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On Matthews' relationship between quasi-metrics and partial metrics: an aggregation perspective

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Abstract J. Borsík and J. Doboš studied the problem of how to merge a family of metric spaces into a single one through a function. They called such functions metric preserving and provided a characterization of them in terms of the so-called triangle triplets. Since then, different papers have extended their study to the case of generalized metric spaces. Concretely, in 2010, G. Mayor and O. Valero provided two characterizations of those functions, called quasi-metric aggregation functions, that allows us to merge a collection of quasi-metric spaces into a new one. In 2012, S. Massanet and O. Valero gave a characterization of the functions, called partial metric aggregation function, that are useful for merging a collection of partial metric spaces into single one as final output.

Inspired by the preceding work, in 2013, J. Martín, G. Mayor and O. Valero addressed the problem of constructing metrics from quasi-metrics, in a general way, using a class of functions that they called metric generating functions. In particular, they solved the posed problem providing a characterization of such functions and, thus, all ways under which a metric can be induced from a quasi-metric from an aggregation viewpoint. Following this idea, we propose the same problem in the framework of partial metric spaces. So, we characterize those functions that are able to generate a quasi-metric from a partial metric, and conversely, in such a way that Matthews' relationship between both type of generalized metrics is retrieved as a particular case. Moreover, we study if both, the partial order and

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the topology induced by a partial metric or a quasi-metric, respectively, are preserved by the new method in the spirit of Matthews. Furthermore, we discuss the relationship between the new functions and those families introduced in the literature, i.e., metric preserving functions, quasi-metric aggregation functions, partial metric aggregation functions and metric generating functions.

Keywords partial metric space · quasi-metric space · aggregation function · quasi-metric generating · partial metric generating

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1 Introduction and Preliminaries

In 1981, J. Borsík and J. Doboš addressed in [3] the issue of how to merge a family of metric spaces into a single one through a function. They solved such a problem characterizing the class of real-valued functions whose composition with each family of metrics provide a single one metric as output. They called such functions metric preserving (metric aggregation functions in [16]).

Let us recall that a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric aggregation function provided that the function $d_\Phi : X \times X \to \mathbb{R}_+$ is a metric for every pair of metric spaces (X_1, d_1) and (X_2, d_2), where $X = X_1 \times X_2$ and $d_\Phi((x,y),(z,w)) = \Phi(d_1(x,z), d_2(y,w))$ for all $(x,y),(z,w) \in X$. Of course the letter $\mathbb{R}_+$ denotes the set of nonnegative real numbers and $\mathbb{R}_+^2 = \{(a,b) : a, b \in \mathbb{R}_+\}$. Notice that we will keep the name of metric preserving function for one dimensional metric aggregation functions.

A first description of metric aggregation functions was made by Borsík and Doboš in [3] (see, also, [5]). In order to introduce such a description, let us recall a few pertinent notions.

According to [3], we will denote by $\mathcal{O}$ the set of all functions $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying: $\Phi(a,b) = 0 \Leftrightarrow a = b = 0$. Moreover, a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is said to be monotone provided that $\Phi(a,b) \leq \Phi(c,d)$ for all $(a,b),(c,d) \in \mathbb{R}_+^2$ with $(a,b) \preceq (c,d)$, where $\preceq$ is defined by $(a,b) \preceq (c,d) \Leftrightarrow a \leq c$ and $b \leq d$. Furthermore, a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is said to be subadditive if $\Phi((a,b)+(c,d)) \leq \Phi(a,b)+\Phi(c,d)$ for all $(a,b),(c,d) \in \mathbb{R}_+^2$, where $+$ stands for the usual addition on $\mathbb{R}_+^2$.

Taking into account the preceding concepts, the aforementioned description is given by the next results.

Proposition 1 *If Φ is a metric aggregation function, then $\Phi \in \mathcal{O}$.*

Proposition 2 *Let $\Phi \in \mathcal{O}$. If Φ is monotone and subadditive, then Φ is a metric aggregation function.*

The preceding results give a method to generate functions that merge two metrics into a new one. They motivate the question of whether the converse of Proposition 2 is true in general and, thus, metric aggregation functions are always monotone and subadditive. However, there are aggregation functions which are not monotone (see, [5,16]). Inspired, in part by the last fact, Borsík and Doboš proved a characterization of metric aggregation functions in terms of the so-called triangle triplets, where a triplet (a, b, c), with $a, b, c \in \mathbb{R}_+$, forms a triangle triplet whenever $a \leq b + c$, $b \leq a + c$ and $c \leq b + a$. The aforementioned characterization can be stated as follows:

Theorem 1 *Let $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$. Then the below assertions are equivalent:*

(1) Φ is a metric aggregation function.
(2) Φ holds the following properties:
(2.1) $\Phi \in \mathcal{O}$.
(2.2) Let $a, b, c, d, f, g \in \mathbb{R}_+$. If (a, b, c) and (d, f, g) are triangle triplets, then so is $(\Phi(a, d), \Phi(b, f), \Phi(c, g))$.

From the preceding result one can immediately obtain the below consequence:

Corollary 1 *Every metric aggregation function is subadditive.*

It must be stressed that, for simplicity, the above result is presented only for two metrics. However, the original Borsík and Doboš result can be stated for a collection (non necessarily finite) of metrics.

Since Borsík and Doboš solved the metric aggregation problem, different papers have extended their study to the case of generalized metric spaces. Concretely, such a study has been explored in the framework of quasi-metric spaces and partial metic spaces.

In 2010, G. Mayor and O. Valero motivated, on the one hand, by the utility of quasi-metrics in several fields of Artificial Intelligence and Computer Science (see, for instance, [6, 7, 10, 18–25] and, on the other hand, by the seminal work of Borsík and Doboš, provided two characterizations of those functions, called quasi-metric aggregation functions, that allows us to merge a collection of quasi-metric spaces into a new one ([16]). Next we recall a few pertinent notions with the aim of introducing such characterizations.

On account of [4], a quasi-metric space is a pair (X, q), where X is a non-empty set and q is a function on $X \times X$ such that, for all $x, y, z \in X$, the following axioms are satisfied:

(Q1) $q(x, y) = 0 = q(y, x) \Leftrightarrow x = y$;
(Q2) $q(x, z) \leq q(x, y) + q(y, z)$.

According to [16], a function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a quasi-metric aggregation function if the function $q_\Phi : X \times X \to \mathbb{R}_+$ is a quasi-metric for every pair of quasi-metric spaces (X_1, q_1) and (X_2, q_2), where $X = X_1 \times X_2$ and

$$q_\Phi((x, y), (z, w)) = \Phi(q_1(x, z), q_2(y, w))$$

for all $(x, y), (z, w) \in X$.

Similar to the metric case the following description of quasi-metric aggregation functions can be provided ([16]).

Theorem 2 *Let $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$. Then the below assertions are equivalent:*

(1) Φ is a quasi-metric aggregation function.
(2) $\Phi \in \mathcal{O}$, and Φ is subadditive and monotone.
(3) Φ holds the following properties:

(2.1) $\Phi \in \mathcal{O}$.
(2.2) *Let* $a, b, c, d, f, g \in \mathbb{R}_+$. *If* $(a, d) \preceq (b, f) + (c, g)$, *then* $\Phi(a, d) \leq \Phi(b, f) + \Phi(c, g)$.

Observe that from the preceding characterizations we obtain that every quasi-metric aggregation function is a metric aggregation function. However, there are metric aggregation functions which are not quasi-metric aggregation functions (see [16]). Again, we only consider two quasi-metrics in the statement of the preceding result but it can be stated for a collection (non necessarily finite) of quasi-metrics. Applications of quasi-metric aggregation functions to asymptotic complexity analysis of algorithms and denotational semantics can be found in [12].

In 1994, S.G. Matthews introduced a new metric notion, which is known as partial metric, in order to provide a mathematical framework to model several computational processes that arise in a natural way in Computer Science ([14]). Nowadays, the applicability of partial metric spaces covers areas like fixed point theory, denotational semantics for programming languages, parallel processing, complexity analysis and logic programming (see, for example, [2,9,10,14,15,17,26]). Recall that, following [14], a partial metric space is a pair (X, p), where X is a non-empty set and p is a function on $X \times X$ such that, for all $x, y, z \in X$, the following axioms are fulfilled:

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $0 \leq p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

In 2012, S. Massanet and O. Valero tackled the partial metric aggregation problem. Thus the notion of partial metric aggregation functions was introduced as follows ([13]):

A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a partial metric aggregation function if the function $p_\Phi : X \times X \to \mathbb{R}_+$ is a partial metric for every pair of partial metric spaces (X_1, p_1) and (X_2, p_2), where $X = X_1 \times X_2$ and

$$p_\Phi((x, y), (z, w)) = \Phi(p_1(x, z), p_2(y, w))$$

for all $(x, y), (z, w) \in X$. A characterization of this new type of functions was provided in [13] as follows:

Theorem 3 *Let* $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. *Then* Φ *is a partial metric aggregation function if and only if it satisfies the following two properties for all* $a, b, c, d, e, f, g, h \in \mathbb{R}_+$.

(1) $\Phi(a, b) + \Phi(c, d) \leq \Phi(e, f) + \Phi(g, h)$ *whenever* $(a, b) + (c, d) \preceq (e, f) + (g, h)$, $(c, d) \preceq (e, f)$ *and* $(c, d) \preceq (g, h)$.
(2) *If* $(c, d) \preceq (a, b)$, $(e, f) \preceq (a, b)$ *and* $\Phi(a, b) = \Phi(c, d) = \Phi(e, f)$, *then* $(a, b) = (c, d) = (e, f)$.

One more time we only consider two partial metrics in the definition of partial aggregation function and in the statement of the preceding result but it can be stated for a collection (non necessarily finite) of partial metrics. Notice that from the preceding characterization we obtain the next consequences:

Corollary 2 *Let* $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. *If* Φ *is a partial metric aggregation function, then the following statements hold:*

(1) Φ is monotone.
(2) If there exists $(a,b) \in \mathbb{R}_+^2$ such that $\Phi(a,b) = 0$, then $a = b = 0$.
(3) Φ is subadditive.

In the light of the above results it seems natural to wonder what is the relationship between partial metric aggregation functions, metric aggregation functions and metric aggregation functions. The response to such a question was given in [13]. Specifically there exist partial metric aggregation functions that are not either metric aggregation functions or quasi-metric aggregation functions and vice versa. However, the next result shows a relationship in a specific case.

Proposition 3 *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a partial metric aggregation function such that $\Phi(0,0) = 0$. Then Φ is a quasi-metric aggregation function and, thus, a metric aggregation function.*

It is a well-known fact that given a quasi-metric q on X, we can symmetrize it by means of two typical constructions. Indeed, we can construct from q the metrics $d_q^{\max}$, d_q^+ on X defined by $d_q^{\max}(x,y) = \max\{q(x,y), q(y,x)\}$ and $d_q^+(x,y) = q(x,y) + q(y,x)$ for each $x, y \in X$. This way of constructing a metric from a quasi-metric is obtained making use of an appropriate aggregation function acting on the numerical values $q(x,y)$ and $q(y,x)$. In fact, $d_q^{\max}(x,y) = \Phi_{\max}(q(x,y), q(y,x))$ and $d_q^+(x,y) = \Phi_+(q(x,y), q(y,x))$, where $\Phi_{\max}(a,b) = \max\{a,b\}$ and $\Phi_+(a,b) = a+b$ for all $a, b \in \mathbb{R}_+$. Taking into account this fact, J. Martín, G. Mayori and O. Valero gave a general method of symmetrization of quasi-metrics based on the use of aggregation in [11]. To this end, they introduced the so-called metric generating functions which are defined as follows:

A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a *metric generating function* if $d_\Phi : X \times X \to \mathbb{R}_+$ is a metric on X for every quasi-metric space (X, q), where the function d_Φ is defined, for each $x, y \in X$, by

$$d_\Phi(x,y) = \Phi(q(x,y), q(y,x)).$$

A characterization of metric generating functions was given in [11] in terms of mixed triplets, where the triplets (a,b,c) and (d,f,g), with $a,b,c,d,f,g \in \mathbb{R}_+$, are called mixed provided that the following inequalities hold:

$$a \leq b + c,\ \ b \leq a + f,\ c \leq g + a,$$

$$d \leq f + g,\ f \leq d + b,\ g \leq c + d.$$

The mentioned characterization can be stated as follows:

Theorem 4 *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. Then the below assertions are equivalent:*

(1) Φ is a metric generating function.
(2) Φ holds the following properties:
(2.1) $\Phi \in \mathcal{O}$.
(2.2) Φ is symmetric, i.e., $\Phi(a,b) = \Phi(b,a)$ for all $(a,b) \in \mathbb{R}_+^2$.
(2.3) $\Phi(a,d) \leq \Phi(b,g) + \Phi(c,f)$ for all $a,b,c,d,f,g \in \mathbb{R}_+$ such that (a,b,c) and (d,f,g) are mixed triplets.

It deserves to be pointed out that the above result gives a general mathematical method to symmetrize quasi-metrics giving as a result a metric and, thus, it solves formally the problem of how to metrize a quasi-metric.

Theorem 3 yields the next consequences:

Corollary 3 *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. If Φ is a metric generating function, then Φ is subadditive.*

The relationship between the class of metric generating functions and the class of quasi-metric aggregation functions were deeply explored in [11]. Concretely the following was obtained.

Proposition 4 *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. If Φ is a symmetric quasi-metric aggregation function, then Φ is a metric generating function.*

Theorem 5 *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a metric generating function. Then the following assertions are equivalent:*

(1) Φ is a quasi-metric aggregation function.
(2) Φ is monotone.

In [14], Matthews introduced a method to generate a quasi-metric from a partial metric and vice versa. Indeed, given a partial metric p on a non-empty set X, then a quasi-metric q_p can be induced on X by $q_p(x,y) = p(x,y) - p(x,x)$ for each $x, y \in X$. Moreover, a technique for the construction of a partial metric from a quasi-metric was also given in [14]. In order to introduce such a technique let us recall that a *weighted quasi-metric space* is a tern (X, q, w_q), where q is a quasi-metric on X and w_q is a function $w_q : X \to \mathbb{R}_+$ satisfying, for each $x, y \in X$, that

$$q(x,y) + w_q(x) = q(y,x) + w_q(y).$$

The mapping w_q is known as the weight function associated to the quasi-metric q. Thus, given a weighted quasi-metric space (X, q, w_q), a partial metric p_{q,w_q} on X can be defined, for each $x, y \in X$, by

$$p_{q,w_q}(x,y) = q(x,y) + w_q(x).$$

In the light of the exposed facts, it seems natural to wonder whether the possible techniques for generating a partial metric from a quasi-metric and vice versa are reduced to the above exposed. So far as we know there are not, except for the Matthews one, any comprehensive study on a general technique to construct partial metrics and quasi-metrics, one from the other. Motivated by this fact, in this paper, we approach the posed question following the ideas presented in [11] and, thus, an aggregation viewpoint. In fact, notice that

$$p_{q,w_q}(x,y) = \Phi_+\left(q(x,y), w_q(x)\right) \text{ and } q_p(x,y) = \Phi_-(p(x,y), p(x,x)),$$

where $\Phi_-(a,b) = a - b$ for each $a, b \in \mathbb{R}_+$.

In particular, we study a general method to generate a quasi-metric form a partial metric and vice versa by means of a real-valued function. To this end, we introduce the notions of quasi-metric generating function and partial metric generating function. Then, we characterize both new type of functions in terms of triplets. Moreover, we also go one step further studying whenever the general method preserves the main structures that quasi-metrics and partial metrics induce on a set (in the spirit of [14]), which are the partially order and the topology. Furthermore, we establish a relationship between the new functions, metric aggregation functions, quasi-metric aggregation functions, partial metric aggregation functions and metric generating functions.

The structure of the paper is as follows. Section 2 is devoted to characterize those functions, that we have called quasi-metric generating functions, that transform a partial metric in a weighted quasi-metric. In the same section, it is showed that such functions preserve the partially order induced by a partial metric, but not the topology. Moreover, a characterization of those quasi-metric generating functions that preserve the topologies is provided. Sections 2 ends with a discussion about the relationship between quasi-metric generating functions, metric aggregation functions, quasi-metric aggregation functions, partial metric aggregation functions and metric generating functions. Whereas, Section 3 is devoted to develop the reciprocal study. So, the partial quasi-metric functions are characterized. Moreover, the facts that partial metric generating functions preserve the partial order induced by a quasi-metric but not the topology are proved. Section 3 ends characterizing those partial metric generating functions that preserve the topology and exploring the relationship between this kind of functions and metric aggregation functions, quasi-metric aggregation functions, partial metric aggregation functions and metric generating functions.

2 A general method for generating quasi-metrics from partial metrics

This section is devoted to provide a general method to generate a quasi-metric from a partial metric in such a way that the technique introduced by Matthews can be retrieved as a particular case . To this end, $\mathbf{D}$ will denote the subset of $\mathbb{R}_+^2$ given by $\mathbf{D} = \{(a,b) \in \mathbb{R}_+^2 : a \geq b\}$.

The next notion will be crucial in order to get the solution to the posed problem from the aggregation perspective.

Definition 1 We will say that a function $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a quasi-metric generating function (briefly, a *qmg*-function) if for each partial metric space (X,p) the function $q_{\Phi,p} : X \times X \to \mathbb{R}_+$ is a quasi-metric on X, where $q_{\Phi,p}(x,y) = \Phi(p(x,y), p(x,x))$ for each $x, y \in X$.

The next example shows that the Matthews technique is a particular case of the exposed approach (see Section 1).

Example 1 Let $\Phi_- : \mathbf{D} \to \mathbb{R}_+$ given by $\Phi_-(a,b) = a - b$. Then, Φ_- is a *qmg*-function. Indeed, given a partial metric space (X,p) we have that $q_{\Phi_-,p}(x,y) = p(x,y) - p(x,x)$ for each $x, y \in X$, which is the well-known weighted quasi-metric q_p induced by the partial metric p.

The next example provides an alternative way of generating a quasi-metric from a partial metric which is based on the use of quasi-metric generating functions.

Example 2 Let $\Phi_{-,\frac{1}{2}} : \mathbf{D} \to \mathbb{R}_+$ given by

$$\Phi_{-,\frac{1}{2}}(x,y) = \begin{cases} 0, & \text{if } x = y = 0 \\ x - \frac{y}{2}, & \text{otherwise} \end{cases}.$$

Then, $\Phi_{-,\frac{1}{2}}$ is a *qmg*-function. Indeed, given a partial metric space (X,p), it is not hard to check that $q_{\Phi_{-,\frac{1}{2}},p}$ is a quasi-metric on X with $q_{\Phi_{-,\frac{1}{2}},p}(x,y) = p(x,y) - \frac{p(x,x)}{2}$ for each $x,y \in X$ with $x \neq y$, and $q_{\Phi_{-,\frac{1}{2}},p}(x,x) = 0$ for each $x \in X$.

Proposition 6, below, also yields a way of building quasi-metrics from partial metrics which differs from the Matthews technique.

The next concept will be play a central role in order to characterize those functions $\Phi : \mathbf{D} \to \mathbb{R}_+$ which are *qmg*-functions.

Definition 2 We will say that $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1, y_2, y_3) \in \mathbb{R}^3_+$ if the following conditions are satisfied:

(i) $x_1 \geq \max\{y_1, y_3\}$, with $x_1 > y_1$ or $x_1 > y_3$, and $x_1 + y_2 \leq x_2 + x_3$;
(ii) $x_2 \geq \max\{y_2, y_1\}$, with $x_2 > y_2$ or $x_2 > y_1$, and $x_2 + y_3 \leq x_3 + x_1$;
(iii) $x_3 \geq \max\{y_3, y_2\}$, with $x_3 > y_3$ or $x_3 > y_2$, and $x_3 + y_1 \leq x_1 + x_2$.

It is not har to check that $(2,1,2)$ is a quadrangular triplet on $(0,1,2)$. Notice that $(0,1,2)$ is not a quadrangular triplet on $(2,1,2)$.

The next theorem provides a characterization of *qmg*-functions by means of quadrangular triplets and, thus, a general method to generate a quasi-metric from a partial metric.

Theorem 6 *Let $\Phi : \boldsymbol{D} \to \mathbb{R}_+$ be a function. The the following assertions are equivalent:*

(1) Φ is a qmg-function.
(2) Φ satisfies:
  (i) $\Phi^{-1}(0) = \{(x,y) \in \boldsymbol{D} : x = y\}$;
  (ii) $\Phi(x_1,y_1) \leq \Phi(x_2,y_1) + \Phi(x_3,y_2)$, whenever $(x_1,x_2,x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1,y_2,y_3) \in \mathbb{R}^3_+$.

Proof (1) $\Rightarrow$ (2). Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a *qmg*-function.

Next we show that Φ satisfies condition (i). Suppose that $\Phi(x,y) = 0$ for some $x,y \in \mathbf{D}$. Consider $(\mathbb{R}, p_y)$ the partial metric space where $p_y(a,b) = |a-b| + y$ for each $a,b \in \mathbb{R}$, where $\mathbb{R}$ stands for the real number set. Taking into account that Φ is a *qmg*-function, then $q_{\Phi,p_y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a quasi metric on $\mathbb{R}$, where $q_{\Phi,p_y}(a,b) = \Phi(p_y(a,b), p_y(a,a))$ for each $a,b \in \mathbb{R}$.

Since $x \geq y$ we have that $p_y(y,x) = p_y(x,y) = |x-y| + y = x$. In addition, $p_y(x,x) = |x-x| + y = y$ and $p_y(y,y) = |y-y| + y = y$.

Attending to the above observations and taking into account our assumptions, we have that $q_{\Phi,p_y}(x,y) = \Phi(p_y(x,y), p_y(x,x)) = \Phi(x,y) = 0$. Moreover, $q_{\Phi,p_y}(y,x) = \Phi(p_y(y,x), p_y(y,y)) = \Phi(x,y) = 0$. Thus, $(Q1)$ implies $x = y$.

Next we show that Φ satisfies condition (ii). To this end, suppose that $(x_1,x_2,x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1,y_2,y_3) \in \mathbb{R}^3_+$.

In the following we construct a partial metric space in order to show that $\Phi(x_1, y_1) \leq \Phi(x_2, y_1) + \Phi(x_3, y_2)$.

Let $X = \{a, b, c\}$ be a set of three points. We define $p : X \times X \to \mathbb{R}_+$ as follows:

$$p(a, c) = p(c, a) = x_1 \text{ and } p(a, a) = y_1;$$

$$p(a, b) = p(b, a) = x_2 \text{ and } p(b, b) = y_2;$$

$$p(b, c) = p(c, b) = x_3 \text{ and } p(c, c) = y_3.$$

It is not hard to check that (X, p) is a partial metric space, since (x_1, x_2, x_3) is a quadrangular triplet on (y_1, y_2, y_3).

By our hypothesis, $q_{\Phi,p} : X \times X \to \mathbb{R}_+$ is a quasi-metric, where $q_{\Phi,p}(u, v) = \Phi(p(u, v), p(u, u))$ for each $u, v \in X$. Then

$$q_{\Phi,p}(a, c) \leq q_{\Phi,p}(a, b) + q_{\Phi,p}(b, c),$$

which is equivalent to

$$\Phi(p(a, c), p(a, a)) \leq \Phi(p(a, b), p(a, a)) + \Phi(p(b, c), p(b, b)).$$

Therefore, by definition of p, we have that

$$\Phi(x_1, y_1) \leq \Phi(x_2, y_1) + \Phi(x_3, y_2).$$

(2) $\Rightarrow$ (1). Assume that $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a function satisfying conditions (i) and (ii). Let (X, p) be a partial metric space, we will show that $q_{\Phi,p}$ is a quasi-metric on X, where $q_{\Phi,p}(x, y) = \Phi(p(x, y), p(x, x))$ for each $x, y \in X$.

First we suppose that $q_{\Phi,p}(x, y) = 0 = q_{\Phi,p}(y, x)$ for some $x, y \in X$. Then,

$$\Phi(p(x, y), p(x, x)) = q_{\Phi,p}(x, y) = 0,$$

and

$$\Phi(p(y, x), p(y, y)) = q_{\Phi,p}(y, x) = 0.$$

Condition (i) implies that $p(x, y) = p(x, x)$ and $p(y, x) = p(y, y)$. Taking into account that p is a partial metric on X we have that $p(x, y) = p(x, x) = p(y, y)$, and so $x = y$. Since Φ satisfies (i) we deduce that $q_{\Phi,p}(x, y) = 0 = q_{\Phi,p}(y, x)$ provided that $x = y$. Thus $q_{\Phi,p}$ satisfies axiom (Q1) of quasi-metrics.

It remains to prove that $q_{\Phi,p}$ fulfils the triangle inequality, i.e., axiom $(Q2)$ of quasi-metrics. With this aim, let $x, y, z \in X$. We sill show that $q_{\Phi,p}(x, z) \leq q_{\Phi,p}(x, y) + q_{\Phi,p}(y, z)$. Observe that the cases $x = y$, $y = z$ or $x = z$ are obvious. So, we assume that $x \neq y$, $x \neq z$ and $y \neq z$. In such a case we obtain:

$$p(x, z) > p(x, x) \text{ or } p(x, z) > p(z, z) \text{ and } p(x, z) \geq \max\{p(x, x), p(z, z)\};$$

$$p(x, y) > p(x, x) \text{ or } p(x, y) > p(y, y) \text{ and } p(x, y) \geq \max\{p(x, x), p(y, y)\};$$

$$p(y, z) > p(y, y) \text{ or } p(y, z) > p(z, z) \text{ and } p(y, z) \geq \max\{p(y, y), p(z, z)\}.$$

Moreover, by axiom $(P4)$ of partial metrics, we have that

$$p(x, z) + p(y, y) \leq p(x, y) + p(y, z);$$

$$p(x, y) + p(z, z) \leq p(x, z) + p(z, y);$$

$$p(y,z) + p(x,x) \leq p(y,x) + p(x,z).$$

Then, $(p(x,z), p(x,y), p(y,z)) \in \mathbb{R}^3_+$ is quadrangular triplet on $(p(x,x), p(y,y), p(z,z)) \in \mathbb{R}^3_+$. Thus, by condition (ii), we have that

$$\Phi(p(x,z), p(x,x)) \leq \Phi(p(x,y), p(x,x)) + \Phi(p(y,z), p(y,y)),$$

and so

$$q_{\Phi,p}(x,z) \leq q_{\Phi,p}(x,y) + q_{\Phi,p}(y,z).$$

According to [8], given a quasi-metric sapce (X,q), then q induces a partial order $\preceq_q$ on X given by $x \preceq_q y \Leftrightarrow q(x,y) = 0$. In [14], Matthews showed that given a partial metric space (X,p), then p also induces a partial order $\preceq_p$ on X given by $x \preceq_p y \Leftrightarrow p(x,y) = p(x,x)$. Moreover, in the same reference, it was proved that $\preceq_{q_p} = \preceq_p$.

In the light of the preceding facts, it seems natural to discuss whether, given a qmg-function $\Phi : \mathbf{D} \to \mathbb{R}_+$ and a partial metric space (X,p), the partial orders $\preceq_{q_{\Phi,p}}$ and $\preceq_p$ are exactly the same on X, i.e., whether a qmg-function preserves the order induced by the partial metric that it transforms. The next result gives a positive answer to the questions under consideration.

Proposition 5 *Let $\Phi : \boldsymbol{D} \to \mathbb{R}_+$ be a qmg-function and let (X,p) be a partial metric space. Then $\preceq_{q_{\Phi,p}} = \preceq_p$.*

Proof Let $x, y \in X$ .On the one hand, we have that $x \preceq_p y \Leftrightarrow p(x,x) = p(x,y)$. On the other hand, we have that $x \preceq_{q_{\Phi,p}} y \Leftrightarrow q_{\Phi,p}(x,y) = 0$. Theorem 6 guarantees that $\Phi^{-1}(0) = \{(x,y) \in \mathbf{D} : x = y\}$ and, thus, that $x \preceq_p y \Leftrightarrow x \preceq_{q_{\Phi,p}} y$ as claimed.

Following [8], each quasi-metric q on X induces a T_0 topology $\mathcal{T}(q)$ on X which has as a base the family of open balls $\{B_q(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x;\epsilon) = \{y \in X : q(x,y) < \epsilon\}$. Moreover, according to [14], each partial metric p on X induces a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open balls $\{B_p(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x;\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$. As in the case of the partial order, Matthews proved that the topology induced by a partial metric p and by the associated quasi-metric q_p coincide, i.e., $\mathcal{T}(p) = \mathcal{T}(q_p)$.

Similarly to the partial order case, it seems natural to ask if the same situation happens in our context, i.e., if a qmg-function Φ preserves the topology induced by the partial metric p which it transforms and, thus, $\mathcal{T}(p) = \mathcal{T}(q_{\Phi,p})$. Nevertheless, the behaviour of qmg-functions regarding the preservation of the topology is slightly different. In fact, the answer to the question posed is negative such as Example 3 reveals.

The next proposition will be crucial to show, by means of Example 3, that the topology induced by a partial metric p on a set X does not coincide, in general, with the topology induced by the generated quasi-metric $q_{\Phi,p}$.

Proposition 6 *Let $\Phi_2 : \boldsymbol{D} \to \mathbb{R}_+$ be the function given by $\Phi(0,0) = 0$ and $\Phi_2(x,y) = \frac{x-y}{x}$ for each $(x,y) \in \boldsymbol{D} \setminus \{(0,0)\}$. Then Φ_2 is a qmg-function.*

Proof First of all, we observe that $\Phi_2(x,y) \geq 0$, for each $(x,y) \in \mathbf{D}$. Now, we show that Φ satisfies conditions (i) and (ii) in the statement of Theorem 6.

Clearly, by definition, $\Phi(0,0) = 0$. Next suppose that $\Phi_2(x,y) = 0$ for some $(x,y) \in \mathbf{D} \setminus \{(0,0)\}$. Then $\frac{x-y}{x} = 0$. The last equality is held if and only if $x = y$. Thus the aforesaid condition (i) is satisfied by Φ.

In order to prove that Φ fulfils condition (ii), assume that $(x_1,x_2,x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1,y_2,y_3) \in \mathbb{R}^3_+$. It remains to show that $\Phi_2(x_1,y_1) \leq \Phi_2(x_2,y_1) + \Phi_2(x_3,y_2)$. First, observe that, by definition of quadrangular triplet $(x_1,x_2,x_3) \neq (0,0,0)$. So, $\Phi_2(x_1,y_1) = \frac{x_1-y_1}{x_1}$, $\Phi_2(x_2,y_1) = \frac{x_2-y_1}{x_2}$ and $\Phi_2(x_3,y_2) = \frac{x_3-y_2}{x_3}$. Now, we distinguish two possible cases:

Case 1. $x_1 \geq \max\{x_2,x_3\}$. On the one hand, we have that

$$\Phi_2(x_1,y_1) = \frac{x_1-y_1}{x_1} \leq \frac{x_1-y_1}{\max\{x_2,x_3\}}$$

and

$$\Phi_2(x_2,y_1) + \Phi_2(x_3,y_2) = \frac{x_2-y_1}{x_2} + \frac{x_3-y_2}{x_3} \geq \frac{x_2-y_1}{\max\{x_2,x_3\}} + \frac{x_3-y_2}{\max\{x_2,x_3\}}.$$

On the other hand $x_1-y_1 \leq x_2-y_1+x_3-y_2$, since (x_1,x_2,x_3) is a quadrangular triplet on (y_1,y_2,y_3). It follows that

$$\frac{x_1-y_1}{\max\{x_2,x_3\}} \leq \frac{x_2-y_1}{\max\{x_2,x_3\}} + \frac{x_3-y_2}{\max\{x_2,x_3\}}$$

and, hence, that

$$\Phi_2(x_1,y_1) \leq \Phi_2(x_2,y_1) + \Phi(x_3,y_2).$$

Case 2. $x_1 < \max\{x_2,x_3\}$. Put $x_1' = \max\{x_2,x_3\}$. It is a routine to check that (x_1',x_2,x_3) is a quadrangular triplet on (y_1,y_2,y_3), since $x_1' > x_1 \geq \max\{y_1,y_3\}$ and $x_1' + y_2 \leq x_2 + x_3$. Therefore, by Case 1, we deduce that $\Phi_2(x_1',y_1) \leq \Phi_2(x_2,y_1) + \Phi_2(x_3,y_2)$. Moreover, we observe that, for each $a,b,\alpha \in \mathbb{R}_+$ with $a \leq b$, we have that $\frac{a}{b} \leq \frac{a+\alpha}{b+\alpha}$.
Letting $\alpha = \max\{x_2,x_3\} - x_1$ and $x_1' = \max\{x_2,x_3\}$ we obtain that

$$\Phi_2(x_1,y_1) = \frac{x_1-y_1}{x_1} \leq \frac{x_1-y_1+\alpha}{x_1+\alpha} = \frac{\max\{x_2,x_3\}-y_1}{\max\{x_2,x_3\}} = \Phi_2(x_1',y_1).$$

Thus, $\Phi_2(x_1,y_1) \leq \Phi_2(x_2,y_1) + \Phi_2(x_3,y_2)$.

Consequently Φ fulfils condition (ii) in Theorem 6. Hence, we deduce that Φ is a *qmg*-function.

As announced before, the next example shows that, in general, a *qmg*-function Φ does not preserve the topology induced by the partial metric p which it transforms and, thus, that $\mathcal{T}(p) \neq \mathcal{T}(q_{\Phi,p})$ in general.

Example 3 Let $(\mathbb{R}_+, p_m)$ be the partial metric space such that $p_m(x,y) = \max\{x,y\}$ for each $x,y \in \mathbb{R}_+$. It is not hard to verify that, for each $x \in \mathbb{R}_+$ and $\epsilon > 0$, the open ball centred at x with radius ϵ is given by $B_{p_m}(x;\epsilon) = [0, x+\epsilon[$.

It is clear that the quasi-metric generated by means of the function Φ_2 (introduced in Proposition 6) from p_m is given by $q_{\Phi_2,p_m}(0,0) = 0$ and $q_{\Phi_2,p_m}(x,y) = \frac{\max\{x,y\}-x}{\max\{x,y\}}$ for each $(x,y) \in \mathbf{D} \setminus \{(0,0)\}$. Then $q_{\Phi_2,p_m}(0,y) = 1$, for each $y \in]0,\infty[$ and, hence, $B_{q_{\Phi_2,p_m}}(0;\epsilon) = \{0\}$ for each $\epsilon \in]0,1[$. Hence, $B_{q_{\Phi_2,p_m}}(0;\epsilon) \notin \mathcal{T}(p_m)$ and, therefore, $\mathcal{T}(p_m) \neq \mathcal{T}(q_{\Phi_2,p_m})$.

On account of the above example, we focus our effort on seeking conditions on *qmg*-functions Φ in order to ensure that, for each partial metric space (X,p), $\mathcal{T}(p)=\mathcal{T}(q_{\Phi,p})$. To this end, we introduce the next concept.

Definition 3 Let $\Phi:\mathbf{D}\to\mathbb{R}_+$ be a *qmg*-function. We will say that Φ is a strongly quasi-metric generating function (*sqmg*-function for short) if for each partial metric space (X,p) we have that $\mathcal{T}(p)=\mathcal{T}(q_{\Phi,p})$.

It must be stressed that the name of strongly quasi-metric generating function has been inspired by strongly metric preserving functions introduced in [5].

In the light of the preceding definition, an instance of *sqmg*-functions is given by the function Φ_- introduced in Examples 1.

The next result will be essential for getting a characterization of those *qmg*-functions that are *sqmg*-functions.

Lemma 1 *Let $\Phi:\boldsymbol{D}\to\mathbb{R}_+$ be a qmg-function. Then, Φ is increasing in the first component, i.e., $\Phi(x,z)\geq\Phi(y,z)$ whenever $x\geq y$.*

Proof Let $(x,z),(y,z)\in D$ with $x\geq y$. It is clear that if $x=y$, then $\Phi(x,z)=\Phi(y,z)$. So we assume that $x>y$. Moreover, we can consider that $y>0$ because otherwise we have that $y=z=0$ and, thus, that $\Phi(x,z)\geq\Phi(y,z)=\Phi(0,0)=0$. Consider the terns $(y,x,x-y),(z,x-y,0)\in\mathbb{R}^3_+$. It is not hard to see that $(y,x,x-y)$ is a quadrangular triplet on $(z,x-y,0)$. Therefore, by Theorem 6, we have that

$$\Phi(y,z)\leq\Phi(x,z)+\Phi(x-y,x-y)=\Phi(x,z).$$

Thus, Φ is increasing in the first component.

Although *qmg*-functions do not preserve the topology of the partial metric that they transform, we have always that the topology induced by the quasi-metric $q_{\Phi,p}$ generated from a *qmg*-function Φ is finer than the topology induced by the partial metric p from which it is constructed. Let us recall a topology $\mathcal{T}_1$ is said to be finer than a topology $\mathcal{T}_2$ provided that each open set in $\mathcal{T}_2$ it is so in $\mathcal{T}_1$ (see, for instance, [1]). From now on, the fact that a topology $\mathcal{T}_1$ is finer that a topology $\mathcal{T}_2$ will be denoted by $\mathcal{T}_2\subseteq\mathcal{T}_1$.

Theorem 7 *Let $\Phi:\boldsymbol{D}\to\mathbb{R}_+$ be a qmg-function and let (X,p) be a partial metric space. Then, $\mathcal{T}(p)\subseteq\mathcal{T}(q_{\Phi,p})$.*

Proof Consider $x\in X$ and the real number $\epsilon>0$. Put $\delta=\Phi(p(x,x)+\epsilon,p(x,x))$. Then $\delta>0$. Next we prove that $B_{q_{\Phi,p}}(x;\delta)\subseteq B_p(x;\epsilon)$. Indeed, let $y\in B_{q_{\Phi,p}}(x;\delta)$. Then $\Phi(p(x,y),p(x,x))<\delta=\Phi(p(x,x)+\epsilon,p(x,x))$. Assume for the purpose of contradiction that $p(x,y)-p(x,x)\geq\epsilon$. By Lemma 1 we obtain that $\Phi(p(x,x)+\epsilon,p(x,x))\leq\Phi(p(x,y),p(x,x))<\Phi(p(x,x)+\epsilon,p(x,x))$ which is not possible. It follows that $p(x,y)-p(x,x)<\epsilon$ and, hence, that $y\in B_p(x;\epsilon)$. Therefore, $\mathcal{T}(p)\subseteq\mathcal{T}(q_{\Phi,p})$ as claimed.

The next theorem characterizes *sqmg*-functions.

Theorem 8 *Let $\Phi:\boldsymbol{D}\to\mathbb{R}_+$ be a qmg-function. Then the following assertions are equivalent:*

(1) Φ is a sqmg-function.
(2) For each $a \in \mathbb{R}_+$, the function $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at 0, where $\Phi_a(x) = \Phi(x+a, a)$ for each $x \in \mathbb{R}_+$.

Proof (1) $\Rightarrow$ (2). Suppose for the purpose of contradiction that there exists $a_0 \in \mathbb{R}_+$ such that Φ_{a_0} is not continuous at 0. Next we see that Φ is not a *sqmg*-function.

Since Φ_{a_0} is not continuous at 0, then there exist $\epsilon_0 > 0$ such that for each $\delta > 0$ we can find $x_\delta \in [0, \delta[$ satisfying $\Phi_{a_0}(x_\delta) \geq \epsilon_0$ (observe that $\Phi_{a_0}(0) = 0$).

Consider the partial metric space $(\mathbb{R}_+, p_m)$ introduced in Example 3 and take $x = a_0$. Clearly, for each $\delta > 0$ we have that $y_\delta = x_\delta + a_0 \notin B_{q_{\Phi,p_m}}(x; \epsilon_0)$, since

$$q_{\Phi,p_m}(x, y_\delta) = \Phi(p_m(x, y_\delta), p_m\{x, x\}) = \Phi(x_\delta + a_0, a_0) = \Phi_{a_0}(x_\delta) \geq \epsilon_0.$$

It follows that $\mathcal{T}(p) \neq \mathcal{T}(q_{\Phi,p})$ because $B_{q_{\Phi,p_m}}(x; \epsilon_0) \in \mathcal{T}(q_{\Phi,p})$ and $B_{q_{\Phi,p_m}}(x; \epsilon_0) \notin \mathcal{T}(p)$. Indeed, for each $\delta > 0$, we have that $p_m(x, y_\delta) - p_m(x, x) = x_\delta + a_0 - a_0 = x_\delta < \delta$ and, thus, $y_\delta \in B_{p_m}(x; \delta)$ but $y_\delta \notin B_{q_{\Phi,p_m}}(x; \epsilon_0)$. Consequently, Φ is not a *sqmg*-function which is a contradiction.

(2) $\Rightarrow$ (1). By Theorem 7 we have, for each partial metric space (X, p), that $\mathcal{T}(p) \subseteq \mathcal{T}(q_{\Phi,p})$. It remains to prove that $\mathcal{T}(q_{\Phi,p}) \subseteq \mathcal{T}(p)$. With this aim we show that, for each $x \in X$, given $\epsilon > 0$ there exists $\delta > 0$ such that $B_p(x; \delta) \subseteq B_{q_{\Phi,p}}(x; \epsilon)$. Indeed, by the continuity of $\Phi_{p(x,x)}$ at 0, there exists $\delta > 0$ such that, for each $\alpha \in [0, \delta[$, we have that $\Phi_{p(x,x)}(\alpha) < \epsilon$. It follows that $B_p(x; \delta) \subseteq B_{q_{\Phi,p}}(x; \epsilon)$. Hence if $y \in B_p(x; \delta)$, then $p(x, y) - p(x, x) < \delta$ and so $p(x, y) - p(x, x) \in [0, \delta[$. Whence, $q_{\Phi,p}(x, y) = \Phi(p(x, y), p(x, x)) = \Phi_{p(x,x)}(p(x, y) - p(x, x)) < \epsilon$ and, therefore, $y \in B_{q_{\Phi,p}}(x; \epsilon)$. This ends the proof.

As a consequence of the previous theorem, we can show that the function $\Phi_{-,\frac{1}{2}}$ introduced in Example 2 also constitutes an instance of *sqmg*-function. Indeed, on the one hand,

$$\Phi_{\left(-,\frac{1}{2}\right)_0}(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{x}{2}, & \text{otherwise} \end{cases},$$

and, on the other hand, for each $a \in]0, \infty[$, $\Phi_{\left(-,\frac{1}{2}\right)_a}(x) = x + \frac{a}{2}$. Then, for each $a \in \mathbb{R}_+$, we have that $\Phi_{\left(-,\frac{1}{2}\right)_a}$ is continuous at 0 and so, by Theorem 8, we conclude that $\Phi_{-,\frac{1}{2}}$ is a *sqmg*-function. However, the function Φ_2, given in Proposition 6, is not a *sqmg*-function because Φ_{2_0} is not continuous at 0.

We finish this section exploring the relationship between *qmg*-functions, (quasi-)metric aggregation functions, partial metric aggregation functions and metric generating functions, whenever all of them are defined on $\mathbf{D}$.

Recall that Theorems 1 and 2 states that metric aggregation functions and quasi-metric aggregations functions belong to $\mathcal{O}$. Proposition 6 shows that there are *qmg*-functions that are neither metric aggregation functions nor quasi-metric aggregations functions because $\Phi_2 \notin \mathcal{O}$. By Theorem 3 we know that metric generating functions also belong to $\mathcal{O}$, and thus, Proposition 6 gives an instance, namely Φ_2, of *qmg*-functions which does not belong to $\mathcal{O}$ and, in addition, it is not a metric generating function. By Corollary 2, partial metric aggregation functions are monotone and, therefore, *qmg*-functions are not, in general, partial metric aggregation functions. Indeed, the aforesaid mapping Φ_2 is not monotone, since $(2, 1) \preceq (2, 2)$ and $\frac{1}{2} = \Phi_2(2, 1) \not\leq \Phi_2(2, 2) = 0$.

Reciprocally we analyze if (quasi-)metric aggregation functions, partial metric aggregation functions and metric generating functions are *qmg*-functions.

The next example shows that there are metric and quasi-metric aggregation functions that are not *qmg*-functions.

Example 4 Consider the function $\Phi_{0,1} : \mathbf{D} \to \mathbb{R}_+$ defined by

$$\Phi_{0,1}(a,b) = \begin{cases} 0 \text{ if } a = b = 0 \\ 1 \text{ otherwise.} \end{cases}$$

Since $\Phi_{0,1}$ is monotone, subadditive and $\Phi_{0,1} \in \mathcal{O}$ we have, by Theorems 1 and 2, that it is a metric and quasi-metric aggregation function. However, $\Phi_{0,1}(1,1) = 1$ and thus, by Theorem 6, it is not a *qmg*-function.

Next we show that there are partial metric aggregation functions that are not *qmg*-functions.

Example 5 Consider the function $\Phi_{+1} : \mathbf{D} \to \mathbb{R}_+$ defined by $\Phi_{+1}(a,b) = a + b + 1$ for all $(a,b) \in \mathbf{D}$. It is clear that the function Φ_{+1} fulfils conditions in the statement of Theorem 3 and, thus, it is a partial metric aggregation function. Nevertheless, $\Phi_{+1}(1,1) = 3$ and thus, by Theorem 6, it is not a *qmg*-function.

The next example gives an instance of metric generating function which is not a *qmg*-function.

Example 6 Consider the function $\Phi_{med} : \mathbf{D} \to \mathbb{R}_+$ given by $\Phi_{med}(a,b) = \frac{a+b}{2}$ for all a, b. It is not har to check that Φ_{med} satisfies Theorem 4 and, hence, that it is a metric generating function. However, $\Phi_{med}(1,1) = \frac{1}{2}$ and, therefore, by Theorem 6 it is not a *qmg*-function.

Despite the above exposed facts, surprisingly, *qmg*-functions can be useful to construct (quasi-)metric preserving functions, i.e., one dimensional (quasi-)metric aggregation functions, such as the next result shows.

Proposition 7 *Let $\Phi : \boldsymbol{D} \to \mathbb{R}_+$ be a qmg-function. Then, for each $a \in \mathbb{R}_+$, the function $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ is a quasi-metric preserving function, where $\Phi_a(x) = \Phi(x + a, a)$ for each $x \in \mathbb{R}_+$*

Proof Fix $a \in \mathbb{R}_+$. Since $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a *qmg*-function we have that $\Phi_a^{-1}(0) = \{0\}$ Lemma 1 ensures that Φ_a is increasing. So, we only need to show that Φ_a is also subadditive. To this end, consider $x, y \in \mathbb{R}_+$ with $x \neq 0$ and $y \neq 0$ (the case $x = 0$ or $y = 0$ is obvious). It is not hard to check that $(x + y + a, x + a, y + a)$ is a quadrangular triplet on (a, a, a). Then, by Theorem 6, we deduce that

$$\Phi_a(x + y) = \Phi(x + y + a, a) \leq \Phi(x + a, a) + \Phi(y + a, a) = \Phi_a(x) + \Phi_a(y).$$

Thus, by Theorem 2, we obtain that Φ_a is a quasi-metric preserving function.

Since every quasi-metric preserving function is a metric preserving function (see Theorems 1 and 2) we obtain immediately from Proposition 7 the next consequence.

Corollary 4 *Let $\Phi : \boldsymbol{D} \to \mathbb{R}_+$ be a qmg-function. Then, for each $a \in \mathbb{R}_+$, the function $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ is a metric preserving function, where $\Phi_a(x) = \Phi(x+a, a)$ for each $x \in \mathbb{R}_+$*

Examples 1 and 2 provide instances of *qmg*-functions that, by Proposition 7 and Corollary 4, allow us to induce (quasi-)metric preserving functions.

Observe that Proposition 7 does not allows us to generate, in general, partial metric preserving functions (one dimensional partial metric aggregation functions). To clarify this assertion it is sufficient that we consider, again, the function Φ_2 given in Proposition 6. Then $\Phi_{2_0}(x) = 1$ for all $x \in \mathbb{R}_+$. By Theorem 3 we have that Φ_{2_0} is not a partial metric preserving function because $1 \leq 3$, $2 \leq 3$ and $\Phi_{2_0}(3) = \Phi_{2_0}(1) = \Phi_{2_0}(2)$.

3 A general method for generating partial metrics from quasi-metrics

The aim of this section is to introduce a general method to generate a partial metric from a quasi-metric in such a way that the technique introduced by Matthews can be recovered as a particular case. With this objective, we introduce the next concept which will be essential for tackling the posed problem.

Definition 4 We will say that a function $\Psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a partial metric generating function (briefly, *pmg*-fucntion) if for each weighted quasi-metric space (X, q, w_q) the function $p_{\Psi,q,w_q} : X \times X \to \mathbb{R}_+$ is a partial metric on X, where $p_{\Psi,q,w_q}(x, y) = \Psi(q(x, y), w_q(x))$ for each $x, y \in X$.

The next example shows that the Matthews technique is a particular case of the exposed approach (see Section 1).

Example 7 Let $\Psi_+ : \mathbb{R}^2_+ \to \mathbb{R}_+$ given by $\Psi_+(a, \alpha) = a+\alpha$ for each $a, \alpha \in \mathbb{R}_+$. Then, Ψ_+ is a *pmg*-function. Indeed, given a weighted quasi-metric space (X, q, w_q) we have that $p_{\Psi_+,q,w_q}(x, y) = q(x, y) + w_q(x)$ for each $x, y \in X$, which is the well-known partial metric p_{q,w_q} induced by the weighted quasi-metric space q.

The next example provides an alternative way of generating a partial metric from a quasi-metric which is based on the use of partial metric generating functions.

Example 8 Define $\Psi_{+1} : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\Psi_{+1}(a, \alpha) = a + \alpha + 1$. It is not hard to check that Ψ_{+1} is a *pmg*-function. Indeed, given a weighted quasi-metric space (X, q, w_q), an easy verification shows that p_{Ψ_{+1},q,w_q} is a partial metric on X with $p_{\Psi_{+1},q,w_q}(x, y) = q(x, y) + w_q(x) + 1$ for each $x, y \in X$.

The so-called upper quasi-metric space will be crucial in order to achieve our target. Let recall that the upper quasi-metric space is the weighted quasi-metric space given by the tern $(\mathbb{R}_+, q_u, w_{q_u})$ such that $q_u(x, y) = \max\{y - x, 0\}$ for each $x, y \in X$ and $w_{q_u}(x) = x$ for each $x \in \mathbb{R}_+$.

The next result will be crucial in order to yield a characterization of *pmg*-functions later on.

Lemma 2 *Let $\Psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a pmg-function and $\alpha \in \mathbb{R}_+$. Then Ψ is increasing in the first component.*

Proof Fix $\alpha \in \mathbb{R}_+$. Since Ψ is a *pmg*-function we have that, given $(\mathbb{R}_+, q_u, w_{q_u})$, the function $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+^2$. Let $a, b \in \mathbb{R}_+$ and assume that $a \leq b$. Then, $q_u(\alpha, a+\alpha) = a$, $q_u(\alpha, \alpha) = 0$, $q_u(\alpha, b+\alpha) = b$ and $q_u(b+\alpha, a+\alpha) = 0$. So, on the one hand, we have that

$$\Psi(a,\alpha) + \Psi(0, b+\alpha) =$$

$$\Psi(q_u(\alpha, a+\alpha), w_{q_u}(\alpha)) + \Psi(q_u(b+\alpha, b+\alpha), w_{q_u}(b+\alpha)) =$$

$$p_{\Psi,q_u,w_{q_u}}(\alpha, a+\alpha) + p_{\Psi,q_u,w_{q_u}}(b+\alpha, b+\alpha).$$

On the other hand, we have that

$$\Psi(b,\alpha) + \Psi(0, b+\alpha) =$$

$$\Psi(q_u(\alpha, b+\alpha), w_{q_u}(\alpha)) + \Psi(q_u(b+\alpha, a+\alpha), w_{q_u}(b+\alpha)) =$$

$$p_{\Psi,q_u,w_{q_u}}(\alpha, b+\alpha) + p_{\Psi,q_u,w_{q_u}}(b+\alpha, a+\alpha).$$

Now, since $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+$ we have that

$$p_{\Psi,q_u,w_{q_u}}(\alpha, a+\alpha) + p_{\Psi,q_u,w_{q_u}}(b+\alpha, b+\alpha) \leq p_{\Psi,q_u,w_{q_u}}(\alpha, b+\alpha) + p_{\Psi,q_u,w_{q_u}}(b+\alpha, a+\alpha).$$

It follows that

$$\Psi(a,\alpha) + \Psi(0, b+\alpha) \leq \Psi(b,\alpha) + \Psi(0, b+\alpha).$$

This last inequality implies $\Psi(a,\alpha) \leq \Psi(b,\alpha)$, as we claimed.

The next theorem provides a characterization of the class of *pmg*-functions and, thus, a general method to generate a partial metric from a (weighted) quasi-metric.

Theorem 9 *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a function. The the following assertions are equivalent:*

(1) Ψ is a pmg-function.
(2) Ψ satisfies, for each $a, b, c, \alpha, \beta \in \mathbb{R}_+$, the following conditions:
 (i) $\Psi(a,\alpha) = \Psi(a+\alpha-\beta, \beta)$, whenever $a+\alpha \geq \beta$;
 (ii) $\Psi(c,\alpha) + \Psi(0,\beta) \leq \Psi(a,\alpha) + \Psi(b,\beta)$, whenever $c \leq a+b$ and $\beta \leq a+\alpha$;
 (iii) If $\Psi(0,\alpha) = \Psi(a,\alpha) = \Psi(0,\beta)$ and $a+\alpha \geq \beta$, then $a = 0$ and $\alpha = \beta$.

Proof (1) $\Rightarrow$ (2).Let $a, b, c, \alpha, \beta \in \mathbb{R}_+$.

In order to prove (i), consider that $a+\alpha \geq \beta$. Notice that the case $a = 0$ and the case $\alpha = \beta$ are obvious. So, we suppose that $a > 0$ or $\alpha \neq \beta$.

Consider the set $X = \{x, y, z\}$ and define the function q on $X \times X$ as follows:

$$q(x,y) = a; \quad q(y,x) = a+\alpha-\beta; \quad q(x,z) = 2a+\alpha-\beta;$$

$$q(z,x) = 2a+2\alpha-\beta; \quad q(y,z) = a+\alpha-\beta; \quad q(z,y) = a+\alpha;$$

$$q(x,x) = q(y,y) = q(z,z) = 0.$$

It is not hard to verify that q is a quasi-metric on X. Besides, if we define the function w_q on X given by

$$w_q(x) = \alpha; \quad w_q(y) = \beta \quad \text{and} \quad w_Q(z) = 0,$$

we have that

$$q(x,y) + w_q(x) = a + \alpha = a + \alpha - \beta + \beta = q(y,x) + w_q(y);$$

$$q(x,z) + w_q(x) = 2a + \alpha - \beta + \alpha = 2a + 2\alpha - \beta = q(z,x) + w_q(z)$$

and

$$q(y,z) + w_q(y) = a + \alpha - \beta + \beta = a + \alpha = q(z,y) + w_q(z).$$

Therefore, (X, q, w_q) is a weighted quasi-metric space.

Now, by hypothesis, p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u,v) = \Psi(q(u,v), w_q(u))$ for each $u, v \in X$. Then, $p_{\Psi,q,w_q}(u,v) = p_{\Psi,q,w_q}(v,u)$, for each $u, v \in X$. Hence

$$\Psi(a,\alpha) = \Psi(q(x,y), w_q(x)) =$$

$$p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,x) =$$

$$\Psi(q(y,x), w_q(y)) = \Psi(a + \alpha - \beta, \beta).$$

Next we show that Φ fulfils (ii). To this end, let $a, b, c, \alpha, \beta \in \mathbb{R}_+$ with $c \leq a+b$ and $\beta \leq a + \alpha$. Lemma 2 ensures that the inequality under consideration is hold for the case $a = 0$ and $\alpha = \beta$, and for the case $b = \beta = 0$. So, we suppose that $a > 0$ or $\alpha \neq \beta$, and $b > 0$ or $\beta > 0$. First, we will prove that the next inequality is fulfilled

$$\Psi(a+b,\alpha) + \Psi(0,\beta) \leq \Psi(a,\alpha) + \Psi(b,\beta).$$

With this purpose we consider the set $X = \{x, y, z\}$. Define the function q on $X \times X$ by:

$$q(x,y) = a; \quad q(y,x) = a + \alpha - \beta; \quad q(y,z) = b; \quad q(z,y) = b + \beta;$$

$$q(x,z) = a + b; \quad q(z,x) = a + b + \alpha;$$

and

$$q(x,x) = q(y,y) = q(z,z) = 0.$$

Then one can verify that q is a quasi-metric on X. Even more, if we define $w_q(x) = \alpha$, $w_q(y) = \beta$ and $w_q(z) = 0$, then

$$q(x,y) + w_q(x) = a + \alpha = a + \alpha - \beta + \beta = q(y,x) + w_q(y);$$

$$q(x,z) + w_q(x) = a + b + \alpha = q(z,x) + w_q(z)$$

and

$$q(y,z) + w_q(y) = b + \beta = q(z,y) + w_q(z).$$

Therefore, (X, q, w_q) is a weighted quasi-metric space.

Since Ψ is a *pmg*-function we have that p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u,v) = \Psi(q(u,v), w_q(u))$ for each $u, v \in X$. From this fact we deduce, on the one hand, that

$$\Psi(a+b,\alpha) + \Psi(0,\beta) = \Psi(q(x,z), w_q(x)) + \Psi(q(y,y), w_q(y)) =$$

$$p_{\Psi,q,w_q}(x,z) + p_{\Psi,q,w_q}(y,y),$$

and, on the other hand, that

$$\Psi(a,\alpha)+\Psi(b,\beta)=\Psi(q(x,y),w_q(x))+\Psi(q(y,z),w_q(y))=p_{\Psi,q}(x,y)+p_{\Psi,q}(y,z).$$

Since $p_{\Psi,q}(x,z)+p_{\Psi,q}(y,y)\leq p_{\Psi,q}(x,y)+p_{\Psi,q}(y,z)$, we obtain that

$$\Psi(a+b,\alpha)+\Psi(0,\beta)\leq\Psi(a,\alpha)+\Psi(b,\beta).$$

Thus, by Lemma 2, we conclude that

$$\Psi(c,\alpha)+\Psi(0,\beta)\leq\Psi(a+b,\alpha)+\Psi(0,\beta)\leq\Psi(a,\alpha)+\Psi(b,\beta).$$

It remains to prove condition (iii). Suppose that $\Psi(0,\alpha)=\Psi(a,\alpha)=\Psi(0,\beta)$ and $a+\alpha\geq\beta$.

Consider the weighted quasi-metric space $(\mathbb{R}_+,q_u,w_{q_u})$. Then, $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+$, where $p_{\Psi,q_u,w_{q_u}}(x,y)=\Psi(q_u(x,y),w_{q_u}(x))$ for each $x,y\in\mathbb{R}_+$.

First, we will see that $\alpha=\beta$. Suppose that $\alpha\geq\beta$ (the proof for the case $\beta\geq\alpha$ runs following similar arguments). In such a case,

$$p_{\Psi,q_u,w_{q_u}}(\alpha,\alpha)=\Psi(q_u(\alpha,\alpha),w_{q_u}(\alpha))=\Psi(0,\alpha);$$

$$p_{\Psi,q_u,w_{q_u}}(\beta,\beta)=\Psi(q_u(\beta,\beta),w_{q_u}(\beta))=\Psi(0,\beta);$$

$$p_{\Psi,q_u}(\alpha,\beta)=\Psi(q_u(\alpha,\beta),w_{q_u}(\alpha))=\Psi(\max\{\beta-\alpha,0\},\alpha)=\Psi(0,\alpha).$$

Since $p_{\psi,q_u,w_{q_u}}$ is a partial metric then $p_{\Psi,q_u w_{q_u}}(\alpha,\beta)=p_{\Psi,q_u,w_{q_u}}(\beta,\alpha)$ and, hence,

$$p_{\Psi,q,w_{q_u}}(\alpha,\alpha)=p_{\Psi,q,w_{q_u}}(\alpha,\beta)=p_{\Psi,q,w_{q_u}}(\beta,\beta),$$

and so $\alpha=\beta$.

Now, for the purpose of contradiction, we assume that $a>0$. Consider the set $X=\{x,y,z\}$ with $x\neq y$. Define the function q on $X\times X$ by:

$$q(x,y)=q(z,y)=q(y,x)=q(y,z)=a;\qquad q(x,z)=q(z,x)=2a;$$

and

$$q(x,x)=q(y,y)=q(z,z)=0.$$

One can verify that q is a quasi-metric on X. Moreover, if we define $w_q(x)=w_q(y)=w_q(z)=\alpha$, we have that

$$q(x,y)+w_q(x)=a+\alpha=a+\alpha=q(y,x)+w_q(y);$$

$$q(x,z)+w_q(x)=2a+\alpha=q(z,x)+w_q(z)$$

and

$$q(y,z)+w_q(y)=a+\alpha=q(z,y)+w_q(z).$$

Therefore, (X,q,w_q) is a weighted quasi-metric space. Then, p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u,v)=\Psi(q(u,v),w_q(u))$ for each $u,v\in X$. Furthermore we have

$$p_{\Psi,q,w_q}(x,x)=\Psi(q(x,x),w_q(x))=\Psi(0,\alpha);$$

$$p_{\Psi,q,w_q}(x,y)=\Psi(q(x,y),w_q(x))=\Psi(a,\alpha);$$

and

$$p_{\Psi,q,w_q}(y,y) = \Psi(q(y,y), w_q(y)) = \Psi(0,\alpha).$$

Since $\Psi(0,\alpha) = \Psi(a,\alpha)$ we obtain that

$$p_{\Psi,q,w_q}(x,x) = p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,y)$$

and, thus, that $x = y$, which is a contradiction.

(2) $\Rightarrow$ (1). Let (X, q, w_q) be a weighted quasi-metric space. Define $p_{\Psi,q,w_q}(x,y) = \Psi(q(x,y), w_q(x))$ for each $x, y \in X$. Next we show that p_{Ψ,q,w_q} is a partial metric on X. To this aim, let $x, y, z \in X$.

Suppose that $p_{\Psi,q,w_q}(x,x) = p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,y)$. By construction of p_{Ψ,q,w_q} we have that

$$\Psi(0, w_q(x)) = \Psi(q(x,x), w_q(x)) = \Psi(q(x,y), w_q(x)) =$$
$$\Psi(q(y,y), w_q(y)) = \Psi(0, w_q(y)).$$

Besides $q(x,y)+w_q(x) \geq w_q(y)$, since $q(x,y)+w_q(x) = q(y,x)+w_q(y)$. Whence we deduce that $q(x,y) = 0$ and $w_q(x) = w_q(y)$, because of Ψ satisfies (iii). Moreover, in such a case we have that $w_q(x) = q(y,x)+w_q(y)$, which implies that $q(y,x) = 0$. Thus, $q(x,y) = q(y,x) = 0$ and so $x = y$. Obviously, if $x = y$ we have that $p_{\Psi,q,w_q}(x,x) = p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,y)$. We conclude that p_{Ψ,q,w_q} satisfies axiom $(P1)$ of partial metrics.

The definition of Ψ gives that $p_{\Psi,q,w_q}(x,x) = \Psi(0, w_q(x)) \geq 0$. Moreover, Lemma 2 guarantees that

$$p_{\Psi,q,w_q}(x,x) = \Psi(0, w_q(x)) \leq \Psi(q(x,y), w_q(x)) = p_{\Psi,q,w_q}(x,y).$$

It follows that p_{Ψ,q,w_q} satisfies axiom $(P2)$ of partial metrics.

Since $q(x,y) + w_q(x) \geq w_q(y)$ and $q(x,y) + w_q(x) = q(y,x) + w_q(y)$ we obtain from condition (i) that

$$p_{\Psi,q,w_q}(x,y) = \Psi(q(x,y), w_q(x)) = \Psi(q(y,x), w_q(y)) = p_{\Psi,q,w_q}(y,x).$$

So p_{Ψ,q,w_q} fulfils axiom $(P3)$ of partial metrics.

Finally we show that p_{Ψ,q,w_q} satisfies axiom $(P4)$ of partial metrics. On the one hand,

$$p_{\Psi,q,w_q}(x,z) + p_{\Psi,q,w_q}(y,y) =$$
$$\Psi(q(x,z), w_q(x)) + \Psi(q(y,y), w_q(y)) =$$
$$\Psi(q(x,z), w_q(x)) + \Psi(0, w_q(y)).$$

On the other hand,

$$p_{\Psi,q,w_q}(x,y) + p_{\Psi,q,w_q}(y,z) = \Psi(q(x,y), w_q(x)) + \Psi(q(y,z), w_q(y)).$$

Since $q(x,z) \leq q(x,y) + q(y,z)$ and $w_q(y) \leq q(x,y) + w_q(x)$, we deduce from condition (ii) that

$$\Psi(q(x,z),w_q(x))+\Psi(0,w_q(y))\leq \Psi(q(x,y),w_q(x))+\Psi(q(y,z),w_q(y)).$$

Hence we conclude that

$$p_{\Psi,q,w_q}(x,z)+p_{\Psi,q,w_q}(y,y)\leq p_{\Psi,q,w_q}(x,y)+p_{\Psi,q,w_q}(y,z).$$

Therefore p_{Ψ,q,w_q} is a partial metric on X, and this ends the proof.

An immediate consequence of the above characterization is given by the following result which will be key in our subsequent discussion.

Corollary 5 *Let $\Psi:\mathbb{R}^2_+\to\mathbb{R}_+$ be a pmg-function and let $\alpha,a\in\mathbb{R}_+$. If $\Psi(a,\alpha)=\Psi(0,\alpha)$, then $a=0$.*

Proof Suppose that $\Psi(a,\alpha)=\Psi(0,\alpha)$. Set $a+\alpha=\beta$. By condition (i) in the statement of Theorem 9 we have that $\Psi(a,\alpha)=\Psi(0,a+\alpha)$. Consequently, $\Psi(a,\alpha)=\Psi(0,\alpha)=\Psi(0,a+\alpha)$ and, by condition $(iiii)$ in the aforesaid theorem we deduce that $a=0$.

Similar to the case of quasi-metric generating functions one can explore whether, given a *pmg*-function $\Phi:\mathbb{R}^2_+\to\mathbb{R}_+$ and a weighted quasi-metric space (X,q,w_q), the partial orders $\preceq_{p_{\Psi,q,w_q}}$ and $\preceq_q$ are exactly the same on X, i.e., whether a *pmg*-function preserves the order induced by the quasi-metric that it transforms. The next result gives a positive answer to the posed inquiry.

Proposition 8 *Let $\Phi:\mathbb{R}^2_+\to\mathbb{R}_+$ be a pmg-function and let (X,q,w_q) a weighted quasi-metric space. Then $\preceq_{p_{\Psi,q,w_q}}=\preceq_q$.*

Proof Let $x,y\in X$. Suppose that $x\preceq_q y$. Then $q(x,y)=0$ and, thus, we have that

$$p_{\Psi,q,w_q}(x,y)=\Psi(q(x,y),w_q(x))=\Psi(0,w_q(x))=\Psi(q(x,x),w_q(x))=p_{\Psi,q,w_q}(x,x).$$

Whence we get that $x\preceq_{p_{\Psi,q,w_q}} y$. Next assume that $x\preceq_{p_{\Psi,q,w_q}} y$. Then $p_{\Psi,q,w_q}(x,y)=p_{\Psi,q,w_q}(x,x)$. Hence we have that $\Psi(q(x,y),w_q(x))=\Psi(q(x,x),w_q(x))=\Psi(0,w_q(x))$. Corollary 5 ensures that $q(x,y)=0$ and, thus, that $x\preceq_q y$.

Similarly to the the *qmg*-functions, it seems natural to wonder if a *pmg*-function Φ preserves the topology induced by the weighted quasi-metric q that it transforms and, hence, $\mathcal{T}(q)=\mathcal{T}(p_{\Psi,q,w_q})$. However, the next example shows that this is not the case.

Example 9 Define $\Psi_1:\mathbb{R}^2_+\to\mathbb{R}_+$ by $\Psi_1(0,0)=0$ and $\Psi_1(a,\alpha)=a+\alpha+1$ otherwise. A straightforward computation gives that Ψ_1 is a *pmg*-function. Consider the weighted quasi-metric $(\mathbb{R}_+,q_u,w_{q_u})$. It is not hard to check that, for each $x\in\mathbb{R}_+$ and $\epsilon>0$, the open ball centered at x with radius ϵ is given by $B_{q_u}(x;\epsilon)=[0,x+\epsilon[$. However, $B_{p_{\Psi_1,q_u,w_{q_u}}}(0;\epsilon)=\{0\}$ for each $\epsilon\in]0,1[$. Hence, $B_{p_{\Psi_1,q_u,w_{q_u}}}(0;\epsilon)\notin\mathcal{T}(q_u)$ and, therefore, $\mathcal{T}(q_u)\neq\mathcal{T}(p_{\Phi_1,q_u,w_{q_u}})$.

In the light of the preceding example we focus our effort on characterizing those *pmg*-functions which preserve the topology of the weighted quasi-metric that it transforms. With this aim we introduce the notion below.

Definition 5 Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a *pmg*-function. We will say that Ψ is a strongly partial metric generating function (*spmg*-function for short) if for each weighted quasi-metric space (X, q, w_q) we have that $\mathcal{T}(q) = \mathcal{T}(p_{\Psi,q,w_q})$.

It is easily seen that Examples 7 and 8 provide instances of *spmg*-functions.

Similar to *qmp*-functions, we have that the topology induced by the partial metric p_{Ψ,q,w_q} generated from a *pmg*-function Ψ is always finer than the topology induced by the weighted quasi-metric q from which it is constructed. The next result states such an affirmation.

Theorem 10 *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function and let (X, q, w_q) be a weighted quasi-metric space. Then, $\mathcal{T}(q) \subseteq \mathcal{T}(p_{\Psi,q,w_q})$.*

Proof Consider $x \in X$ and the real number $\epsilon > 0$. Put $\delta = \Psi(\epsilon, w_q(x)) - \Psi(0, w_q(x))$. Corollary 5 guarantees that $\delta > 0$. Next we show that $B_{p_{\Psi,q,w_q}}(x;\delta) \subset B_q(x;\epsilon)$. Let $y \in B_{p_{\Psi,q,w_q}}(x;\delta)$. Then, $p_{\Psi,q,w_q}(x,y) < p_{\Psi,q,w_q}(x,x) + \delta$. Then

$$\Psi(q(x,y), w_q(x)) < \Psi(0, w_q(x)) + \delta = \Psi(\epsilon, w_q(x)).$$

Now, by Lemma 2, we deduce that $q(x,y) < \epsilon$. Therefore, $y \in B_q(x;\epsilon)$.

The next theorem characterizes *spmg*-functions.

Theorem 11 *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function. Then the following assertions are equivalent:*

(1) Ψ is a spmg-function.
(2) For each $\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at 0, where $\Psi_\alpha(a) = \Psi(a, \alpha)$ for each $a \in \mathbb{R}_+$.

Proof (1) $\Rightarrow$ (2). Suppose for the purpose of contradiction that there exists $\alpha_0 \in \mathbb{R}_+$ such Ψ_{α_0} is not continuous at 0. Next we show that Ψ is not a *spmg*-function.

Since Ψ_{α_0} is not continuous at 0, then there exist $\epsilon_0 > 0$ such that for each $\delta > 0$ we can find $a_\delta \in [0, \delta[$ satisfying $\Psi_{\alpha_0}(a_\delta) - \Psi_{\alpha_0}(0) \geq \epsilon_0$ (observe that $\Psi_{\alpha_0}(0) \leq \Psi_{\alpha_0}(a_\delta)$ by Lemma 2).

Consider the weighted quasi-metric space $(\mathbb{R}_+, q_u, w_{q_u})$. Take $x = \alpha_0$. Then $\mathcal{T}(q_u) \neq \mathcal{T}(p_{\Psi,q_u,w_{q_u}})$, since we have that $B_{p_{\Psi,q_u,w_q}}(x;\epsilon_0) \in \mathcal{T}(p_{\Psi,q_u,w_{q_u}})$ but $B_{p_{\Psi,q_u,w_q}}(x;\epsilon_0) \notin \mathcal{T}(q_u)$. Indeed, for each $\delta > 0$, put $y_\delta = a_\delta + \alpha_0$. Then $y_\delta \in B_{q_u}(x;\delta)$, since $q_u(x, y_\delta) = \max\{y_\delta - x, 0\} = a_\delta < \delta$. Nevertheless, we have that $y_\delta \notin B_{p_{\Psi_{\alpha_0},q_u,w_{q_u}}}(x;\epsilon_0)$ because

$$p_{\Psi,q_u,w_{q_u}}(x, y_\delta) = \Psi(q_u(x, y_\delta), w_{q_u}(x)) = \Psi(a_\delta, \alpha_0) = \Psi_{\alpha 0}(a_\delta) \geq$$

$$\Psi_{\alpha_0}(0) + \epsilon_0 = \Psi(0, w_{q_u}(\alpha_0)) + \epsilon_0 = p_{\Psi,q_u,w_{q_u}}(x,x) + \epsilon_0.$$

Therefore, Ψ is not a *spmg*-function which is a contradiction.

(2) $\Rightarrow$ (1). Let (X, q, w_q) be a weighted quasi-metric space. Suppose that, for each $\alpha \in \mathbb{R}_+$, the function Ψ_α is continuous at 0. By Theorem 10 we deduce that $\mathcal{T}(q) \subseteq \mathcal{T}(p_{\Psi,q,w_q})$. So we just need to show that $\mathcal{T}(p_{\Psi,q,w_q}) \subseteq \mathcal{T}(q)$. Next we prove that, given $x \in X$ we have that, for each real number $\epsilon > 0$ there exist a real number $\delta > 0$ such that $B_q(x;\delta) \subseteq B_{p_{\Psi,q,w_q}}(x;\epsilon)$. Since $\Psi_{w_q(x)}$ is continuous at 0, there exists $\delta > 0$ such that, for each $a \in [0, \delta[$ we have that $\Psi_{w_q(x)}(a) - \Psi_{w_q(x)}(0) <$

ϵ. In such a case, $B_q(x;\delta) \subseteq B_{p_{\Psi,q,w_q}}(x;\epsilon)$. Indeed, if $y \in B_q(x;\delta)$, then $q(x,y) < \delta$ and, in addition, $p_{\Psi,q,w_q}(x,y) - p_{\Psi,q,w_q}(x,x) = \Psi_{w_q(x)}(q(x,y)) - \Psi_{w_q(x)}(0) < \epsilon$. Therefore $y \in B_{p_{\Psi,q,w_q}}(x;\epsilon)$.

Note that the *pgm*-function Ψ_1 introduced in Example 9 fulfils that the function Ψ_0 is not continuous at 0, which agrees with the fact that Ψ_1 is not a *spgm*-function.

We conclude this section discussing the relationship between *pmg*-functions, (quasi-)metric aggregation functions, partial metric aggregation functions and metric generating functions.

Example 8 shows that there are *pmg*-functions that are neither (quasi-)metric aggregation functions nor metric generating functions because the function Ψ_{+1} introduced in Example 8 fulfils that $\Psi_{+1} \notin \mathcal{O}$. Let us recall that Theorems 1, 2 and 4 state that (quasi-)metric aggregation functions and metric generating functions belong to $\mathcal{O}$.

Nevertheless, the next result shows that every *pmg*-function is always a partial metric aggregation function.

Proposition 9 *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function. Then Ψ is a partial metric aggregation function.*

Proof We prove that Ψ satisfies (1) and (2) in Theorem 3. To this end, assume that $(x_1,x_2),(y_1,y_2),(z_1,z_2),(w_1,w_2) \in \mathbb{R}_+^2$, with $(x_1,x_2)+(y_1,y_2) \preceq (z_1,z_2)+(w_1,w_2)$, $(y_1,y_2) \preceq (z_1,z_2)$ and $(y_1,y_2) \preceq (w_1,w_2)$. Next we show that

$$\Psi(x_1,x_2)+\Psi(y_1,y_2) \leq \Psi(z_1,z_2)+\Psi(w_1,w_2).$$

Set $c = x_1+x_2$, $a = z_1+z_2$, $b = w_1+w_2-y_1-y_2$, $\alpha = 0$ and $\beta = y_1+y_2$. Then, $c,a,b,\alpha,\beta \in \mathbb{R}_+$, $c \leq a+b$ and $a+\alpha \geq \beta$. So, by (*ii*) in Theorem 9, we have that

$$\Psi(x_1+x_2,0)+\Psi(0,y_1+y_2) \leq \Psi(z_1+z_2,0)+\Psi(w_1+w_2-y_1-y_2,y_1+y_2).$$

Besides, by (*i*) in Theorem 9, we have that $\Psi(x_1+x_2,0) = \Psi(x_1,x_2)$, $\Psi(0,y_1+y_2) = \Psi(y_1,y_2)$, $\Psi(z_1+z_2,0) = \Psi(z_1,z_2)$ and $\Psi(w_1,w_2) = \Psi(w_1+w_2-y_1-y_2,y_1+y_2)$. Therefore we obtain that $\Psi(x_1,x_2)+\Psi(y_1,y_2) \leq \Psi(z_1,z_2)+\Psi(w_1,w_2)$ and, thus, that (1) in Theorem 3 is satisfied.

Next we show that Ψ satisfies (2) in Theorem 3. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathbb{R}_+^2$, with $(x_1,x_2) \preceq (z_1,z_2)$ and $(y_1,y_2) \preceq (z_1,z_2)$, such that $\Psi(x_1,x_2) = \Psi(y_1,y_2) = \Psi(z_1,z_2)$. We claim that $(x_1,x_2) = (y_1,y_2) = (z_1,z_2)$.

Consider $a = z_1+z_2-x_1-x_2$, $\alpha = x_1+x_2$ and $\beta = y_1+y_2$. Then $a,\alpha,\beta \in \mathbb{R}_+$ and $a+\alpha \geq \beta$. By (*i*) in Theorem 9, we have that $\Psi(0,\alpha) = \Psi(0,x_1+x_2) = \Psi(x_1,x_2)$, $\Psi(a,\alpha) = \Psi(z_1+z_2-x_1-x_2,x_1+x_2) = \Psi(z_1,z_2)$ and $\Psi(0,\beta) = \Psi(0,y_1+y_2) = \Psi(y_1,y_2)$. Then $\Psi(0,\alpha) = \Psi(a,\alpha) = \Psi(0,\beta)$. Hence, by (*iii*) in Theorem 9, we obtain that $z_1+z_2 = x_1+x_2 = y_1+y_2$. Moreover one can show easily that $z_1 = x_1 = y_1$ and $z_2 = x_2 = y_2$, since $z_1 \geq \max\{x_1,y_1\}$ and $z_2 \geq \max\{x_2,y_2\}$. Therefore, $(x_1,x_2) = (y_1,y_2) = (z_1,z_2)$ as we claimed.

Conversely we analyze if (quasi-)metric aggregation functions, partial metric aggregation functions and metric generating functions are *pmg*-functions.

The next example shows that there are metric and quasi-metric aggregation functions that are not *qmg*-functions.

Example 10 Consider the function $\Psi_{0,1} : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$\Psi_{0,1}(a,b) = \begin{cases} 0 \text{ if } a = b = 0 \\ 1 \text{ otherwise.} \end{cases}$$

Clearly $\Psi_{0,1}$ is monotone, subadditive and $\Psi_{0,1} \in \mathcal{O}$. By Theorems 1 and 2 we have that it is a metric and quasi-metric aggregation function. However, $\Psi_{0,1}$ is not a *pmg*-function. Indeed, $\Psi_{0,1}(0,1) = \Psi_{0,1}(2,1) = \Psi_{0,1}(0,\frac{1}{2}) = 1$ and, in addition, $\frac{1}{2} \leq 1+2$ but $2 \neq 0$. So, $\Psi_{0,1}$ does not satisfy condition (iii) in the statement of Theorem 9.

Next we show that there are partial metric aggregation functions that are not *pmg*-functions.

Example 11 Consider the function $\Psi_{\frac{1}{2}} : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by $\Psi_{\frac{1}{2}}(a,b) = a + \frac{b}{2}$ for all $(a,b) \in \mathbb{R}_+^2$. It is clear that the function $\Psi_{\frac{1}{2}}$ fulfils the conditions in the statement of Theorem 3 and, thus, it is a partial metric aggregation function. However, $\Psi_{\frac{1}{2}}$ does not satisfies condition (iii) in the statement of Theorem 9 and, hence, it is not a *pmg*-function. Notice that $0 \leq 1+1$ and that $\Psi_{\frac{1}{2}}(1,1) \neq \Psi_{\frac{1}{2}}(2,0)$.

The next example gives an instance of metric generating function which is not a *pmg*-function.

Example 12 Consider the function $\Psi_{max} : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by $\Psi_{max}(a,b) = \max\{a,b\}$ for all $a,b \in \mathbb{R}_+$. It is not har to check that Ψ_{max} satisfies Theorem 4 and, hence, that it is a metric generating function. Moreover, $0 \leq 1+1$ and $\Psi_{\max}(1,1) \neq \Psi_{\max}(2,0)$. This $\Psi_{\max}$ does not satisfy condition (iii) in Theorem 9 and, therefore, it is not a *pmg*-function.

Similar to the (quasi-)metric preserving approach, a method for generating partial metric preserving functions from *pmg*-functions can be obtained such as the next result shows.

Theorem 12 *Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a pmg-function. Then for each $\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a partial metric preserving function, where $\Psi_\alpha(a) = \Psi(a,\alpha)$ for each $a \in \mathbb{R}_+$.*

Proof Fix $\alpha \in \mathbb{R}_+$. First we show that Ψ_α satisfies condition (1) in statement of Theorem 3. Let $x,y,z,w \in \mathbb{R}_+$ such that $x+y \leq z+w$ and $y \leq \min\{z,w\}$. Set $a = z$, $b = w-y$, $c = x$ and $\beta = y+\alpha$. It follows that that $c \leq a+b$ and $\beta \leq a+\alpha$ and, by condition (i) in Theorem 9, that $\Psi(0,\beta) = \Psi(y,\alpha)$ and $\Psi(b,\beta) = \Psi(w,\alpha)$. By condition (ii) in Theorem 9 we deduce that

$$\Psi_\alpha(x) + \Psi_\alpha(y) = \Psi(c,\alpha) + \Psi(0,\beta) \leq \Psi(a,\alpha) + \Psi(b,\beta) = \Psi_\alpha(z) + \Psi_\alpha(w).$$

Next we prove that Ψ_α satisfies condition (ii) in statement of Theorem 3. Let $x,y,z \in \mathbb{R}_+$, with $x \geq \max\{y,z\}$, and suppose that $\Psi_\alpha(x) = \Psi_\alpha(y) = \Psi_\alpha(z)$. Set $a = x-y$, $\alpha' = y+\alpha$ and $\beta = z+\alpha$. In such a case, we have that $\Psi(a,\alpha') = \Psi(x,\alpha)$, $\Psi(0,\alpha') = \Psi(y,\alpha)$ and $\Psi(0,\beta) = \Psi(z,\alpha)$ because of Ψ satisfies condition (i) in Theorem 9. Then we obtain that $\Psi(a,\alpha') = \Psi(0,\alpha') = \Psi(0,\beta)$ and $a+\alpha' \geq \beta$. So, $a = 0$ and $\alpha' = \beta$ due to Ψ satisfies condition (iii) in Theorem 9. Hence, $x = y = z$. This ends the proof.

As an immediate consequence of Theorem 12 and Propositoin 3 we obtain a method for generating (quasi-)metric metric preserving functions from *pmg*-functions.

Corollary 6 *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function such that $\psi(0,0) = 0$. Then for each $\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a quasi-metric preserving function and, thus, a metric preserving function, where $\Psi_\alpha(a) = \Psi(a, \alpha)$ for each $a \in \mathbb{R}_+$.*

Notice that Examples 8 gives an instance of *pmg*-functions that allows us to induce, following Proposition 12, a partial metric aggregation function. In addition, Examples 7 and 9 yield instances of *pmg*-functions which are able to generate, according to Corollary 6, partial metric aggregation functions that are at the same time (quasi-)metric preserving functions.

4 Further work

According to the method introduced by Matthews in [14] (see Section 1), given a partial metric space (X, p), the function q_p is a quasi-metric on X where $q_p(x, y) = p(x, y) - p(x, x)$ for each $x, y \in X$. Clearly, the induced quasi-metric q_p is weighted with weight function w_{q_p} given by $w_{q_p}(x) = p(x, x)$. Of course, the preceding method has been generalized in Section 2 by means of *qmg*-functions and, in addition, a characterization of such functions has been given in the same section. However, such a general method does not produce in general weighted quasi-metrics. Indeed if we consider the mapping Ψ_2 introduced in Proposition 6 and the partial metric space $(\mathbb{R}_+, p_{\max})$, then it is not hard to verify that the induced quasi-metric space $(\mathbb{R}_+, q_{\Phi_2, p_{\max}})$ is not weighted. Therefore, it remains, as an open question, to characterize those *qmg*-functions that generate always weighted quasi-metrics.

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