Structural Dynamics

Tutorial - Modelling in structural dynamics

Detailed solutions

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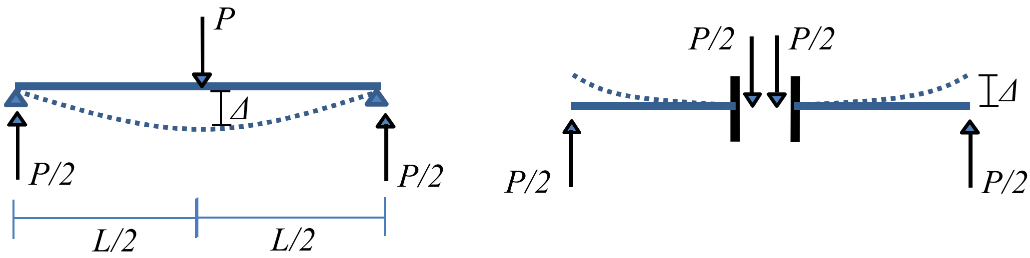
1. **Question 1**
We realise that the deformed shape of a simply supported beam loaded in the middle will have zero slope at the mid-point and that half-beam is analogous to a cantilever beam:



Remember, we need to ask whether the "force" taken by both cantilevers *has to* be the same or if it is rather the "displacement/deformation" that *has to* be equal. We realise that in this case the two cantilevers will *necessarily* have the same displacement from support to tip. Therefore, the simply supported beam may be viewed as a parallel system consisting of two identical cantilevers. Each cantilever has a stiffness of:

$$k_i = \frac{3EI}{(L/2)^3} = \frac{24EI}{L^3}$$

For a parallel system we add the stiffnesses as:

$$k_e = 2k_i = \frac{48EI}{L^3}$$

2. **Question 2**
The position (configuration) of all systems can be described by only one coordinate. Therefore all systems are **single degree-of-freedom structures**.

- a)

$$k = \frac{3EI}{h^3}$$
$$M\ddot{u} + ku = 0$$
$$\omega = \sqrt{\frac{3EI}{Mh^3}}$$

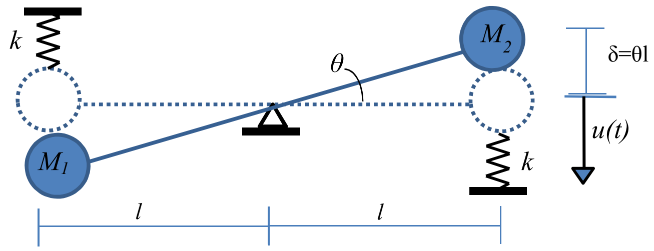
- b) The key here is to recognise that all columns will be displaced by the same amount and hence their stiffnesses act in parallel. Also the lateral stiffness of a double fixed column is $12EI/h^3$. Therefore:

$$k = 2\frac{12EI}{h^3} + \frac{3EI}{h^3} = \frac{27EI}{h^3}$$
$$M\ddot{u} + ku = 0$$
$$\omega = \sqrt{\frac{27EI}{Mh^3}}$$

- c) First, the stiffness of the structural systems is obtained through statics $F = k\Delta_{st}$. Therefore:

$$k = \frac{3EI}{2l^3}$$
$$M\ddot{u} + ku = 0$$
$$\omega = \sqrt{\frac{3EI}{2Ml^3}}$$

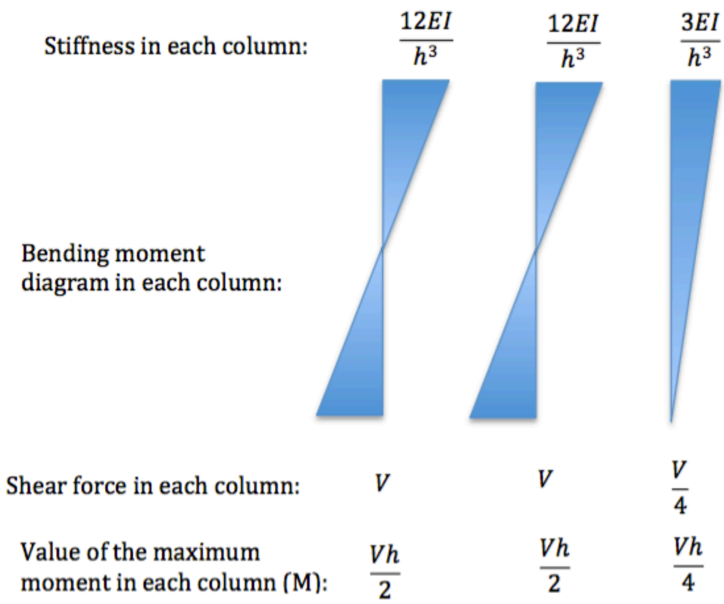
- d) A rotational degree of freedom θ is chosen as seen in the figure.



A rotation θ will cause the following inertial forces: $M_1\ddot{\theta}l$ acting at a distance l from the pinned support and $M_2\ddot{\theta}l$ acting at a distance l from the pinned support. Similarly, a rotation θ will cause spring forces of: $k\theta l$ acting at distances of l from the pinned support. Therefore:

$$(M_1 + M_2)l^2\ddot{\theta} + 2kl^2\theta = 0$$
$$\omega = \sqrt{\frac{2k}{M_1 + M_2}}$$

3. **Question 3**
It is important to realise that since the column are acting *in parallel* the horizontal force (V) will be distributed among them in direct proportion to their relative stiffness. Therefore we have:



The maximum moment will happen at the top of each column. The governing strength condition will be related to the maximum among the peak moments. The moments can be calculated by knowing the bending moment diagram and the value of shear (lateral force) that each column must resist. Both are depicted in the figure above.

The total shear resisted by the frame is:

$$V_T = V + V + V/4 = \frac{9V}{4}$$

Strength-based design will require to satisfy a maximum allowable stress (σ^*) at the most critical section (top of the left or mid column). Therefore, assuming a constant column section of width (d) and inertia (I):

$$\frac{Md}{2I} = \frac{Vh}{2}\frac{d}{2I} = \frac{Vhd}{4I} \leq \sigma^*$$

and

$$I_{strength} \geq \frac{Vhd}{4\sigma^*}$$

4. **Question 4**
A limit peak displacement of u^* is a typical condition of *motion-based design*. Given the characteristics of the frame of exercise 2(b) we should assume that the top displacement should be the same for all columns (system in parallel). Therefore by applying the elastic relationship between force-stiffness-displacement at the frame storey level we have:

$$\frac{9V}{4} = \frac{27EI}{h^3}u^*$$

and

$$\frac{Vh^3}{12EI} \leq u^*$$

$$I_{serviceability} \geq \frac{Vh^3}{12Eu^*}$$

5. **Question 5**
From the previous two questions we can calculate the *stiffness ratio,* r, as a way of comparing both approaches to design:

$$r = \frac{I_{serviceability}}{I_{strength}} = \frac{\sigma^*}{3E}\frac{h}{d}\frac{h}{u^*}$$

and the value of h/u^* for which motion (flexibility) criteria will control the design will be:

$$\frac{h}{u^*} = \frac{3E}{\sigma^*}\frac{d}{h}$$

6. **Question 6**
The two beams contribute to the overall stiffness of the system in a parallel manner. This implies that the system stiffness can be found from direct addition of the stiffnesses of the individual beams. The stiffness of the cantilever is:

$$k_C = \frac{3EI}{L_C^3}$$

The stiffness of the simple beam is:

$$k_B = \frac{48EI}{L_B^3}$$

The total stiffness is therefore:

$$k = k_B + k_C = \frac{3EI}{L_C^3} + \frac{48EI}{L_B^3} = 3EI\left[\frac{1}{L_C^3} + \frac{16}{L_B^3}\right]$$

An expression for the natural frequency can be found from:

$$\omega = \sqrt{\frac{3EI\left[\frac{1}{L_C^3} + \frac{16}{L_B^3}\right]}{M}}$$

The natural period of the system can be found from this natural frequency using the expression:

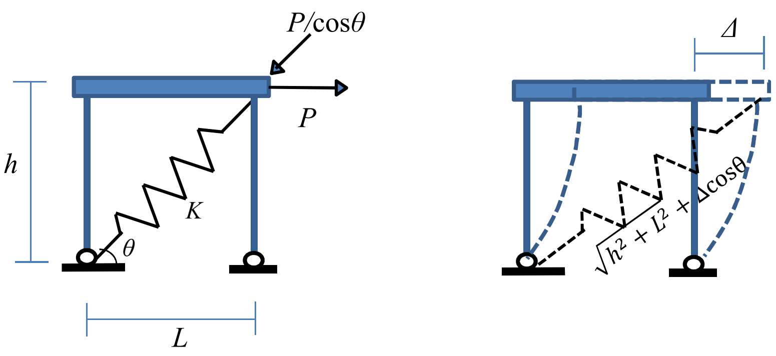
$$T = \frac{2\pi}{\omega_0}$$

7. **Question 7**

(a) The key is recognizing that this is a system with stiffnesses in series. Therefore:

$$k_e = \frac{\frac{3EIK}{l^3}}{K + \frac{3EI}{L^3}}$$

(b) This is a system in parallel which has contributions from the 2 pinned-fixed columns (each with a $k_c = 3EI/h^3$ **plus** the horizontal component of the stiffness of the diagonal spring member.
In order to calculate the horizontal component of the stiffness related to the spring we recognise that for any force P applied to the mass, the system displaces laterally by $u = \Delta$ and the the corresponding axial deformation in the spring is $\Delta \cos\theta$



If the axial deformation in the spring corresponding to a horizontal displacement of Δ is $\Delta\cos\theta$, then we can find the "horizontal" stiffness of the spring by:

$$K = \frac{\frac{P}{\cos\theta}}{\Delta\cos\theta}$$

$$k_{horizontal} = \frac{P}{\Delta} = K\cos^2\theta$$

where

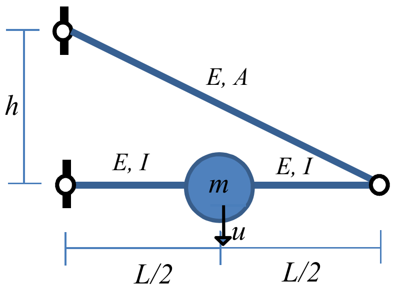
$$\cos\theta = \frac{L}{\sqrt{L^2 + h^2}}$$

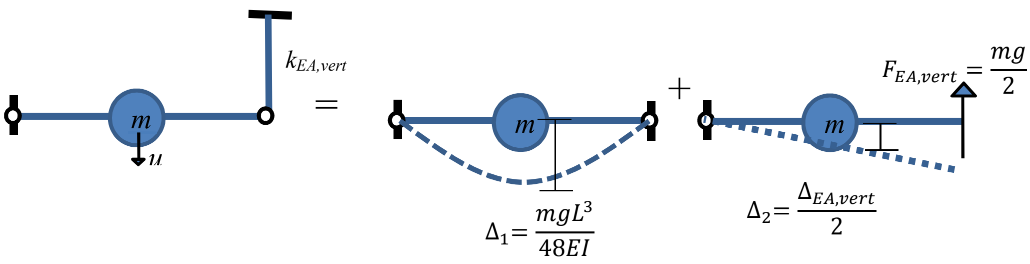
$$\cos^2\theta = \frac{L^2}{L^2 + h^2}$$

and therefore the system stiffness is:

$$k_e = \frac{6EI}{h^3} + \frac{KL^2}{L^2 + h^2}$$

(c) Similar to the previous case but now the stiffnesses are acting in series as shown in the figure.





The first deformation component corresponds to a simple supported beam due to the weight of the mass m applied at mid-span ($\Delta_1 = \dfrac{mgL^3}{48EI} \Rightarrow k_1 = \dfrac{48EI}{L^3}$).

The second deformation component comes from the vertical deformation of the diagonal bar. In this case, it is important to consider that the degree of freedom is the displacement of the mass **at the middle** of the horizontal beam. Therefore, in order to determine the stiffness corresponding to the second displacement component (Δ_2) we must consider the vertical component of the force acting on the diagonal ($F_{EA,vert}$) which deformation is **double** the deformation in the middle of the horizontal bar:

$$\frac{P}{2} = \frac{mg}{2} = k_{EA,vert} 2\Delta_2$$

this means that the equivalent stiffness k_2 corresponding to the deformations at the mid-span is:

$$k_2 = 4k_{EA,vert}$$

So the question now is how to calculate $k_{EA,vert}$ which is the vertical stiffness component of the diagonal bar. In order to do so we recall the solution of the previous question (Question 5(b)) where we treated the lateral/horizontal component of an inclined spring. In that case the horizontal stiffness was found as:

$$k_{horizontal} = K\cos^2\theta$$

Making the analogy with the inclined bar, now we have to calculate the vertical component of stiffness ($K_{EA,vert}$), which will be:

$$k_{EA,vert} = K\cos^2\theta$$

But in our case K is the axial stiffness of the inclined bar which depends on the elastic modulus, E, its area A and its length $\sqrt{L^2h^2}$ such that:

$$k_{EA,vert} = K\cos^2\theta = \frac{EA}{\sqrt{L^2+h^2}} \cdot \cos^2\theta$$

with $\cos\theta = \frac{h}{\sqrt{L^2+h^2}}$. Therefore:

$$k_{EA,vert} = K\cos^2\theta = \frac{EA}{(L^2+h^2)^{1/2}} \cdot \frac{h^2}{(L^2+h^2)} = \frac{EAh^2}{(L^2+h^2)^{3/2}}$$

and k_2 is (as found above)

$$K_2 = 4k_{EA,vert} = \frac{4EAh^2}{(L^2+h^2)^{3/2}}$$

Finally, combining k_1 and k_2 in series we have:

$$k_e = \frac{\left(\frac{4AEh^2}{(L^2+h^2)^{3/2}}\right)\frac{48EI}{L^3}}{\frac{4AEh^2}{(L^2+h^2)^{3/2}} + \frac{48EI}{L^3}}$$