} \quad & 4\sum_{e \in E} w_e y_e + \boldsymbol{e}^T\boldsymbol{Q}\boldsymbol{e} + 2\boldsymbol{q}^T\boldsymbol{e} \\ \text{sub. to} \quad & \boldsymbol{y} : \text{a 0-1 cut vector of } G, \end{aligned} \quad (13)$$

or equivalently

$$\begin{aligned} \text{min} \quad & \sum_{ij \in E} w_{ij}(x_i - x_j)^2 + \boldsymbol{e}^T\boldsymbol{Q}\boldsymbol{e} + 2\boldsymbol{q}^T\boldsymbol{e} \\ \text{sub. to} \quad & \boldsymbol{x} \in \{-1, 1\}^{N+1}. \end{aligned} \quad (14)$$

It is easy to see that problem (11) and (14) are equivalent each other. Here, we denote, without loss of generality, $x_N = 1$ in (14). Since, as we pointed out that minimum cut problem with negative coefficients belongs to the class of NP-hard, there will no hope to develop efficient algorithms to solve (11) directly.

3. SDP RELAXATION AND LP RELAXATION

Following Lu [9], we reformulate (11) as

$$\begin{aligned} \text{min} \quad & \boldsymbol{Q} \bullet \boldsymbol{X} + 2\boldsymbol{q}^T\boldsymbol{x} \\ \text{sub. to} \quad & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T = \boldsymbol{O}, \\ & \boldsymbol{x} \in \{-1, 1\}^N \end{aligned} \quad (15)$$

where $Q \bullet X = \sum_{i,j} Q_{ij} X_{ij}$. Then, since $X_{ii} = x_i^2 = 1$ ($i = 0, \ldots, N-1$) we obtain an relaxation problem of (11) by

$$\begin{array}{ll} \min & \boldsymbol{Q} \bullet \boldsymbol{X} + 2\boldsymbol{q}^T \boldsymbol{x} \\ \text{sub. to} & X_{ii} = 1 \; (i = 0, \ldots, N-1), \\ & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O}. \end{array} \quad (16)$$

Here, $A \succeq \boldsymbol{O}$ denotes that A is positive semidefinite. Between a cut vector $\boldsymbol{y} \in \{0, 1\}^E$ of the graph G and $(\boldsymbol{x}, \boldsymbol{X})$ that satisfies (15), the following equations hold:

$$\left\{ \begin{array}{l} x_j = -2y_{0j} + 1 \; (j = 0, \ldots, N-1), \\ X_{ij} = -2y_{ij} + 1 \; (0 \le i < j \le N-1). \end{array} \right. \quad (17)$$

We can use software to solve SDP in polynomial times, for example, SeDuMi [14].

It is easily verified that the next triangle inequalities hold [15]:

$$\left. \begin{array}{r} x_i + x_j + X_{ij} \ge -1, \\ x_i - x_j - X_{ij} \ge -1, \\ -x_i - x_j + X_{ij} \ge -1, \\ -x_i + x_j - X_{ij} \ge -1. \end{array} \right\} \quad (18)$$

Here, we denote $0 \le i < j \le N-1$. And, for $0 \le i < j < k \le N-1$,

$$\left. \begin{array}{r} X_{ij} + X_{ik} + X_{jk} \ge -1, \\ X_{ij} - X_{ik} - X_{jk} \ge -1, \\ -X_{ij} - X_{ik} + X_{jk} \ge -1, \\ -X_{ij} + X_{ik} - X_{jk} \ge -1 \end{array} \right\} \quad (19)$$

hold. (18) is the triangle inequalities for the vertex set $\{i, j, N\}$, and (19) is the triangle inequalities for the vertex set $\{i, j, k\}$. Therefore, the next optimization problem

$$\begin{array}{ll} \min & \boldsymbol{Q} \bullet \boldsymbol{X} + 2\boldsymbol{q}^T \boldsymbol{x} \\ \text{sub. to} & X_{ii} = 1 \; (i = 0, \ldots, N-1), \\ & (18), (19), \\ & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O} \end{array} \quad (20)$$

strengthen the problem (16).

Since it is easier to solve LP problems than SDP problem, we will relax the SDP relaxation problem to LP relaxation problem. The following minimization problem

$$\begin{array}{ll} \min & 2\sum_{i<j} Q_{ij} X_{ij} + 2\boldsymbol{q}^T \boldsymbol{x} + \sum_{i=0}^{N-1} Q_{ii} \\ \text{sub. to} & (18), \\ & -1 \le x_i \le 1 \; (i = 0, \ldots, N-1) \end{array} \quad (21)$$

is a LP relaxation problem with a bounded optimal solution. Adding (19) as constraints,

$$\begin{array}{ll} \min & 2\sum_{i<j} Q_{ij} X_{ij} + 2\boldsymbol{q}^T \boldsymbol{x} + \sum_{i=1}^{N-1} Q_{ii} \\ \text{sub. to} & (18), (19), \\ & -1 \le x_i \le 1 \; (i = 0, \ldots, N-1) \end{array} \quad (22)$$

we have a strengthened problem for (21).

4. COMPARISON OF THE RELAXATION PROBLEMS

From the theoretical view point, SDP relaxation with triangle inequalities are stronger than SDP relaxation, LP relaxation, or LP relaxation with triangle inequalities. However, SDP relaxation with triangle inequalities is a rather heavy relaxation for programming techniques and computation. Hence, we simply compare the SDP relaxation, LP relaxation and, LP relaxation with triangle inequalities through numerical experiments.

In the numerical experiments, the specification of the filter design problem is basically same as Lu [9], hence, FIR filter is an odd degree and even symmetric linear phase lowpass filter. The design specification is as follows: the normalized passband is $[0, \omega_p] = [0, 0.225]$, stopband is $[\omega_s, 1] = [0.275, 1]$, $W(\omega) = 1$ on $[0, \omega_p]$, $W(\omega) = 500$ on $[\omega_s, 0.5]$, $L = 12$. And, we set each $m_k = 2$. The CPU used is mobile Pentium III 650 MHz, memory is 192 Mbytes. All problems are solved by SeDuMi (Ver.1.03) [14]. The CPU time contains only the execution time of SeDuMi.

By the numerical experiments, we found (1) all the solutions of LP with triangle inequalities (22) are optimal solutions of the original problem (11), that is, all the solutions of (22) are $\{-1, 1\}$-integer solutions and automatically optimal solutions for (11). This reveals that the triangle inequalities are very important. At now, all triangle inequalities are included in the relaxation problems, we can also develop an algorithm that has triangle inequalities as cutting planes.

How to obtain an $\{-1, 1\}$-solution from the solution of the relaxation problems is as follows:
(1) For the SDP relaxation problem, let the solution of the relaxation problem $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{X}})$,
(1-1) sign($\tilde{\boldsymbol{x}}$).
(1-2) let $\boldsymbol{v}_i$ $(i = 0, \ldots, N-1)$ be the eigen vectors of $\tilde{\boldsymbol{X}}$, and set sign($\boldsymbol{v}_i$) $(i = 0, \ldots, N-1)$.
(1-3) Use the Goemans-Williamson's randomized algorithm [16].
Select the best solution of the above solutions. (Lu [9] exploits (1-1) and (1-2) for the maximal eigen vectors, However, we recommend the above, since it does not take so much CPU time, and may improve the solution.)
(2) For LP relaxation problems, we exploited the sign vector $\boldsymbol{v}_i$ of the solution of the relaxation problems $\tilde{\boldsymbol{x}}$.

5. CONCLUSION

In this paper, we proposed LP based relaxation techniques to solve the design problem of FIR filters with SP2 coefficients under MLE criterion. And, we compared this relaxation technique and the SDP relaxation through numerical experiments. By the numerical example, LP based relaxation technique seems to work fairly good.

Table 1. Comaparison of upper bounds ($*$ is the best solution).

N	WLS	LP	LP+Tri	SDP
7	0.29413	0.42797$*$	0.42797$*$	0.42797$*$
13	0.10269	0.10543$*$	0.10543$*$	0.10543$*$
19	0.02473	0.08771$*$	0.08771$*$	0.08771$*$
25	0.01169	0.02655	0.02606$*$	0.02606$*$
31	0.00288	0.01258$*$	0.01258$*$	0.01415
37	0.00141	0.06321	0.06231$*$	0.06263
43	0.00035	0.15770	0.08531$*$	0.08531$*$
49	0.00018	0.05953	0.05781$*$	0.05781$*$
55	0.00004	0.04949	0.04946$*$	0.04979
57	0.00004	0.05035$*$	0.05035$*$	0.05035$*$
59	0.00002	0.05359	0.05345$*$	0.05350
61	0.00002	0.05425	0.05409$*$	0.05442

Table 2. Computational Time (sec)

n	LP	LP+Tri	SDP
7	0.04	0.05	0.06
13	0.08	0.09	0.08
19	0.08	0.15	0.11
25	0.10	0.31	0.11
31	0.19	0.67	0.13
37	0.19	1.34	0.16
43	0.25	2.51	0.19
49	0.31	4.80	0.20
55	0.43	8.85	0.26
57	0.45	9.78	0.26
59	0.48	14.46	0.28
61	0.50	14.94	0.30

6. REFERENCES

[1] D.M. Kodek and K. Steiglitz, ”Finite-length wordlength tradeoffs in FIR digital filter design,” *IEEE Trans. Acoustics, Speech, Signal Processing,* vol. ASSP-28, pp.739-744, December 1980.

[2] Y.C. Lim and S.R. Parker, ”FIR filter design over a discrete power-of-two coefficient space,” *IEEE Trans. Acoustics, Speech, Signal Processing,* vol. ASSP-31, pp.583-591, June 1983.

[3] Y.C. Lim and S.R. Parker, ”Discrete coefficient FIR digital filter design based upon LMS criteria,” *IEEE Trans. Circuits, Syst.,* vol. CAS-30, pp.723-739, Oct. 1983.

[4] P. Siohan and A. Benslimane, ”Finite precision design of optimal linear phase 2-D FIR digital filters,” *IEEE Trans. Circuits Syst.,* vol. 36, pp. 11-22, January 1989.

[5] M. Suk and S.K. Mitra, ”Computer-aided design of digital filters with finite wordlength,” *IEEE Trans. Audio Electroacoust.,* vol. AU-20, pp. 356-366, December 1972.

[6] P.P. Vaidyanathan, ”Efficient and multiplierless design of FIR filters with very sharp cutoff via maximally flat building blocks,” *IEEE Transaction on Circuits and Systems*, vol. 32, pp. 236-244, March 1985.

[7] J.T. Yli-Kaakinen and T.A. Saramaki, ”An Algorithm for the design of multiplierless approximately linear-phase lattice wave digital filters,” *Proc. ISCAS’2000,* vol. II, pp. 77-80, Geneva, June 2000.

[8] C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization,* Prentice-Hall, 1982.

[9] W.-S. Lu, ”Design of FIR Filters with discrete coefficients: A Semidefinite Programming Relaxation Approach”, ISCAS-2001.

[10] T. Fujie and M. Kojima, ”Semidefinite relaxation for nonconvex programs,” *Journal of Global Optimization,* vol. 10, pp. 367-380,1997.

[11] M.Kojima,S.Shindoh, and S.Hara, ”Interior-point methods for the monotone semidefinite linear complementarity problems,” *SIAM Journal on Optimization,* Vol.7, pp.86-125, 1997.

[12] S. Kim and M. Kojima, ”Second order cone programming relaxation of nonconvex quadratic optimization problems,” Technical Report, Department of Mathematical and Computing Sciences, Tokyo institute of Technology, 2000.

[13] Y.C. Lim, R. Yang, D. Li and J. Song, ”Signed power-of-two (SPT) term allocation scheme for the design of digital filters,” *Proc. ISCAS,* Monterey, CA., June 1998.

[14] J.F. Sturm, *SeDuMi Version 1.03,* September 1999. (Download link: http://www2.Unimaas.nl/ sturm/sofware/sedumi.html).

[15] M. Muramatsu and T Suzuki, ”A new second-order cone programming relaxation,” Technical Report, CS-01-01, Department of computer Science, the Universigy of Electro-Communications, Tokyo, Japan, 2001.

[16] M.X. Goemans nad S.P. Williamson, ”Improves approximation algorithms for maximum cut and satisfiability programs using semidefinite programming,” *Journal of Assoc. Comut. Math.,* vol. 42, pp. 1115-1145, 1995.