Lecture 1: Modelling in structural dynamics

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Learning outcomes

By the end of this session you should be able to:

- Derive the equation of motion for simple systems
- Understand the concept of degrees-of-freedom and identify the conditions for the determination of the appropriate number of degrees-of-freedom required in common problems
- Determine the equivalent stiffness of conventional structural systems
- Estimate the natural frequency of systems characterized by a single degree-of-freedom and their corresponding frequency response functions

1.1 Introduction

The aim of engineering is to provide satisfactory solutions to some of the most pressing human problems balancing economy and safety. In order to arrive to an efficient and effective answer we, as engineers, often introduce a number of assumptions and idealizations aimed at making the problem mathematically manageable. To this end, we make use of *mathematical models* which provide the link between the *real world* and the *mathematically feasible* expressions we conceive as analogies of that real world [CRA43,HES66].

As we have seen in the previous lecture, a problem in structural dynamics differs from a static problem firstly in terms of the nature of the loads. This implies that we will search not for a unique solution but for a (temporal) succession of solutions corresponding to all points of interest. Another important difference between dynamic and static problems is that when a structure is subjected to dynamic actions, its resulting displacements are now also associated with the inertial forces generated within the structure itself. That is, **the internal actions in a dynamically-acted structure must equilibrate not only the externally acting load but also the inertial forces resulting from the accelerations caused in the structure.** This is in fact the principal characteristic of a dynamic problem. So much so that in some cases, when the accelerations caused by the dynamic load are slow enough to be neglected, the problem can be treated as a static one, even though the load and response may be varying in time.

In this part of the course we will deal only with Single-Degree-of-Freedom (SDOF) structural systems such as that presented in Figure 1.1.

We can define the **number of Degrees of Freedom** of a system as the number of independent parameters necessary to specify its configuration at any time.
More precisely, in structural engineering (and in attention to the direct relationship between deformations and structural damage) we usually refer to **degrees of freedom as the number of coordinates** that is sufficient to specify the position of the mass(es) of the structure.

Consequently, a *Single Degree-of-freedom System (SDOF)* will be any structure modelled as a system with a single displacement/deformation coordinate.

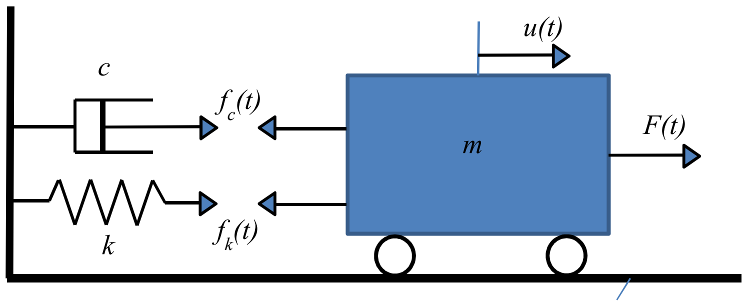


Figure 1.1: Single Degree-of-Freedom System

1.2 Formulation of the equation of motion

Equation of Motion: A mathematical expression defining the dynamic displacement of a structural system *which solution provides a complete description of the system's response in time.*

1.2.1 Newton's second law

The net sum of the forces acting upon a body corresponds to the rate of change of momentum in that body.

$$F = \frac{dI}{dt} = \frac{d}{dt}(m\dot{u}) = m\ddot{u} \tag{1.1}$$

$$-f_k(t) - f_c(t) + F(t) = m\ddot{u}(t) \tag{1.2}$$

introducing the spring force $f_k(t)$ and the damping force $f_c(t)$:

$$f_k(t) = k \cdot u(t) \tag{1.3}$$

$$f_c(t) = c \cdot \dot{u}(t) \tag{1.4}$$

Equation 1.2 can be written as:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F(t) \tag{1.5}$$

and we can define the circular frequency of the structural system as:

$$\omega = \sqrt{\frac{k}{m}} \tag{1.6}$$

we will deal more thoroughly with the circular frequency of structures in the next lecture.

1.2.2 D'Alembert principle

The principle is based on the idea of a fictitious inertia force that is equal to the product of the mass times its acceleration, and acts in the opposite direction as the acceleration. The mass is at all times in equilibrium under the resultant force F and the inertia force $T = -m\ddot{u}$, such that:

$$F + T = 0 \tag{1.7}$$

Example:

Consider the system depicted in Figure 1.2. Formulate its equation of motion.

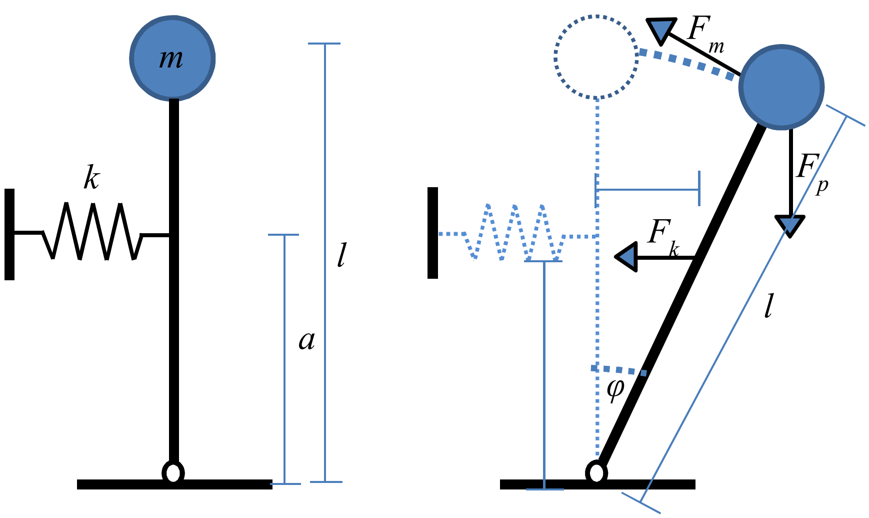


Figure 1.2: Inverted pendulum

Spring force:

$$F_k = \tag{1.8}$$

Inertial force:

$$F_m = \tag{1.9}$$

External force:

$$F_p = \tag{1.10}$$

Equilibrium:

$$\tag{1.11}$$

$$\tag{1.12}$$

$$\tag{1.13}$$

Circular frequency:

$$\omega = \sqrt{\frac{K_E}{M_E}} = \tag{1.14}$$

What if the system of the Example above (Figure 1.2) was rotated by -90 degrees? what would be the corresponding equation of motion?

1.3 Modelling Damping

We can categorize damping according to their source:

- External: external contact non-structural elements, energy radiation through the ground, etc
- Internal:

  Contact: Relative movement between the parts of the structures (e.g. bearing, joints, etc)

  Material: viscous, friction, yielding, etc

These various mechanisms are often load-dependant and distributed in different ways throughout the system. More importantly, most of them are NOT velocity-dependent. Therefore, our decision to characterize the damping force F_c in Equation 1.5 as a velocity-dependent force based on a viscous damping approximation with a constant damping coefficient c is a **modelling decision** done for practical purposes alone!

Let's assume that no external force is present in the structure $F(t) = 0$. Therefore, Equation 1.5 becomes:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \tag{1.15}$$

dividing by mass:

$$\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{k}{m}u(t) = 0 \tag{1.16}$$

and defining the damping ratio as:

$$\xi = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega m} \tag{1.17}$$

we can express 1.16 as:

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = 0 \tag{1.18}$$

The damping coefficient (c) is very difficult to estimate but one can make reasonable inferences about the damping ratio (ξ) without evaluating it strictly according to its definition. Once we approach the modelling of damping in this simplified manner we can also observe that structures of particular materials and typical geometries, tend to have values of damping ratios that are somewhat predictable. Table 1.1 and Figures 1.4 and **??** summarize a database of period and damping values for typical European buildings.

Material	Damping ratio $\left(\xi = \frac{c}{2m\omega}\right)$
Reinforced concrete (uncracked)	0.007-0.010
Reinforced concrete (cracked)	0.010-0.040
Reinforced concrete (pre-stressed)	0.004-0.007
Composite components	0.002-0.003
Steel	0.001-0.002

Table 1.1: Typical values of damping in structures. From [BAC95]

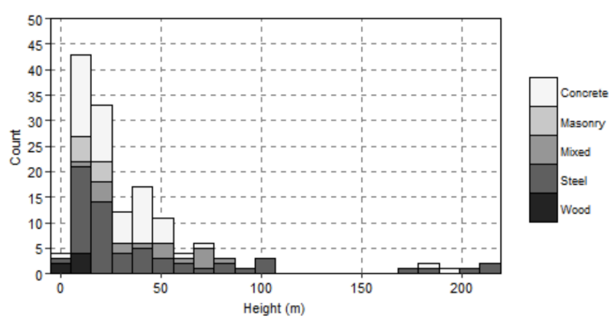


Figure 1.3: Histogram of building heights used by Cruz and Miranda grouped by material. From [CRUZ18]

Sometimes we use **supplemental damping** for enhancing the dynamic behaviour of structures against extreme loading. A typical viscous damper configuration is presented in Figures 1.5 while Figure 1.6 presents a summary of an experimental evaluation on the viscous damper.

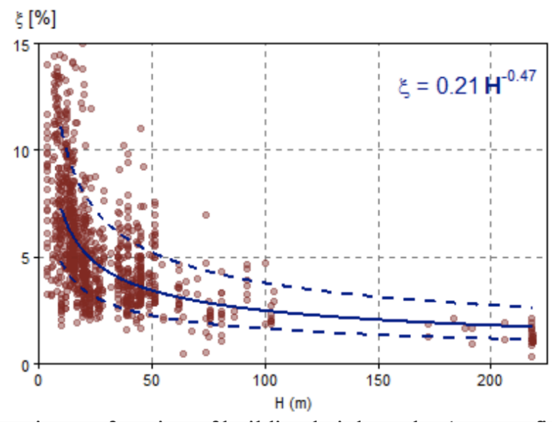


Figure 1.4: Damping ratio as a function of building height and $\pm 1\sigma$ confidence intervals. From [CRUZ18]

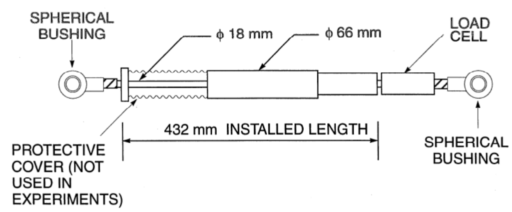


Figure 1.5: Scheme of a viscous damper. From [KC99]

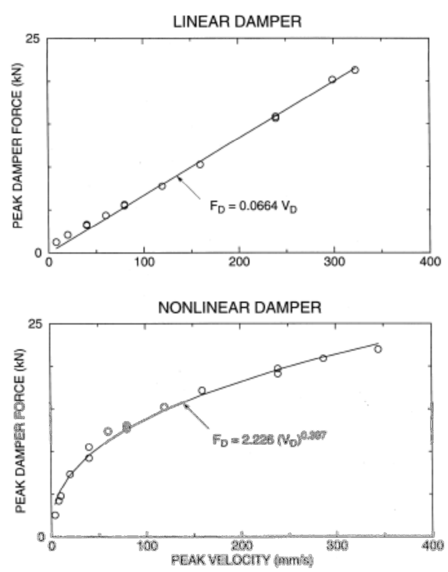
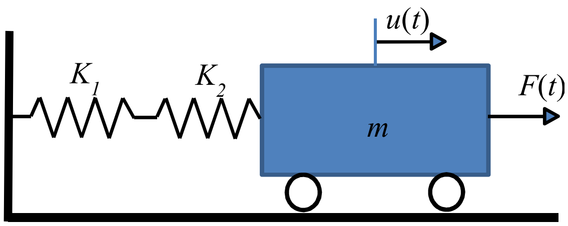


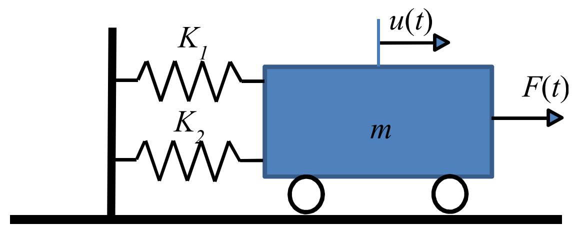
Figure 1.6: Relation between peak damping force and peak velocity. From [KC99]

1.4 Modelling - Stiffness

When considering the equation of motion, we will always require an estimate of the stiffness of the system. For most common cases, we will be able to obtain it directly, but in other cases we will need to apply any of the generic methods that have been studied in previous years (e.g. virtual work, flexibility, etc). The key to deriving the stiffness of a system is to appreciate the difference between stiffness contributions acting in series or in parallel.



(a) System with springs in series



(b) System with springs in parallel

Figure 1.7: Equivalent stiffness.

What would be the equivalent lateral lateral stiffness of these structures/sub-assemblages?

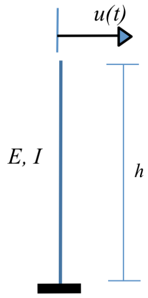


Figure 1.8: cantilever

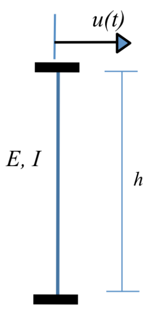


Figure 1.9: Double-fixed column

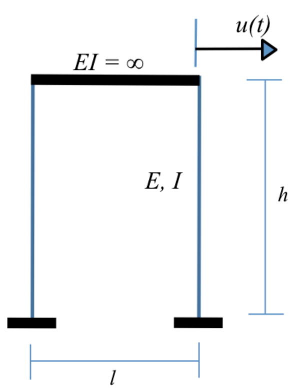


Figure 1.10: Frame

What is the lateral stiffness of these structures?

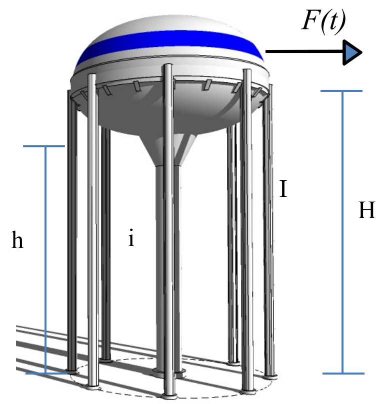


Figure 1.11: Elevated water tank

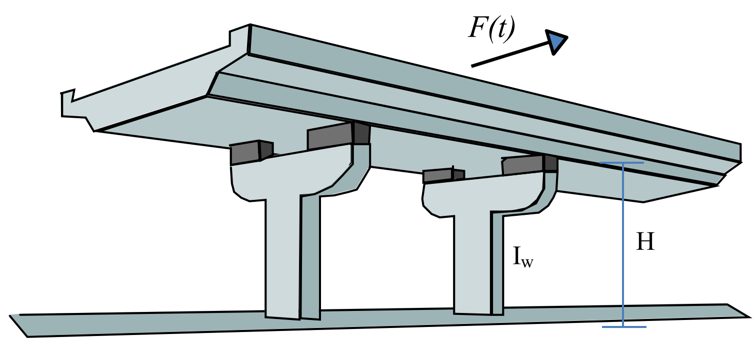


Figure 1.12: Bridges and urban viaducts

Slide 1

Appendix A - Energy formulations

Let's define:

- *Kinetic energy, T* which is the work that an external force provides to a mass.
- *Deformation energy, U* which is defined as the work that an external force needs to provide in order to generate a deformation
- *Potential energy of external forces, V* which is defined with respect to the equilibrium position

For conservative systems, $dE/dt = 0$:

$$E = T + U + V = To + Uo + Vo = constant \tag{1.19}$$

In the case of the inverted pendulum considered in the previous Example (Figure 1.2), we have the following: Energy in the spring

$$U = E_{deformation,k} = \frac{1}{2}k\,(a \cdot \sin(\phi))^2 = \frac{1}{2} \cdot k \cdot (a \cdot \phi)^2 \tag{1.20}$$

Energies in the mass

$$T = E_{kinetic,m} = \frac{1}{2} \cdot m \cdot \left(\dot{\phi} \cdot l\right)^2 \tag{1.21}$$

$$V = E_{potential,m} = -(m \cdot g) \cdot (1 - \cos(\phi)) \cdot l \tag{1.22}$$

Since $\cos(\phi) = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + ...$ we assume, for a small angle, $\cos(\phi) = 1 - \frac{\phi^2}{2}$ therefore $\frac{\phi^2}{2} = 1 - \cos(\phi)$ and the potential energy of the mass becomes:

$$E_{potential,m} = -\left(m \cdot g \cdot 0.5 \cdot l \cdot \phi^2\right) \tag{1.23}$$

and applying energy conservation:

$$E_{total} = constant = E_{deformation,k} + E_{kinetic,m} + E_{potential,m} \tag{1.24}$$

$$E_{total} = \frac{1}{2}\left(m \cdot l^2\right)\dot{\phi}^2 + \frac{1}{2}\left(k \cdot a^2 - m \cdot g \cdot l\right) \cdot \phi^2 \tag{1.25}$$

Derivative of energy with respect to time:

$$\frac{E_{total}}{dt} = \left(m \cdot l^2\right) \cdot \dot{\phi} \cdot \ddot{\phi} + \left(k \cdot a^2 - m \cdot g \cdot l\right) \cdot \phi \cdot \dot{\phi} = 0 \tag{1.26}$$

and after cancelling the velocity term $\dot{\phi}$ we have

$$m \cdot l^2 \cdot \ddot{\phi} + (a^2 \cdot k - m \cdot g \cdot l) \cdot \phi = 0 \tag{1.27}$$

The whole process has been formalized in what is called the Euler-Lagrange formulation. To this end, the **Lagrangian, L** is defined as:

$$L = T - V \tag{1.28}$$

An the **Euler-Lagrange equation** is:

$$\frac{d}{dt}\frac{\partial T(u,\dot{u})}{\partial \dot{u}} - \frac{\partial T(u,\dot{u})}{\partial u} + \frac{\partial V(u)}{\partial u} - p(u,\dot{u}) = 0 \tag{1.29}$$

where $p(u,\dot{u})$ represents the external and dissipative forces.

Lagrange's equations are applicable to all systems which kinetic energy can be expressed in terms of generalized coordinates and their first time derivatives, and which potential energy can be expressed in terms of the generalized coordinates alone. This includes linear and nonlinear systems. This makes Lagrange's formulations more powerful than Newtonian ones over a vast number of cases.

References

[CRA43] K. CRAICK (1943) *The Nature of Explanation* Cambridge University Press, UK.

[HES66] M. HESSE (1966) *Models and Analogies in Science* Notre Dame University Press, US.

[BAC95] H. BACHMANN, F. SEIBLE, W.J. AMMANN, F. DEISCHL, J. EISENMANN, I. FLOEGL, G.H. HIRSCH, G.K. KLEIN, G.J. LANDE, O. MAHRENHOLTZ, H.G. NATKE, H. NUSSBAUMER,*Vibration Problems in Structures*, BirkhÃď user, 1995.

[PSC96] M.J.N. PRIESTLEY, F. SEIBLE and G.M. CALVI, *Seismic Design and Retrofit of Bridges*, John Wiley & Sons, 1996.

[SOL05] G. SOLARI *Mechanical Properties of Buildings - Experimental Database*, Italy, 2005.

[CRUZ18] C. CRUZ, E. MIRANDA, *First mode damping ratios inferred from the seismic response of buildings*, Eleventh U.S. National Conference on Earthquake Engineering, 2018

[KC99] A. KASALANATI and C. CONSTANTINOU *Experimental Study of Bridge Elastomeric and Other Isolation and Energy Dissipation Systems with Emphasis on Uplift Prevention and High Velocity Near-Source Seismic Excitation*, State University of New York, Department of Civil, Structural and Environmental Engineering, Technical Report MCEER-99-0004, 1999.