[microreview]

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The metric tensor pullback

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Abstract

This is an article on differential geometry that connects metric tensor with its pullback [1].

keywords: metric pullback, differential geometry, quantum gravity, black hole

The most updated version of this white paper is available at

<https://osf.io/puhzw/download>
<https://zenodo.org/record/4840029>

Introduction

1. The motivation for this article is to try to fully understand the **pull-back** of a **metric tensor** for potential applications in **quantum gravity**, more specifically in the **black hole spacetime** physics [2].
2. In the next section, we present a roadmap with the main concepts and definitions connecting metric and its pullback.
3. Some of the prerequisites needed are listed in the Supplemental Material and in [3].

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Pullback metric tensor

4. We have included here a list summarizing the mathematical definitions in order to have a **global view** of what is a **metric tensor pullback**.
5. We are considering a Riemannian manifold of two dimensions, embedded in $\mathbb{R}^3$.
6. In this article, the word *smooth* means C^∞ , or infinitely differentiable.
7. Instead of the dot product, let us replace it (due to Riemann) by an arbitrary inner product on each tangent plane of an abstract surface M .
8. The metric tensor g is a function on all ordered pairs $(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors at the point $\mathbf{p} = (p_1, p_2, p_3)$ of $M \subset \mathbb{R}^3$.
9. In mathematical symbols, $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $g_p(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle_p$, where $\langle \mathbf{v}, \mathbf{w} \rangle_p$ is the inner product at $\mathbf{p} \in \mathbb{R}^3$ [1].
10. The metric tensor is a geometric structure with a collection of inner products.
11. g can also be considered a smooth, covariant, 2-tensor field [4].
12. Consider an abstract surface A (see [1]) and a geometric surface B .
13. A geometric surface is an abstract surface composed by an inner product on each of its tangent planes [1].
14. It can be summarized as “surface + metric tensor = geometric surface” [1].
15. Let $F : A \longrightarrow B$ be a smooth map between the surfaces A and B .
16. The map $F^* : \mathbb{T}^k(B) \longrightarrow \mathbb{T}^k(A)$ is the *pullback* by F , where the pullback by $\eta \in \mathbb{T}^k(B)$ is a smooth covariant k -tensor field $F^*\eta \in \mathbb{T}^k(A)$.

17. $F^*\eta : A \longrightarrow T^k(A)$, where each $p \in A$ is associated with $(F^*\eta)(p) \in T^k(T_pA)$.

18. Thus, for any vectors $v_1, v_2, \dots, v_r \in T_pA$, we define the covariant k -tensor

$$(F^*\eta)(p) \in T^k(T_pA)$$

by

$$(F^*\eta)(p)(v_1, v_2, \dots, v_r) = \eta(F(p))(dF(p)v_1, \dots, dF(p)v_r).$$

19. For $k = 2$, the map $F^* : \mathbb{T}^2(B) \longrightarrow \mathbb{T}^2(A)$ is the *pullback* by F .

20. Let g be a metric tensor on B , we consider $g \in \mathbb{T}^2(B)$.

21. Therefore, $F^*g \in \mathbb{T}^k(A)$ denotes the *pullback* of g and it is defined by

$$(F^*g)(p)(v_1, v_2) = g(F(p))(dF(p)v_1, dF(p)v_2).$$

22. That is,

$$\langle v_1, v_2 \rangle_A = \langle dF(p)v_1, dF(p)v_2 \rangle_B,$$

for any $v_1, v_2 \in T_pA$.

23. How can we visualize (19) graphically?

Final Remarks

24. We hope we provided all the necessary steps to understand the **pull-back** of a **metric tensor**.

25. If the “**hole in the black hole conjecture**” [2] is correct, perhaps the pullback map $F^*(g)$ could provide us the fundamental means to understand the formation of a black hole.

Open Invitation

Review, add content, and co-author this white paper [5, 6].

*Join the **Open Mathematics Collaboration**.*

Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [7, 8].

How to cite this paper?

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Agreement

All authors agree with [6].

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Supplemental Material

Definitions

Points, dot product, norm, distance

- 26. $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ are **points** of $\mathbb{R}^3$.
- 27. The **dot product** of $\mathbf{p}$ and $\mathbf{q}$ is the number $\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3$.
- 28. The **norm** of $\mathbf{p}$ is the number $\|\mathbf{p}\| = (\mathbf{p} \cdot \mathbf{p})^{1/2} = (p_1^2 + p_2^2 + p_3^2)^{1/2}$.
- 29. $\mathbf{p} - \mathbf{q} = (p_1 - q_1, p_2 - q_2, p_3 - q_3)$.
- 30. The **Euclidean distance** from $\mathbf{p}$ to $\mathbf{q}$ is the number $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = ((p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2)^{1/2}$.

Relation, function

- 31. Consider the sets A and B .
- 32. A set $R \subseteq A \times B$ is a relation from A to B .
- 33. $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$ is the Cartesian product.
- 34. A relation $f \subseteq A \times B$ is a **function** from A to B , written $f : A \rightarrow B$, if $\forall a \in A \exists! b \in B((a, b) \in f)$.
- 35. f is **one-to-one** if $\neg \exists a_1 \in A \exists a_2 \in A (f(a_1) = f(a_2) \wedge a_1 \neq a_2)$.
- 36. f is **onto** if $\forall b \in B \exists a \in A (f(a) = b)$.
- 37. f is a **bijection** if it is one-to-one and onto.
- 38. if $f : A \rightarrow B$ is bijective, then $f^{-1} : B \rightarrow A$ is the **inverse** of f .

Vector space, inner product

- 39. $\mathbb{R}^3$ is a **vector space** over the real numbers for it satisfies the axioms for a vector space [9].
- 40. An **inner product** is a function that assigns two vectors to a number.
- 41. The number $\langle \mathbf{v}, \mathbf{w} \rangle$ is an inner product between the pair of vectors $\mathbf{v}, \mathbf{w}$ in the vector space V .

Open subset, neighborhood, one-to-one mapping

- 42. $D \subseteq \mathbb{R}^2$ is an **open subset** of $\mathbb{R}^2$ iff $\forall d \in D, \exists c > 0, c \in \mathbb{R}$ such that the interval $(d - c, d + c) \subseteq D$.
- 43. c is called a *neighborhood* of d .
- 44. $\mathbf{x} : D \rightarrow \mathbb{R}^3$ is a **one-to-one mapping (function)** called a *coordinate patch*.
- 45. A *proper patch* is a *coordinate patch* that has the inverse function $\mathbf{x}^{-1}$.

Surface, differentiability

- 46. A **surface** in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ that is smooth and two-dimensional, i.e., that satisfies (44) and (45).
- 47. Let a surface be a set M , a collection of points.
- 48. Consider a one-to-one function $\mathbf{x} : D \rightarrow M$.
- 49. (48) is an abstract patch in M .
- 50. Let $\mathcal{P}$ be a collection of abstract patches.

51. The *covering axiom* states that the images of the patches in $\mathcal{P}$ cover M .
52. The *smooth overlap axiom* states that $\forall \mathbf{x}, \mathbf{y} \in \mathcal{P}$, $\mathbf{y}^{-1}\mathbf{x}$ and $\mathbf{x}^{-1}\mathbf{y}$ are Euclidean differentiable and defined on open sets of $\mathbb{R}^2$.
53. **Euclidean differentiable** means **differentiable** on Euclidean space.

Ordered triples, Euclidean space

54. The set of all **ordered triples** of real numbers $\mathbf{p} = (p_1, p_2, p_3)$, called points, is a Euclidean 3-space $\mathbb{R}^3$.
55. Let f be a real-valued function on $\mathbb{R}^3$.
56. **Real-valued function** means that its codomain is a **cartesian product** of $\mathbb{R}$.
57. f is differentiable if all partial derivatives of f , of all orders, exist and are continuous.
58. f is said to be infinitely differentiable, or smooth, or of class C^∞ .

Metric tensor

59. “A metric tensor is a type of function which takes as input a pair of tangent vectors v and w at a point of a surface (or higher dimensional differentiable manifold) and produces a real number scalar $g(v, w)$ in a way that generalizes many of the familiar properties of the dot product of vectors in Euclidean space.” [10]

Tensors and Tensor Fields

60. Consider a finite-dimensional real vector space E .

61. A covariant k -tensor on E is a real-valued multilinear function of k elements of E

$$T : \underbrace{E \times \cdots \times E}_{k \text{ times}} \longrightarrow \mathbb{R}.$$

62. $T^k(E)$ denotes the vector space of all covariant k -tensors on E .

63. Let A be a smooth abstract surface.

64. We define the bundle of covariant k -tensors on A by

$$T^k A = \prod_{p \in A} T^k(T_p A),$$

this bundle is called tensor bundle on A .

65. A smooth section of a tensor bundle is called a smooth tensor field on A [4].

66. $\mathbb{T}^k(A) = \{\text{smooth sections of } T^k A\}$ denotes the real vector space of smooth sections of the bundle $T^k A$.

67. If $\eta \in \mathbb{T}^k(A)$, then η is a map

$$\eta : A \longrightarrow T^k A,$$

called smooth covariant k -tensor field on A , where each $p \in A$ is associated with the covariant k -tensor $\eta(p) \in T^k(T_p A)$.

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