CIVE96008: Dynamics of Structures **Spring 2020/2021**

Lecture 2: Free vibrations and impulse loads

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Learning outcomes

By the end of this session you should be able to:

- Understand the solution for the displacement response of a system under free vibration conditions
- Characterise the extent to which damping influences the free vibration response
- Understand how the logarithmic decrement may be used to infer damping characteristics
- Understand how the response to some arbitrary transient loading can be found from considering the free-vibration response of a SDOF

2.1 Free vibrations

A structure undergoes **free vibrations** when it is brought out of its static equilibrium and can then oscillate **without any external dynamic excitation**

Free vibrations can be damped or un-damped as expressed by the following equations:

$$m\ddot{u} + ku = 0$$

$$m\ddot{u} + c\dot{u} + ku = 0$$

In the following sections we will solve these equations using different formulations of the harmonic (e.g. amplitude and phase, exponential functions and trigonometric functions)

2.2 Undamped free vibrations

$$m\ddot{u} + ku = 0 \tag{2.1}$$

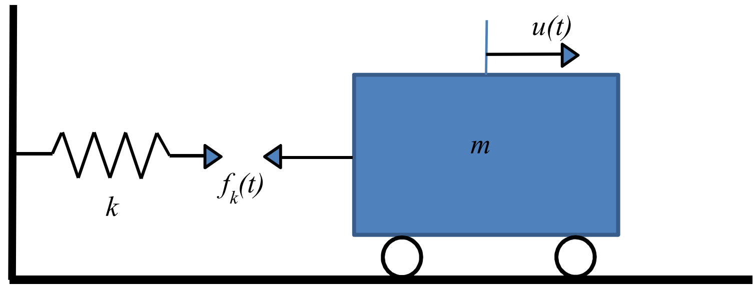


Figure 2.1: Un-damped single degree-of-freedom system

2.2.1 Formulation A: Amplitude and phase angle

Ansatz:

$$u(t) = Acos(\omega_n t - \phi) \tag{2.2}$$

$$\ddot{u}(t) = -A\omega_n^2 cos(\omega_n t - \phi) \tag{2.3}$$

By substituting Equations 2.2 and 2.3 into 2.1

$$A(-\omega_n^2 m + k)cos(\omega_n t - \phi) = 0 \tag{2.4}$$

$$-\omega_n^2 m + k = 0 \tag{2.5}$$

Natural circular frequency **of the structural system** [1]:

$$\omega_n = \sqrt{k/m} \tag{2.6}$$

Relationships:

Angular velocity [rad/s]:

$$\omega_n = \sqrt{k/m} \tag{2.7}$$

Number of revolutions per time [1/s],[Hz]:

$$f_n = \frac{\omega_n}{2\pi} \tag{2.8}$$

Time required per revolution [s]:

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \tag{2.9}$$

Transformation of the equation of motion:

$$\ddot{u}(t) + \omega_n^2 u(t) = 0 \tag{2.10}$$

[1] Please note that from now on we will be calling the natural circular frequency of the structure ω_n to differentiate it from the circular frequency of the excitation, ω

The unknowns A and ϕ in Equation 2.2 can be obtained by considering the **static equilibrium disturbed** by the initial displacement $\mathbf{u(0)} = u_0$ and initial velocity $\dot{\mathbf{u}}\mathbf{(0)} = v_0$. Therefore, solving the system of two equations (Eqs. 2.2 and 2.3) and two unknowns we have:

$$A = \sqrt{u_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \quad ; \quad tan\phi = \frac{v_0}{u_0\omega_n} \tag{2.11}$$

2.2.2 Formulation B: Trigonometric functions

Ansatz:

$$u(t) = A_1 cos(\omega_n t) + A_2 sin(\omega_n t) \tag{2.12}$$

$$\ddot{u}(t) = -A_1\omega_n^2 cos(\omega_n t) - A_2\omega_n^2 sin(\omega_n t) \tag{2.13}$$

By substituting Equations 2.12 and 2.13 into Equation 2.1:

$$A_1(-\omega_n^2 m + k)cos(\omega_n t) + A_2(-\omega_n^2 m + k)sin(\omega t) = 0 \tag{2.14}$$

$$-\omega_n^2 m + k = 0 \quad ; \quad \omega_n = \sqrt{k/m} \tag{2.15}$$

The unknowns A_1 and A_2 in Equation 2.12 can be obtained by considering the **static equilibrium disturbed** by the initial displacement $\mathbf{u(0)} = u_0$ and initial velocity $\dot{\mathbf{u}}\mathbf{(0)} = v_0$. Therefore, solving the system of two equations and two unknowns we have:

$$A_1 = u_0 \quad ; \quad A_2 = \frac{v_0}{\omega_n} \tag{2.16}$$

2.2.3 Formulation C: Exponential functions

Ansatz:

$$u(t) = e^{\lambda t} \tag{2.17}$$

$$\ddot{u}(t) = \lambda^2 e^{\lambda t} \tag{2.18}$$

By substituting Equations 2.17 and 2.18 into Equation 2.1:

$$\left(m\lambda^2 + k\right) e^{\lambda t} = 0 \tag{2.19}$$

$$m\lambda^2 + k = 0 \tag{2.20}$$

$$\lambda^2 = -\frac{k}{m} \tag{2.21}$$

$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n \tag{2.22}$$

The complete solution of the Ordinary Differential Equation is:

$$u(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \tag{2.23}$$

by means of Euler's formulas (see Lecture 01):

$$e^{i\alpha} = cos(\alpha) + isin(\alpha) \tag{2.24}$$

$$e^{-i\alpha} = cos(\alpha) - isin(\alpha) \tag{2.25}$$

Equation 2.23 can be transformed as follows:

$$u(t) = (C_1 + C_2)cos(\omega t) + i(C_1 - C_2)sin(\omega t) \tag{2.26}$$

Which corresponds to Equation 2.12!

$$u(t) = A_1 cos(\omega_n t) + A_2 sin(\omega_n t) \tag{2.27}$$

2.3 Damped free vibrations

In reality vibrations subside because damping exists. In the previous lecture we have seen that damping arises due to a number of mechanisms (e.g.friction between structural and non-structural elements, intrinsic viscous damping, inelastic behaviour, elastic hysteresis, micro-cracks, etc). We have also discussed that it is virtually impossible to model damping exactly because all these different mechanisms act in different ways and locations throughout the structure and are dependent on various different parameters. From a mathematical point of view, **viscous damping** is easy to treat (i.e. by means of a **damping constant: c** [N.s/m]). So what we normally do is lump all damping together into an **equivalent viscous damper.**

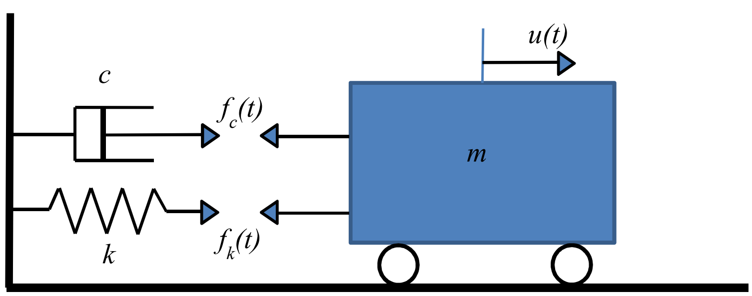


Figure 2.2: Damped single degree-of-freedom system

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{2.28}$$

2.3.1 Formulation C: Exponential functions

Ansatz:

$$u(t) = e^{\lambda t} \tag{2.29}$$

$$\dot{u}(t) = \lambda e^{\lambda t} \tag{2.30}$$

$$\ddot{u}(t) = \lambda^2 e^{\lambda t} \tag{2.31}$$

By substituting Equations 2.29 - 2.31 into Equation 2.28:

$$(\lambda^2 m + \lambda c + k)e^{\lambda t} = 0 \tag{2.32}$$

$$\lambda^2 m + \lambda c + k = 0 \tag{2.33}$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{2.34}$$

Critical Damping when $c^2 - 4km = 0$

$$c_{cr} = 2\sqrt{km} \tag{2.35}$$

Damping ratio:

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m} \tag{2.36}$$

Transformation of the equation of motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \tag{2.37}$$

$$\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{k}{m}u(t) = 0 \tag{2.38}$$

$$\ddot{u}(t) + 2\xi\omega_n\dot{u}(t) + \omega_n^2 u(t) = 0 \tag{2.39}$$

Types of vibration:

- Under-damped free vibrations: $\xi = \frac{c}{c_{cr}} < 1$
- Critically damped free vibrations: $\xi = \frac{c}{c_{cr}} = 1$
- Over-damped free vibrations: $\xi = \frac{c}{c_{cr}} > 1$

2.3.1.1 Under-damped free vibrations

By substituting $\xi = \dfrac{c}{2\omega_n m}$ and $\omega_n^2 = \dfrac{k}{m}$ in Equation 2.34:

$$\begin{aligned}\lambda &= -\xi\omega_n \pm \sqrt{\omega_n^2\xi^2 - \omega_n^2} \\ &= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \\ &= -\xi\omega_n \pm i\omega_n\sqrt{1 - \xi^2}\end{aligned} \tag{2.40}$$

Damped circular frequency:

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{2.41}$$

$$\lambda = -\xi\omega_n \pm i\omega_d \tag{2.42}$$

The complete solution of the ODE is:

$$u(t) = C_1 e^{(-\xi\omega_n + i\omega_d)t} + C_2 e^{(-\xi\omega_n - i\omega_d)t} \tag{2.43}$$

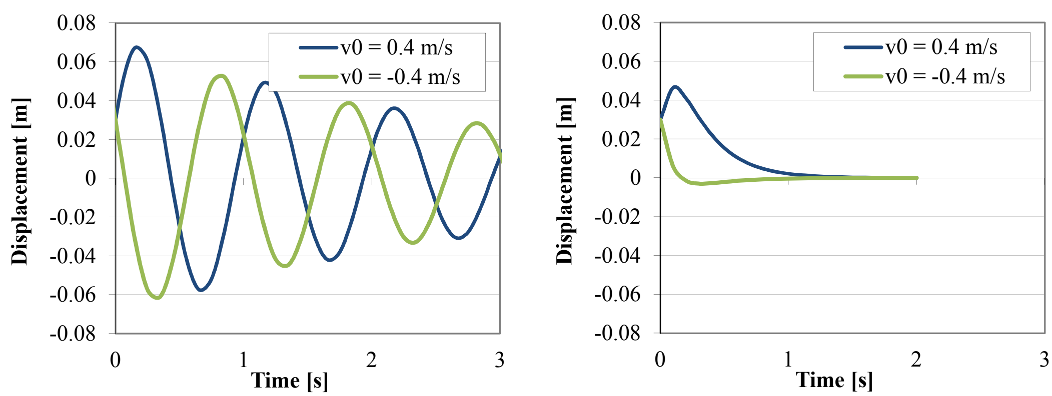
$$u(t) = e^{-\xi\omega_n t}(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}) \tag{2.44}$$

$$u(t) = e^{-\xi\omega_n t}(A_1 cos(\omega_d t) + A_2 sin(\omega_d t)) \tag{2.45}$$

The determination of the **unknowns** A_1 **and** A_2 is carried out as usual by means of the **initial conditions for displacement (**$u(0) = u_0$**) and velocity (**$\dot{u} = v_0$**)**:

$$A_1 = u_0 \tag{2.46}$$

$$A_2 = \frac{v_0 + \xi\omega_n u_0}{\omega_d} \tag{2.47}$$



(a) Under-damped free vibrations, $\xi = 0.05$

(b) Over-damped free vibrations, $\xi = 1.10$

Figure 2.3: Types of free vibration for a SDOF with $T = 1$ s and $u(0) = 0.03$m

2.3.2 Formulation A: Amplitude and phase angle

Equation 2.45 can be rewritten in terms of the motion's amplitude and phase angle:

$$u(t) = Ae^{-\xi\omega_n t} cos(\omega_d t - \phi) \tag{2.48}$$

with:

$$A = \sqrt{u_0^2 + \left(\frac{v_0 + \xi\omega_n u_0}{\omega_d}\right)^2} \tag{2.49}$$

$$tan\phi = \frac{v_0 + \xi\omega_n u_0}{\omega_d u_0} \tag{2.50}$$

- The motion is a sinusoidal vibration with circular frequency ω_d and decreasing amplitude $Ae^{-\xi\omega_n t}$.
- Note that the period of the damped vibration is longer (i.e. the vibration is slower) $T_d = \frac{T_n}{\sqrt{1-\xi^2}}$.
- The envelope of the vibration is represented by $Ae^{-\xi\omega_n t}$.

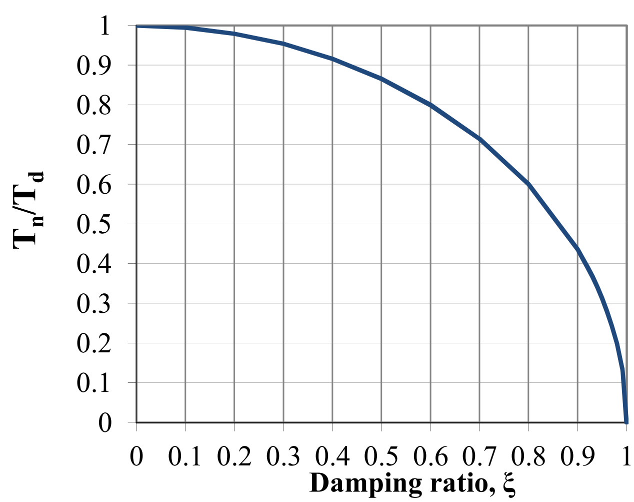


Figure 2.4: Relationship between period ratio(T_n/T_d) and damping ratio (ξ)

2.4 The logarithmic decrement

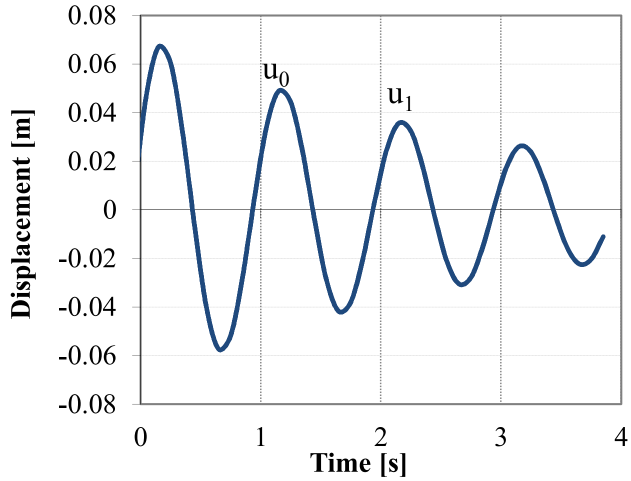


Figure 2.5: Free vibration. Amplitude decrement

Amplitude of two consecutive cycles

$$\frac{u_0}{u_1} = \frac{Ae^{-\xi\omega_n t} cos(\omega_d t - \phi)}{Ae^{-\xi\omega_n (t+T_d)} cos\left(\omega_d (t+T_d) - \phi\right)} \tag{2.51}$$

with

$$e^{-\xi\omega_n (t+T_d)} = e^{-\xi\omega_n t} e^{-\xi\omega_n T_d} \tag{2.52}$$

$$cos\left(\omega_d (t+T_d) - \phi\right) = cos(\omega_d t + \omega_d T_d - \phi) = cos(\omega_d t - \phi) \tag{2.53}$$

Replacing Equations 2.52 and 2.53 in Equation 2.51:

$$\frac{u_0}{u_1} = \frac{1}{e^{-\xi\omega_n T_d}} = e^{\xi\omega_n T_d} \tag{2.54}$$

Logarithmic decrement:

$$\delta = ln\left(\frac{u_0}{u_1}\right) = \xi\omega_n T_d = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \approx 2\pi\xi \tag{2.55}$$

If ξ is small, likewise, if ξ is small, the damping ratio becomes:

$$\xi \approx \frac{\delta}{2\pi} \tag{2.56}$$

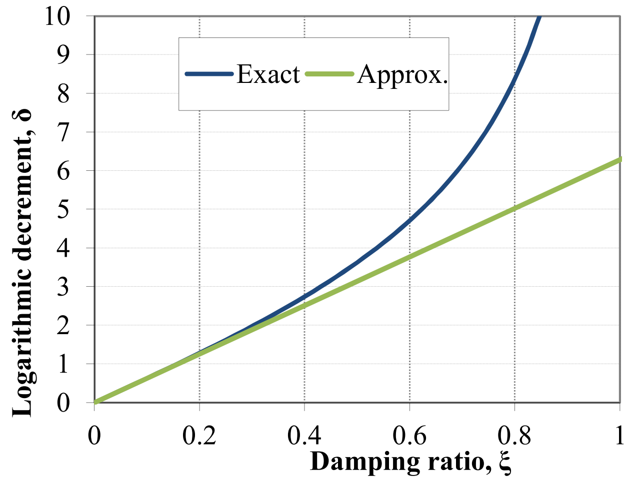


Figure 2.6: Relationship between exact damping equation and its approximation

2.5 Transient loads

2.5.1 Impulse

An impulsive load is a load which is applied during a short period of time (Figure 2.7). The corresponding **magnitude** of this type of load ($\hat{F}$) is defined as the product of the force and the time of its duration (mathematically defined as the integral of some forcing function over a period of time in [Ns]:

$$\hat{F} = \int_{t}^{t+\Delta t} F\,dt \tag{2.57}$$

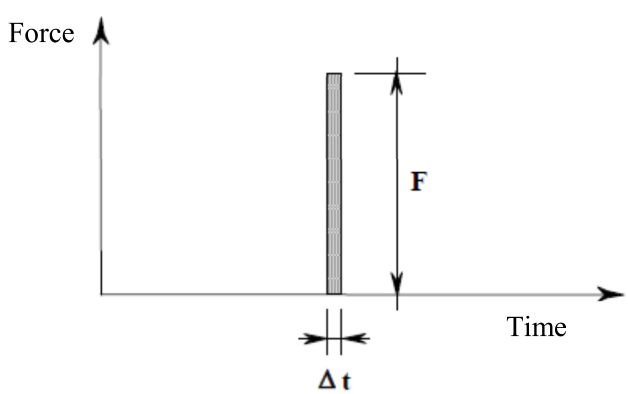


Figure 2.7: Impulse

It is evident from Figure 2.1 that the same impulse magnitude can be achieved by having a large-amplitude load over a short period of time or a smaller load over a longer period.

If we treat the force F as the net sum of forces acting on a body, then by means of Newton's Second Law we have:

$$F = ma = m\frac{dv}{dt} \tag{2.58}$$

by expressing the derivative of velocity as a differential and reordering:

$$F\Delta t = m\Delta v \tag{2.59}$$

Therefore, the impulse magnitude $\hat{F} = F\Delta t$ is equivalent to the mass multiplied by the change in velocity $m\Delta v$.

2.5.2 SDOF response to an impulse

If we apply an impulsive load, as described above, to the mass of an elastic SDOF system, we will be inducing a change in its velocity that is equivalent to (from Equation 2.3):

$$\Delta v = \frac{F\Delta t}{m} = \frac{\hat{F}}{m} \tag{2.60}$$

This is analogous to impose the following initial conditions:

$$u_0 = 0 \quad ; \quad v_0 = \dot{u_0} = \frac{\hat{F}}{m} \tag{2.61}$$

- Therefore, the response for any time t of an **un-damped SDOF** to an impulsive load of magnitude $\hat{F}$ will be (From Equations 3.12 and 3.16):

$$u(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) = \frac{\hat{F}}{m\omega_n}\sin(\omega_n t) \tag{2.62}$$

- Similarly, the response for any time t of an **damped SDOF** to an impulse $\hat{F}$ will be (From Equations 3.45 to 3.47):

$$u(t) = e^{-\xi\omega_n t}\left(\frac{v_0}{\omega_d}\sin(\omega_d t)\right) \tag{2.63}$$

$$u(t) = e^{-\xi\omega_n t}\left(\frac{\hat{F}}{m\omega_d}\sin(\omega_d t)\right) \tag{2.64}$$

$$u(t) = \frac{\hat{F}}{m\omega_n\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\left(\sin(\omega_n\sqrt{1-\xi^2}t)\right) \tag{2.65}$$

From an engineering point of view "large" force over a "short" time seem imperfect appreciations of a given phenomena. But we only need to express $\hat{F}$ in terms of the original derivatives of Newton's Second Law : $\hat{F} = Fd\tau$.

Therefore, the response of a SDOF to an Impulse applied during any time $d\tau$ (from Equations 2.62 and 2.65) can be expressed as:

un-damped SDOF:

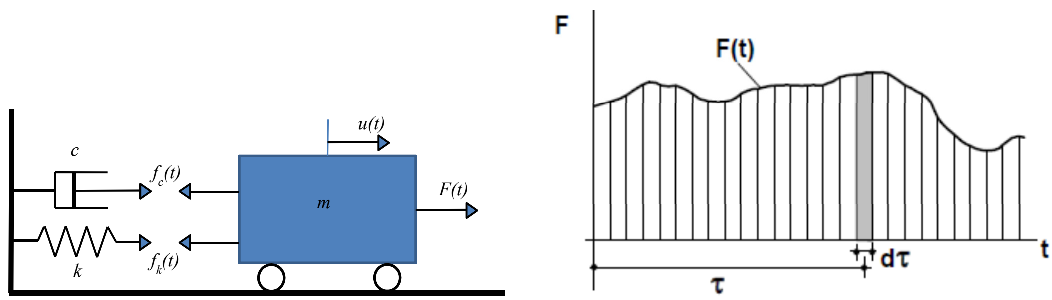
$$du(t) = \frac{F(\tau)}{m\omega_n} \sin(\omega_n(t-\tau))d\tau \tag{2.66}$$

damped SDOF

$$du(t) = \frac{F(\tau)}{m\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n(t-\tau)} \left(\sin(\sqrt{1-\xi^2}\omega_n(t-\tau)\right) d\tau \tag{2.67}$$

2.5.3 Response of a SDOF to an arbitrary loading

Given a load $F(t)$ of arbitrary form, it is possible to subdivide the force $F(t)$ into a series of impulses acting on the structure at τ with a duration $d\tau$ as presented in Figure 2.8



(a) Damped SDOF (b) Arbitrary load

Figure 2.8: Un-damped forced harmonic vibrations for SDOF systems

Integrating the effects of each one of those differential impulses varying τ, we obtain:

un-damped SDOF:

$$u(t) = \int_0^t du(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin\left(\omega_n(t-\tau)\right) d\tau \tag{2.68}$$

damped SDOF:

$$u(t) = \int_0^t du(t) = \frac{1}{m\omega_n\sqrt{1-\xi^2}} \int_0^t F(\tau) e^{-\xi\omega_n(t-\tau)} \sin\left(\sqrt{1-\xi^2}\omega_n(t-\tau)\right) d\tau \tag{2.69}$$

These integrals are know as the **Convolution Integrals** or **Duhamel's Integrals**. In most cases it is much more effective to numerically evaluate them.

The name *Convolution Integrals* expresses the fact that the total response of the system at time t is the sum of the responses to all impulses such that

$$u(t) = \int_0^t p(\tau)h(t-\tau)d\tau \tag{2.70}$$

where $h(t-\tau)$ is the **unit impulse-response function.**

Example:

Consider the case of a constant force of magnitude F_0 applied suddenly to an un-damped elastic oscillator at time $t = 0$. If the oscillator was at rest at the time of the application of the load. What is the response?