ON LONG TIME ASYMPTOTICS OF THE VLASOV–POISSON–BOLTZMANN EQUATION

Laurent Desvillettes
ECOLE NORMALE SUPERIEURE
45, Rue d'Ulm
75230 Paris Cédex 05

Jean Dolbeault
UNIVERSITE PARIS IX–DAUPHINE
CEREMADE
Place du Maréchal de Lattre de Tassigny
75016 Paris

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Abstract

We deal with the long time asymptotics of the Vlasov–Poisson–Boltzmann equation. We prove existence and uniqueness for the equation giving the electric potential at the limit.

1 Introduction

Plasmas (gases of charged particles) can be described using kinetic equations. To simplify the analysis, we shall only consider jelliums, which are plasmas containing only one species of particles (of mass m and charge e). Let us denote $f(t,x,v)$ the density of particles, which at time t and point x, move with velocity v. We assume that the particles remain in a bounded domain Ω and, since we are not interested in relativistic phenomena, the velocity v belongs to $I\!R^3$.

In the mean field approximation, when particles interact only through electromagnetic forces, the density f solves the Vlasov equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \tag{1}$$

where F is proportional to the Lorentz force created by the mean electromagnetic field.

Next, we shall assume that magnetic forces are negligeable, and therefore

$$F = -\frac{e}{m} \nabla_x \phi, \tag{2}$$

where the electric potential ϕ obeys Poisson's law

$$-\triangle_x \phi = \frac{e}{\epsilon_0} \int_{v \in I\!R^3} f \, dv, \tag{3}$$

and where ϵ_0 is the electric permeability of the vacuum. The system (1) – (3) is called the Vlasov–Poisson system.

Now, to give a more realistic description of the plasma, we take the elastic collisions between particles into account. Denoting by (v, v_1) and (v', v_1') the velocities of the particles respectively before and after a collision, the conservation of momentum and energy gives

$$\begin{aligned} v' + v_1' = & \quad v + v_1, && (4)\\ |v'|^2 + |v_1'|^2 = & \quad |v|^2 + |v_1|^2, && (5)\\ & && (6) \end{aligned}$$

which can be parametrized under the form

$$\begin{aligned} v' = & \quad v - ((v - v_1) \cdot \omega)\, \omega, && (7)\\ v_1' = & \quad v_1 + ((v - v_1) \cdot \omega)\, \omega, && (8) \end{aligned}$$

where ω is a unit vector of $I\!R^3$.

Assuming that there is no correlation between particles before and after a collision, we now have to solve the Vlasov–Poisson–Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{e}{m} \nabla_x \phi \cdot \nabla_v f = Q(f,f), \tag{9}$$

where Q is the Boltzmann quadratic collision kernel (acting only on velocities),

$$Q(f,f)(v) = \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \Big\{ f(v')f(v_1') - f(v)f(v_1) \Big\} B(v - v_1, \omega)\, d\omega dv_1, \tag{10}$$

and B is a collision cross section (Cf. [Ce], [Ch, Co] and [Tr, Mu]).

We prove in this work the convergence towards equilibrium for a regular solution of the Vlasov–Poisson–Boltzmann equation in a bounded domain with appropriate boundary conditions. Note that such a result is already known for the Boltzmann equation, even for renormalized solutions (Cf. [A] and [De 1]). The reader will find a survey on this subject in [De 2].

Moreover, we give a description of the Maxwellian steady states for such a plasma, when the mass and energy of the particles are fixed. Note that D. Gogny and P-L. Lions have already given such a description when the mass and temperature are fixed (Cf. [G, L]). For the study of the stationnary solutions of these equations, we refer also to [Do].

In section 2, we prove that in the long time asymptotics, the density of particles satisfying the Vlasov–Poisson–Boltzmann equation converges to a Maxwellian with zero bulk velocity, and uniform temperature. Moreover, we give an equation for the electric potential in this limit.

A formula for the temperature and density of the Maxwellian is given in section 3, using conservation of mass and energy.

Existence and uniqueness for the equation satisfied by the electric potential is proved in section 4.

Finally, we give in section 5 some additional results. Namely, we extend our methods to several models of collision kernels, and explain how to modify our analysis to take into account magnetic forces. We give also some indications about a model with several species of particles.

2 Long time asymptotic behavior of the Vlasov–Poisson–Boltzmann equation

In this section, we are interested in the long time asymptotic behavior of regular solutions of the Vlasov–Poisson–Boltzmann equation and in the form of their stationary limit.

Let us assume that the cross section B satisfies the

Assumption 1: *The cross section B is strictly positive a.e., depends only on $|v-v_1|$ and $|(v-v_1)\cdot\omega|$, and satisfies for some $K>0$ the following bound,*

$$B(v-v_1,\omega) \le K(1+|v|+|v_1|). \tag{11}$$

Note that the cross sections B coming out of hard potentials with the angular cut–off of Grad (Cf. [Gr]) satisfy assumption 1.

We shall denote $n(x)$ the outward normal to $\partial\Omega$ at point x. We assume the specular reflexion of each particle against the wall:

For every t in $[0,+\infty[$, x in $\partial\Omega$, v in $I\!R^3$, such that $v\cdot n(x)\le 0$,

$$f(t,x,v) = f(t,x,Rv), \tag{12}$$

where

$$Rv = v - 2(v\cdot n(x))\,n(x). \tag{13}$$

Moreover, we suppose that the boundary $\partial\Omega$ of the domain is a perfect conductor and therefore:

For every t in $[0,+\infty[$, x in $\partial\Omega$,

$$\phi(t,x) = 0. \tag{14}$$

Finally, we prescribe initial data:

$$f(0,x,v) = f_0(x,v), \tag{15}$$

and

$$\phi(0,x) = \phi_0(x), \tag{16}$$

satisfying the following assumption:

Assumption 2: *The initial datum* f_0 *is nonnegative. It is compatible with the initial datum* ϕ_0,

$$-\triangle_x \phi_0 = \int_{v \in I\!R^3} f_0 \, dv, \tag{17}$$

and the total mass, energy and entropy are finite,

$$\int_{x \in \Omega} \int_{v \in I\!R^3} f_0 \left(1 + |v|^2 + |\log f_0| \right) dv dx + \frac{\epsilon_0}{m} \int_{x \in \Omega} |\nabla_x \phi_0|^2 dx < +\infty. \tag{18}$$

Note that R.J. DiPerna and P-L. Lions proved in [DP, L 3] that when the position x belongs to $I\!R^3$ and under suitable assumptions on f_0 and B, there exists a global nonnegative renormalized solution to the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f). \tag{19}$$

This result has been extended by K. Hamdache in [Ha] to the case when x lies in a subset of $I\!R^3$ and f satisfies appropriate boundary conditions (like specular reflexion).

Moreover, R.J. DiPerna and P-L. Lions have also proved in [DP, L 2] a theorem of existence for the Vlasov–Poisson equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{e}{m} \nabla_x \phi \cdot \nabla_v f = 0, \tag{20}$$

$$-\triangle_x \phi = \frac{e}{\epsilon_0} \int_{v \in I\!R^3} f dv, \tag{21}$$

when x belongs to $I\!R^3$ and for a large class of initial data. The existence of strong solutions for this system is also investigated in [Pf] and [Sc].

It is possible to mix the results on the Boltzmann equation and on the Vlasov–Poisson equation when x belongs to $I\!R^3$, in order to obtain solutions to the Vlasov–Poisson–Boltzmann problem (3), (9) (Cf. [L]). However, there is no result for the time being on this equation when boundary conditions like (12) and (14) are required for the solution.

Therefore, we shall from now on consider solutions of equations (3) – (10) and (12) – (14) whose existence we do not know. More precisely, we shall consider such solutions (f, ϕ) satisfying the following assumption:

Assumption 3: *The density* f *is nonnegative, bounded and uniformly continuous on* $[0, +\infty[\times \overline{\Omega} \times I\!R^3$, *it also belongs to* $L^\infty([0,T] \times \overline{\Omega}; L^1(I\!R^3))$.

Moreover, the potential ϕ belongs to $C^2([0,+\infty[\times\overline{\Omega})$ and its derivatives up to second order are bounded and uniformly continuous.

The main result of this section is the following:

Theorem 1: *Let Ω be a regular (of class C^2) bounded and simply connected open set of $I\!R^3$ such that $\partial\Omega$ is not a surface of revolution, and assume that (f,ϕ) is a solution of equations (3) – (10) and (12) – (14) under assumptions 1, 2, 3. Let t_n be a sequence of real numbers going to infinity, and T be a strictly positive real number. We define $f^n(t,x,v) = f(t+t_n,x,v)$ and $\phi^n(t,x,v) = \phi(t+t_n,x,v)$.*

Then, there exist a subsequence t_{n_k}, a function $\psi(x)$ in $C^2(\overline{\Omega})$, $\rho \geq 0$ and $\theta > 0$ such that $f^{n_k}(t,x,v)$ converges uniformly on every compact sets of $[0,T]\times\overline{\Omega}\times I\!R^3$ to

$$g(t,x,v) = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} \exp\Big\{-\frac{e\psi(x)}{m\theta} - \frac{v^2}{2\theta}\Big\}, \tag{22}$$

and $\phi^{n_k}(t,x)$ converges in $C^2([0,T]\times\overline{\Omega})$ to $\psi(x)$. Moreover, g,ψ satisfy the boundary conditions (12) – (14), and the potential ψ is such that

$$-\triangle_x\psi = \rho\frac{e}{\epsilon_0}\exp\Big\{-\frac{e\psi(x)}{m\theta}\Big\}. \tag{23}$$

Proof of theorem 1: Assumption 3 ensures that the sequence f^n is equicontinuous. Moreover, it ensures that for a given (t,x,v) in $[0,T]\times\overline{\Omega}\times I\!R^3$, the sequence $f^n(t,x,v)$ is bounded. According to Ascoli's theorem, it is possible to extract from f^n a subsequence f^{n_k} such that f^{n_k} converges uniformly on every compact sets of $[0,T]\times\overline{\Omega}\times I\!R^3$ to a bounded and continuous function $g(t,x,v)$.

Moreover, assumption 3 also ensures that the same manipulation holds for ϕ^n and its derivatives up to second order. Therefore, it is possible to extract from ϕ^n a subsequence ϕ^{n_k} such that ϕ^{n_k} converges in $C^2([0,T]\times\overline{\Omega})$ to a function $\psi(t,x)$.

The proof of identities (22) and (23) is divided in four steps.

Note that we shall need in the sequel the conservation of mass

$$\int_{x\in\Omega}\int_{v\in I\!R^3} f(t,x,v)\,dvdx = \int_{x\in\Omega}\int_{v\in I\!R^3} f_0(x,v)\,dvdx, \tag{24}$$

and the conservation (or at least the decrease) of energy

$$\int_{x\in\Omega}\int_{v\in I\!R^3} f(t,x,v)\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega} |\nabla_x\phi(t,x)|^2\,dx$$

$$\leq \int_{x\in\Omega}\int_{v\in I\!R^3} f_0(x,v)\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega} |\nabla_x\phi_0(x)|^2\,dx. \qquad (25)$$

These estimates will be proved in section 3.

First step: *The functions g, ψ still satisfy equations (3) – (10) and (12) – (14).*

Proof of the first step: It is clear that $\partial_t f^{n_k} + v\cdot\nabla_x f^{n_k}$ tends to $\partial_t g + v\cdot\nabla_x g$ in the sense of distributions.

Moreover,

$$\nabla_x\phi^{n_k}\cdot\nabla_v f^{n_k} = \nabla_v\cdot(f^{n_k}\nabla_x\phi^{n_k}), \qquad (26)$$

and $f^{n_k}\nabla_x\phi^{n_k}$ converges uniformly on every compact sets of $[0,T]\times\overline{\Omega}\times I\!R^3$ to $g\nabla_x\psi$. Therefore, $\frac{e}{m}\nabla_x\phi^{n_k}\cdot\nabla_v f^{n_k}$ converges to $\frac{e}{m}\nabla_x\psi\cdot\nabla_v g$ in the sense of distributions.

The conservation (or decrease) of energy (25) (which will be stated in section 3) ensures that for every t in $[0,+\infty[$,

$$\int_{x\in\Omega}\int_{v\in I\!R^3} f(t,x,v)\,|v|^2\,dvdx \leq E_0, \qquad (27)$$

where E_0 is the initial energy. Therefore, $\frac{e}{\epsilon_0}\int_{v\in I\!R^3} f^{n_k}dv$ tends to $\frac{e}{\epsilon_0}\int_{v\in I\!R^3} gdv$ in the sense of distributions.

Moreover, it is clear that $-\triangle_x\phi^{n_k}$ tends to $-\triangle_x\psi$ in the sense of distributions.

Therefore, it only remains to pass to the limit in the sense of distributions in the collision term $Q(f^{n_k}, f^{n_k})$.

According to estimate (27),

$$\int_{x\in\Omega}\int_{|v|\geq R} f(t,x,v)\,dvdx \leq \frac{E_0}{R^2}. \qquad (28)$$

Therefore, according to assumption 1, when R is large enough,

$$\int_{x\in\Omega}\int_{|v|^2+|v_1|^2\geq R^2}\int_{\omega\in S^2} f^{n_k}(t,x,v)f^{n_k}(t,x,v_1)\,B(v-v_1,\omega)\,d\omega dv_1 dvdx$$

$$\leq \int_{x\in\Omega}\int_{|v|^2\geq R^2/2}\int_{|v_1|^2\geq R^2/2} f^{n_k}(t,x,v)f^{n_k}(t,x,v_1)\,K(1+|v|+|v_1|)\,dv_1dvdx$$

$$\leq 3\|f\|_{L^\infty([0,+\infty[\times\Omega;L^1(I\!R^3))}\frac{2KE_0}{R}. \tag{29}$$

Moreover,

$$\int_{x\in\Omega}\int_{|v'|^2+|v_1'|^2\geq R^2}\int_{\omega\in S^2} f^{n_k}(t,x,v')f^{n_k}(t,x,v_1')\,B(v-v_1,\omega)\,d\omega dv_1dvdx$$

$$= \int_{x\in\Omega}\int_{|v|^2+|v_1|^2\geq R^2}\int_{\omega\in S^2} f^{n_k}(t,x,v)f^{n_k}(t,x,v_1)\,B(v-v_1,\omega)\,d\omega dv_1dvdx. \tag{30}$$

Using estimates (29), (30) and the dominated convergence theorem, we get the convergence of $Q(f^{n_k},f^{n_k})$ towards $Q(g,g)$ in the sense of distributions.

Finally, we can pass to the limit in equations

$$\frac{\partial f^{n_k}}{\partial t}+v\cdot\nabla_x f^{n_k}-\frac{e}{m}\nabla_x\phi^{n_k}\cdot\nabla_v f^{n_k}=Q(f^{n_k},f^{n_k}), \tag{31}$$

and

$$-\triangle_x\phi^{n_k}=\frac{e}{\epsilon_0}\int_{v\in I\!R^3} f^{n_k}\,dv. \tag{32}$$

We obtain

$$\frac{\partial g}{\partial t}+v\cdot\nabla_x g-\frac{e}{m}\nabla_x\psi\cdot\nabla_v g=Q(g,g), \tag{33}$$

and

$$-\triangle_x\psi=\frac{e}{\epsilon_0}\int_{v\in I\!R^3} g\,dv. \tag{34}$$

Moreover, it is clear that the boundary conditions (12) and (14) still hold for g and ψ. Therefore, for every t in $[0,T]$, x in $\partial\Omega$, v in $I\!R^3$, such that $v\cdot n(x)\leq 0$,

$$g(t,x,v)=g(t,x,Rv), \tag{35}$$

and

$$\psi(t,x)=0. \tag{36}$$

Second step: *There exists a nonnegative measurable density $r(t,x)$, a strictly positive measurable temperature $T(t,x)$ and a measurable bulk velocity $u(t,x)$ such that g is the Maxwellian of parameters r, u and T,*

$$g(t,x,v)=\frac{r(t,x)}{(2\pi T(t,x))^{3/2}}e^{-\frac{(v-u(t,x))^2}{2T(t,x)}}. \tag{37}$$

Moreover, the function g satisfies the Vlasov equation,

$$\frac{\partial g}{\partial t} + v \cdot \nabla_x g - \frac{e}{m} \nabla_x \psi \cdot \nabla_v g = 0. \tag{38}$$

Proof of the second step: We proceed as in [De 1]. According to Boltzmann's H theorem and to the proof in [DP, L 1] and [Ha], the function f satisfies the following estimate:

$$\sup_{t \in [0,+\infty[} \int_{x \in \Omega} \int_{v \in I\!R^3} f(v) \,|\log f(v)|\, dx dv$$

$$+ \int_{t=0}^{+\infty} \int_{x \in \Omega} \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \Big\{ f(v')f(v_1') - f(v)f(v_1) \Big\}$$

$$\Big\{ \log\Big(f(v')f(v_1') \Big) - \log\Big(f(v)f(v_1) \Big) \Big\}\, B(v - v_1, \omega)\, d\omega dv_1 dv dx dt < +\infty. \tag{39}$$

Therefore, the entropy dissipation

$$\int_0^T \int_{x \in \Omega} \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \Big\{ f^{n_k}(v')f^{n_k}(v_1') - f^{n_k}(v)f^{n_k}(v_1) \Big\}$$

$$\Big\{ \log\Big(f^{n_k}(v')f^{n_k}(v_1') \Big) - \log\Big(f^{n_k}(v)f^{n_k}(v_1) \Big) \Big\}\, B(v - v_1, \omega)\, d\omega dv_1 dv dx dt \tag{40}$$

tends to 0 as k tends to infinity.

But the function

$$\Theta(x,y) = (x-y)(\log x - \log y) \tag{41}$$

is nonnegative, and therefore, using assumption 1, we can extract from f^{n_k} a subsequence still denoted by f^{n_k} such that for a.e. (t, x, v, v_1, ω) in $[0,T] \times \Omega \times I\!R^3 \times I\!R^3 \times S^2$,

$$\Theta\Big(f^{n_k}(t,x,v')f^{n_k}(t,x,v_1'),\; f^{n_k}(t,x,v)f^{n_k}(t,x,v_1) \Big) \underset{k \to +\infty}{\longrightarrow} 0. \tag{42}$$

But this quantity tends also uniformly on every compact sets of $[0,T] \times \overline{\Omega} \times I\!R^3 \times I\!R^3 \times S^2$ to

$$\Theta\Big(g(t,x,v')g(t,x,v_1'),\; g(t,x,v)g(t,x,v_1) \Big). \tag{43}$$

Therefore, for a.e. (t, x, v, v_1, ω) in $[0,T] \times \Omega \times I\!R^3 \times I\!R^3 \times S^2$,

$$g(t,x,v')\, g(t,x,v_1') \;=\; g(t,x,v)\, g(t,x,v_1). \tag{44}$$

Using a result of [Tr, Mu] for example, we see that g is almost everywhere a Maxwellian function of v (whose parameters may depend on t, x). Therefore, we can write

$$g(t,x,v) = \frac{r(t,x)}{(2\pi T(t,x))^{3/2}} e^{-\frac{(v-u(t,x))^2}{2T(t,x)}}, \tag{45}$$

where r, u and T are measurable. But g is locally bounded, therefore the temperature T is strictly positive (and different from $+\infty$) for a.e. (t, x). Moreover, g is nonnegative, therefore $r \geq 0$ for a.e. (t, x). Finally, $Q(g, g) = 0$, since g is a Maxwellian, and according to eq. (33), g satisfies the Vlasov equation (38).

Third step: *The density r defined in the second step is strictly positive (except in the trivial case when $f_0 = 0$). Moreover, the parameters r, u and T defined in the second step are continuous.*

Proof of the third step: The conservation of mass (24) (which will be proved in section 3) and estimate (27) ensures that g still satisfies,

$$\int_{x\in\Omega}\int_{v\in I\!R^3} g(t,x,v)\, dvdx = \int_{x\in\Omega}\int_{v\in I\!R^3} f_0(x,v)\, dvdx. \tag{46}$$

Except when $f_0 = 0$, we obtain for all t in $[0, T]$ the existence of (x_0, v_0) in $\Omega \times I\!R^3$ such that $g(t, x_0, v_0) > 0$. But eq. (45) ensures that $r(t, x_0) > 0$, and therefore for all v in $I\!R^3$, $g(t, x_0, v) > 0$. Because of eq. (38), $g(t+\tau, x_0, v) = g(t, x_0 + v\tau + o(\tau), v(\tau))$ and therefore, for every τ small enough, and every x in Ω, one can find v such that $g(t+\tau, x, v) = g(t, x_0, v(\tau))$ (note that if Ω is not convex, one has to carry out this analysis several times). Therefore, $g(t + \tau, x, v) > 0$ and $r(t + \tau, x) > 0$. Finally, the function r is strictly positive. But r is also the local density

$$r(t,x) = \int_{v\in I\!R^3} g(t,x,v)\, dv, \tag{47}$$

therefore r is continuous, and $\frac{1}{r}$ is also continuous. Finally, $r\,u$ and $r\,T$ can be obtained as moments of g, and as a consequence u and T are also continuous.

Fourth step: *The functions g and ψ satisfy identities (22) and (23).*

Proof of the fourth step: Note first that because of the averaging lemmas introduced in [DP, L 3], eq. (34), eq. (38) and eq. (45) together with the boundary conditions ensure that the macroscopic parameters r, u and T are smooth. Because of the strict positivity of r and T, the quantities $\log r$, $\log T$ and $\frac{1}{T}$ are also smooth.

We recall the properties of $\log g$ and ψ collected in the previous steps of the proof,

1. *The quantity* $\log g$ *satisfies the Vlasov equation,*

$$\partial_t \log g + v \cdot \nabla_x \log g - \frac{e}{m} \nabla_x \psi \cdot \nabla_v \log g = 0. \tag{48}$$

2. *The potential* ψ *satisfies the Poisson equation (34).*

3. *The boundary conditions (35) and (36) hold for* $\log g$ *and* ψ*.*

4. *The function* $\log g$ *is the logarithm of a Maxwellian function of* v*,*

$$\log g = \log r - \frac{3}{2} \log(2\pi T) - \frac{|v-u|^2}{2T}. \tag{49}$$

Then, we inject eq. (49) in eq. (48) and get a polynomial in $v-u$ of degree 3 which is always equal to 0. Looking at the term of highest degree, we get the equation,

$$\nabla_x \Big\{ \frac{1}{T} \Big\} = 0. \tag{50}$$

Therefore, the temperature $T(t,x)$ depends only on t. We write

$$T(t,x) = T_0(t). \tag{51}$$

Then, looking at the term of degree 2, we can see that for any vector h of modulus 1,

$$\Big(\frac{1}{T_0}\Big)' + \nabla_x u(h,h) = 0. \tag{52}$$

According to [De 1], there exist a skew–symmetric tensor Λ depending on t and a vector C depending also on t such that

$$u(t,x) = \Big(\frac{1}{T_0}\Big)'(t)\, x + \Lambda(t)(x) + C(t). \tag{53}$$

But boundary condition (35) ensures that for all t in $[0,T]$, x in $\partial\Omega$,

$$u(t,x) \cdot n(x) = 0. \tag{54}$$

Therefore, if we define a curve in $I\!R^3$ by

$$\frac{dx}{ds}(s) = \quad u(t, x(s)), \tag{55}$$

$$x(0) \in \qquad \partial\Omega, \tag{56}$$

the point $x(s)$ will remain in $\partial\Omega$ for every s in $I\!R$.

Then, we proceed as in [De 1]. Since $\partial\Omega$ is bounded and is not a surface of revolution, we get

$$\Big(\frac{1}{T_0}\Big)'(t) = 0, \quad C(t) = 0, \quad \Lambda(t) = 0. \tag{57}$$

Therefore,

$$u(t, x) = 0. \tag{58}$$

Moreover, we can define $\theta > 0$ by

$$T_0(t) = \theta. \tag{59}$$

Note that eq. (49) injected in eq. (48) also yields

$$\frac{\partial r}{\partial t} = 0, \tag{60}$$

and

$$\nabla_x \log r + \frac{e}{m\theta} \nabla_x \psi = 0. \tag{61}$$

Therefore, r does not depend on t. We denote:

$$r(t, x) = r_0(x). \tag{62}$$

Then, g can be written as

$$g(t, x, v) = \frac{r_0(x)}{(2\pi\theta)^{3/2}} \exp\Big\{ -\frac{v^2}{2\theta} \Big\}, \tag{63}$$

and it clearly does not depend on t. But according to eq. (34) and eq. (36), the potential ψ does not depend on t either. Therefore, according to eq. (61), there exists $\rho > 0$ such that

$$r_0(x) = \rho \exp\Big\{ -\frac{e\psi(x)}{m\theta} \Big\}, \tag{64}$$

which concludes the proof of theorem 1.

We now investigate the global conservations for the Vlasov–Poisson–Boltzmann equation.

3 Conservations of mass and energy, applications

We begin this section with the (formal) proof of conservation of mass and energy for system (3) – (10), (12) – (14). Note first that (according to [Tr, Mu] for example) when $\chi(v) = 1, v$, or $|v|^2$,

$$\int_{v \in I\!R^3} Q(f,f)(v)\, \chi(v)\, dv = 0. \tag{65}$$

Note that this property immediately yields the conservation of mass

$$\frac{\partial}{\partial t} \int_{x\in\Omega} \int_{v \in I\!R^3} f\, dv dx = 0. \tag{66}$$

Moreover, multiplying eq. (9) by $|v|^2$ and integrating over x and v, one gets

$$\frac{\partial}{\partial t}\Big\{ \int_{x\in\Omega} \int_{v \in I\!R^3} f\, |v|^2\, dv dx \Big\} - \int_{x\in\Omega} \int_{v \in I\!R^3} \frac{e}{m} \nabla_x \phi \cdot \nabla_v f\, |v|^2\, dv dx = 0. \tag{67}$$

Denoting by $d\sigma(x)$ the natural measure on $\partial\Omega$, we get after an integration by parts with respect to x and v,

$$\frac{\partial}{\partial t}\Big\{ \int_{x\in\Omega} \int_{v \in I\!R^3} f |v|^2 dv dx \Big\} - 2 \int_{x\in\Omega} \frac{e}{m} \phi\, \nabla_x \cdot \Big\{ \int_{v \in I\!R^3} f\, v\, dv \Big\}\, dx$$

$$+2 \int_{x\in\partial\Omega} \int_{v \in I\!R^3} \frac{e}{m} \phi\, f\, (v \cdot n(x))\, dv d\sigma(x) = 0. \tag{68}$$

Using now the local conservation of mass,

$$\frac{\partial}{\partial t}\Big\{ \int_{v \in I\!R^3} f\, dv \Big\} + \nabla_x \cdot \Big\{ \int_{v \in I\!R^3} f\, v\, dv \Big\} = 0, \tag{69}$$

and the boundary condition (14), estimate (68) becomes

$$\frac{\partial}{\partial t}\Big\{ \int_{x\in\Omega} \int_{v \in I\!R^3} f\, |v|^2\, dv dx \Big\} + 2 \int_{x\in\Omega} \frac{e}{m}\, \phi\, \frac{\partial}{\partial t}\Big\{ \int_{v \in I\!R^3} f\, dv \Big\}\, dx = 0. \tag{70}$$

Using now Poisson's equation (3) and integrating by parts with respect to x, we get

$$\frac{\partial}{\partial t}\Big\{ \int_{x\in\Omega} \int_{v \in I\!R^3} f\, |v|^2\, dv dx \Big\} + \frac{\epsilon_0}{m} \int_{x\in\Omega} 2\, \nabla_x \phi\, \frac{\partial}{\partial t} (\nabla_x \phi)\, dx$$

$$+ \frac{\epsilon_0}{m} \int_{x\in\partial\Omega} 2\, \phi \Big\{ - n(x) \cdot \nabla_x \Big(\frac{\partial \phi}{\partial t}\Big) \Big\}\, d\sigma(x) = 0. \tag{71}$$

Using once again the boundary condition (14), we get at last the conservation of energy,

$$\frac{\partial}{\partial t}\Big\{\int_{x\in\Omega}\int_{v\in I\!R^3} f\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega}|\nabla_x\phi|^2\,dx\Big\} = 0. \tag{72}$$

For a solution of the Vlasov–Poisson–Boltzmann system (3) – (10), (12) – (14), satisfying assumptions 1, 2, 3 of theorem 1, the a–priori estimate (66) holds, and it yields identity (24). Identity (72) also holds, although perhaps only as an inequality, whence eq. (27). estimates (27) and (39) ensure that one can pass to the limit in eq. (24) when the time t tends to infinity (this property has already been used). However, it is not possible to pass to the limit in eq. (25) without a stronger assumption. Therefore, we shall from now on deal with solutions of (3) – (10), (12) – (14), which satisfy the following property,

Assumption 4: *There exists* $\epsilon > 0$ *such that*

$$\sup_{t\in[0,+\infty[}\int_{x\in\Omega}\int_{v\in I\!R^3} f\,|v|^{2+\epsilon}\,dvdx + \sup_{t\in[0,+\infty[}\int_{x\in\Omega}|\nabla_x\phi|^{2+\epsilon}dx < +\infty. \tag{73}$$

If f_0 is an initial datum, we denote now by M_0 the initial mass,

$$M_0 = \int_{x\in\Omega}\int_{v\in I\!R^3} f_0(x,v)\,dvdx, \tag{74}$$

by E_0 the initial energy,

$$E_0 = \int_{x\in\Omega}\int_{v\in I\!R^3} f_0(x,v)\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega}|\nabla_x\phi_0(x)|^2dx, \tag{75}$$

by χ the constant

$$\chi = \frac{4mE_0\epsilon_0}{9M_0^2e^2}, \tag{76}$$

and by F the function

$$F(t) = 1 + \sqrt{1+\chi t^2}\,. \tag{77}$$

Moreover, we define the three following constants $C_i, (i = 1,2,3)$, which depend only upon the physical constants of the problem,

$$C_1 = M_0\Big(\frac{3M_0}{4\pi E_0}\Big)^{3/2}, \quad C_2 = \frac{3M_0}{4E_0}, \quad C_3 = \frac{2mE_0}{3M_0e}. \tag{78}$$

Theorem 2: *Let Ω be a regular (of class C^2) bounded and simply connected open set of $I\!R^3$ such that $\partial\Omega$ is not a surface of revolution. Assume that (f,ϕ) is a solution of (3) – (10), (12) – (14) satisfying assumptions 1,2,3 and 4. Then there exists a function $V(x)$ in $C^2(\overline{\Omega})$ such that for every strictly positive T, $f^\tau(t,x,v) = f(t+\tau,x,v)$ converges uniformly on every compact sets of $[0,T]\times\overline{\Omega}\times I\!R^3$ when τ tends to infinity to*

$$g(x,v) = \frac{C_1}{\int_{x\in\Omega} e^{-V}\,dx}\left(F\Big(\|\nabla_x V\|_{L^2(\Omega)}\Big)\right)^{3/2} e^{-V(x)-C_2 F(\|\nabla_x V\|_{L^2(\Omega)})|v|^2}, \tag{79}$$

and $\phi^\tau(t,x) = \phi(t+\tau,x)$ converges in $C^2([0,T]\times\overline{\Omega})$ to

$$\psi(x) = \frac{C_3}{F(||\nabla_x V||_{L^2(\Omega)})} V(x). \tag{80}$$

Moreover, the potential V satisfies the following equation,

$$-\frac{3}{2}\quad\frac{\chi\,\Delta_x V}{F(\|\nabla_x V\|_{L^2(\Omega)})} = \frac{e^{-V}}{\int_{x\in\Omega} e^{-V}\,dx}, \tag{81}$$

and $V=0$ on $\partial\Omega$.

Proof of theorem 2: We define as in theorem 1 for all sequence t_n going to infinity the functions $f^n(t,x,v) = f(t+t_n,x,v)$, and $\phi^n(t,x,v) = \phi(t+t_n,x,v)$. Theorem 1 ensures that we can extract from t_n a subsequence t_{n_k} such that f^{n_k} converges uniformly on every compact sets of $[0,T]\times\overline{\Omega}\times I\!R^3$ to g defined by (22) for any $T>0$, and ϕ^{n_k} converges in $C^2([0,T]\times\overline{\Omega})$ to ψ.

But according to estimates (66), (72) and (73), we get for all $t\in[0,T]$,

$$\int_{x\in\Omega}\int_{v\in I\!R^3} f^{n_k}(t,x,v)\,dvdx = M_0, \tag{82}$$

and

$$\int_{x\in\Omega}\int_{v\in I\!R^3} f^{n_k}(t,x,v)\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega}|\nabla_x\phi^{n_k}(t,x)|^2 dx = E_0. \tag{83}$$

We can pass to the limit in eq. (82) because of estimates (27) and (39) and obtain the conservation of mass for g,

$$\int_{x\in\Omega}\int_{v\in I\!R^3} g(t,x,v)\,dvdx = M_0. \tag{84}$$

Moreover, estimates (39) and (73) ensure that the passage to the limit in (83) holds. It yields

$$\int_{x\in\Omega}\int_{v\in I\!R^3} g(t,x,v)\,|v|^2\,dvdx + \frac{\epsilon_0}{m}\int_{x\in\Omega}|\nabla_x\psi(t,x)|^2dx = E_0. \tag{85}$$

Injecting (22) in (84) and (85), one obtains,

$$\rho\int_{x\in\Omega}\exp\Big\{-\frac{e\psi(x)}{m\theta}\Big\}\,dx = M_0, \tag{86}$$

and

$$3\rho\theta\int_{x\in\Omega}\exp\Big\{-\frac{e\psi(x)}{m\theta}\Big\}\,dx + \frac{\epsilon_0}{m}\int_{x\in\Omega}|\nabla_x\psi|^2dx = E_0. \tag{87}$$

We denote

$$V(x) = \frac{e\psi(x)}{m\theta}. \tag{88}$$

Equations (86) and (87) become,

$$\rho\int_{x\in\Omega}\exp\Big\{-V(x)\Big\}\,dx = M_0, \tag{89}$$

and

$$3M_0\theta + \frac{m\epsilon_0}{e^2}\theta^2\int_{x\in\Omega}|\nabla_x V|^2\,dx = E_0. \tag{90}$$

We can solve eq. (89) and eq. (90) (knowing that θ must be strictly positive), and get,

$$\rho = \frac{M_0}{\int_{x\in\Omega}\exp\{-V(x)\}dx}, \tag{91}$$

and

$$\theta = \frac{2E_0}{3M_0F(||\nabla_x V||_{L^2(\Omega)})}. \tag{92}$$

Therefore, we obtain for g and ψ the identities given in (79) and (80). Moreover, theorem 1 provided for ψ eq. (23), therefore, V satisfies (81). It remains to prove that the whole families $(f^\tau)_{\tau\geq 0}$ and $(\phi^\tau)_{\tau\geq 0}$ converge to g and ψ. This is due to the uniqueness of the limit of f^{n_k} and ϕ^{n_k}. Indeed, V is defined in a unique way by (81) and the fact that $V = 0$ on $\partial\Omega$ (as it will be seen in section 4, theorem 3).

4 Existence, uniqueness and regularity results

We are now interested in the properties of eq. (81),

$$-\frac{3}{2} \quad \frac{\chi \, \Delta_x V}{1+\sqrt{1+\chi \int_{x\in\Omega} |\nabla_x V|^2 \, dx}} = \frac{e^{-V}}{\int_{x\in\Omega} e^{-V} \, dx} \tag{93}$$

with the Dirichlet boundary condition (imposed by theorem 2)

$$V = 0 \quad \text{on} \quad \partial\Omega. \tag{94}$$

The main statement of this section is the following:

Theorem 3: *Let Ω be a regular (of class C^2) open set of $I\!R^3$. For any $\chi > 0$, eq. (93) with boundary condition (94) has one and only one solution V. This solution is strictly positive on Ω and of class $C^2(\overline{\Omega}) \cap C^\infty(\Omega)$.*

Proof of theorem 3: First, we prove the existence of a positive solution of eq. (93) with boundary condition (94).

Let us define the functional J on $H_0^1(\Omega)$ by setting

$$J(U) = \phi\Big(||\nabla_x U||_{L^2(\Omega)}\Big) + \log\left(\int_{x\in\Omega} e^{-U} dx\right), \tag{95}$$

where ϕ is defined on $I\!R^+$ and

$$\phi(t) = \frac{3}{2}\left(\sqrt{1+\chi t^2} - \log\left(1+\sqrt{1+\chi t^2}\right)\right). \tag{96}$$

The following lemma proves that J is bounded below.

Lemma 1 : *Let Ω be a regular (of class C^2) open set of $I\!R^3$. There exists a continuous function F defined on $[0, +\infty[$ such that*

$$F(t) = -2\log t + O(1) \quad \textit{when} \quad t \to +\infty, \tag{97}$$

and

$$\forall U \in H_0^1(\Omega), \quad \log\left(\int_{x\in\Omega} e^{-U} \, dx\right) \geq F\Big(\|\nabla_x U\|_{L^2(\Omega)}\Big). \tag{98}$$

We shall give a more general version and a proof of the lemma in the appendix.

Now, we use the same method as in [Do]. Let us consider a minimizing sequence $(V_n)_{n\in I\!N}$, i.-e. a sequence satisfying

$$\lim_{n\to+\infty} J(V_n) \quad = \quad \inf_{U\in H_0^1(\Omega)} J(U). \tag{99}$$

It is not restrictive to assume that for every $n \in I\!N$, we have

$$J(V_n) \quad \leq \quad J(0) \quad = \quad \log|\Omega|. \tag{100}$$

Now, one can easily prove that when t goes to infinity, $\phi(t)$ is equivalent to $\frac{3}{2}\sqrt{\chi}\, t$. According to lemma 1, the sequence $(\|\nabla_x V_n\|_{L^2(\Omega)})_{n\in I\!N}$ remains bounded. Moreover it is clear that

$$\forall\, U \ \in H_0^1(\Omega), \quad J(U^+) \ \leq \ J(U), \tag{101}$$

since ϕ is increasing. Therefore, we can also suppose that

$$\forall n \in I\!N, \quad V_n \geq 0. \tag{102}$$

There exists a nonnegative function V of $H_0^1(\Omega)$ such that, after extraction of a subsequence if necessary, $\nabla_x V_n \rightharpoonup \nabla_x V$ in $L^2(\Omega)$, and $V_n \to V$ a.e. The lower semi-continuity of the L^2-norm ensures that

$$\|\nabla_x V\|_{L^2(\Omega)} \quad \leq \quad \liminf_{n\to+\infty} \|\nabla_x V_n\|_{L^2(\Omega)}, \tag{103}$$

and since ϕ is increasing,

$$\phi\Big(\|\nabla_x V\|_{L^2(\Omega)}\Big) \quad \leq \quad \liminf_{n\to+\infty} \phi\Big(\|\nabla_x V_n\|_{L^2(\Omega)}\Big). \tag{104}$$

Using Lebesgue's theorem of dominated convergence, we obtain

$$\lim_{n\to+\infty} \int_{x\in\Omega} e^{-V_n}\, dx \quad = \quad \int_{x\in\Omega} e^{-V}\, dx, \tag{105}$$

and therefore

$$J(V) \quad = \quad \inf_{U\in H_0^1(\Omega)} J(U). \tag{106}$$

Now, let us consider the functional J^+ defined on $H_0^1(\Omega)$ by

$$J^+(U) = \phi\Big(\|\nabla_x U\|_{L^2(\Omega)}\Big) \;+\; \log\left(\int_{x\in\Omega} e^{-U^+}\, dx\right). \tag{107}$$

We have

$$\forall\, U \in H_0^1(\Omega), \quad J(U^+) \leq J^+(U) \leq J(U), \tag{108}$$

and therefore

$$J^+(V) \quad = \inf_{U\in H_0^1(\Omega)} J^+(U). \tag{109}$$

But J^+ is a functional of class C^1 on $H_0^1(\Omega)$, and for all $U \in H_0^1(\Omega)$,

$$dJ^+(U) = -\frac{3}{2} \quad \frac{\chi\,\Delta_x U}{1+\sqrt{1+\chi\int_{x\in\Omega}|\nabla_x U|^2\,dx}} - \frac{e^{-U^+}}{\int_{x\in\Omega} e^{-U^+}\,dx}. \tag{110}$$

Therefore

$$dJ^+(V) = 0$$

$$\Longleftrightarrow \quad -\frac{3}{2} \quad \frac{\chi\,\Delta_x V}{1+\sqrt{1+\chi\int_{x\in\Omega}|\nabla_x V|^2\,dx}} = \frac{e^{-V^+}}{\int_{x\in\Omega} e^{-V^+}\,dx}$$

$$\Longleftrightarrow \quad -\frac{3}{2} \quad \frac{\chi\,\Delta_x V}{1+\sqrt{1+\chi\int_{x\in\Omega}|\nabla_x V|^2\,dx}} = \frac{e^{-V}}{\int_{x\in\Omega} e^{-V}\,dx} \tag{111}$$

since $V \geq 0$ a.e.

Finally, V is a nonnegative solution of eq. (93) with boundary condition (94).

We have now to prove the uniqueness of the solution. Let us define the convex cone H^+ by setting

$$H^+ = \{U \in H_0^1(\Omega) / \quad U \geq 0 \quad \text{a.e on } \Omega\ \}. \tag{112}$$

According to the maximum principle, every solution of eq. (93) with boundary condition (94) is nonnegative and belongs therefore to H^+. As a consequence, every solution V satisfies $dJ^+(V) = 0$. But $J^+(U) = J(U)$ for all $U \in H^+$, and J is strictly convex. Indeed, ϕ is increasing and strictly convex, and the L^2-norm is strictly convex. If we consider two functions U_1 and U_2 of $H_0^1(\Omega)$ and $t \in [0,1]$, we get

$$\log\left(\int_{I\!R^N} \rho_0 e^{-\left(tU_1+(1-t)U_2\right)}\,dx\right) = \log\left(\int_{I\!R^N} \left(\rho_0 e^{-U_1}\right)^t \left(\rho_0 e^{-U_2}\right)^{(1-t)}\,dx\right)$$

$$\leq t \ \log\left(\int_{I\!R^N} \rho_0 e^{-U_1}\, dx\right) + (1-t)\ \log\left(\int_{I\!R^N} \rho_0 e^{-U_2}\, dx\right), \qquad (113)$$

according to Hölder's inequality.

The functional J^+ is therefore strictly convex on H^+, which ensures that the solution of $dJ^+(V) = 0$ is unique on H^+. Finally, the solution of eq. (93) with boundary condition (94) is unique.

It is easy to prove that the solution is strictly positive on Ω with the maximum principle, and of class $C^2(\overline{\Omega}) \cap C^\infty(\Omega)$ with an elliptic bootstraping argument.

Because of the uniqueness of the solution, one can also prove that V is radially symmetric if Ω is a ball. Indeed, the solution composed with every rotation is still a solution.

5 Some additional results

5.1 Others collision kernels

First note that in the analysis of sections 2, 3 and 4, the only algebraic properties required on the kernel Q is the existence of an entropy, and the conservations of mass and energy.

More precisely, the kernel Q must satisfy:

$$\int_{v\in I\!R^3} Q(f,f)\, dv = 0, \qquad (114)$$

$$\int_{v\in I\!R^3} Q(f,f)\, |v|^2\, dv = 0, \qquad (115)$$

$$\int_{v\in I\!R^3} Q(f,f)\ \log f\, dv \leq 0, \qquad (116)$$

and

$$\int_{v\in I\!R^3} Q(f,f)\ \log f\, dv = 0 \qquad (117)$$

if and only if f is a Maxwellian function of v.

Note that these assumptions are satisfied by the B.G.K. kernel (Cf. [Bh, Gr, Kr]),

$$Q(f,f) = M_f - f, \qquad (118)$$

where M_f is the Maxwellian having the same moments as f.

They are also satisfied by the Fokker–Planck–Landau collision kernel (Cf. [Li, Pi]),

$$Q(f,f) = \nabla_v \cdot \int_{v_1 \in I\!R^N} \Theta(|v-v_1|) \Big\{ Id - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2} \Big\}$$

$$\Big\{ f(v_1)\nabla_v f(v) - f(v)\nabla_{v_1} f(v_1) \Big\}\, dv_1, \tag{119}$$

where Θ is a strictly positive function going from $I\!R$ to $I\!R$.

Therefore, the analysis of sections 2, 3 and 4 still holds (at least formally) if $Q(f,f)$ is one of these kernels.

In fact the most accurate collision kernel to consider in the case of a plasma may be a sum of Boltzmann's kernel and Fokker–Planck–Landau's kernel.

5.2 Influence of a magnetic field

We can also take in account the presence of a magnetic field. In such a model, it is still possible to write a Vlasov–Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{e}{m}\Big(E + v \times B\Big) \cdot \nabla_v f = Q(f,f), \tag{120}$$

where E and B are respectively the electric and magnetic fields, satisfying the Maxwell system,

$$-\epsilon_0 \mu_0 \frac{\partial E}{\partial t} + \text{ Curl }_x B = e\mu_0 \int_{v \in I\!R^3} f\, v\, dv, \tag{121}$$

$$\frac{\partial B}{\partial t} + \text{ Curl }_x E = 0, \tag{122}$$

$$\text{Div }_x E = \frac{e}{\epsilon_0} \int_{v \in I\!R^3} f\, dv, \tag{123}$$

$$\text{Div }_x B = 0, \tag{124}$$

and ϵ_0 is the electric permeability of vacuum and μ_0 is the magnetic permittivity of vacuum. Note that the existence of a global renormalized solution to this system is now proved when L^2 bounds are imposed on the initial data (Cf. [DP, L 2]). According to [J], it is possible to find a potential ϕ and a vector potential A such that

$$E = -\nabla_x \phi - \frac{1}{\epsilon_0 \mu_0} \frac{\partial A}{\partial t}, \tag{125}$$

and

$$B = \text{ Curl }_x A. \tag{126}$$

Then, we take the following boundary conditions:

For every t in $[0, +\infty[$, x in $\partial\Omega$, v in $I\!R^3$, such that $v \cdot n(x) \leq 0$,

$$f(t, x, v) = f(t, x, Rv), \tag{127}$$

$$\phi(t, x) = 0, \tag{128}$$

$$A(t, x) = A_0, \tag{129}$$

where A_0 is a constant vector potential imposed on the boundary. It is easy to prove that the limit of the magnetic field $B(t, x)$ when t tends to infinity is 0. Therefore, all the results of section 3 still hold, except that the total energy E_0 is now given by

$$E_0 = \int_{x\in\Omega}\int_{v\in I\!R^3} f_0\,|v|^2\,dv + \frac{\epsilon_0}{m}\int_{x\in\Omega}|e_0|^2\,dx + \frac{1}{m\mu_0}\int_{x\in\Omega}|b_0|^2\,dx, \tag{130}$$

where e_0 and b_0 are the initial electric and magnetic fields.

5.3 The case of several species

Finally, we give some results on a model where different species interact. For every species i, we define the density $f_i(t, x, v)$, the mass m_i and the charge e_i of one particle. We assume that the densities f_i satisfy the Vlasov–Poisson–Boltzmann equation,

$$\frac{\partial f_i}{\partial t} + v \cdot \nabla_x f_i - \frac{e_i}{m_i}\nabla_x\phi \cdot \nabla_v f_i = \sum_{j=1}^{n} Q_{ij}(f_i, f_j), \tag{131}$$

where Q_{ij} is a Boltzmann kernel describing the collisions between the particles of the species i and j. Note that for such a model (we assume that the cross sections B_{ij} associated to Q_{ij} are not identically equal to 0 for all $i, j = 1, 2, ..., n$), the equilibrium is reached when the densities f_i are Maxwellian functions of v having the same temperature θ and the same bulk velocity.

Moreover, we can write the Poisson equation for the potential,

$$-\triangle_x\phi = \sum_{i=1}^{n}\frac{e_i}{\epsilon_0}\int_{v\in I\!R^3} f_i\,dv. \tag{132}$$

We still require the boundary conditions (12) for each species i and (14) for the potential ϕ. The analysis of section 2 leads to the following identities for the limits g_i and ψ of f_i and ϕ when the time t goes to infinity,

$$g_i(t,x,v) = \frac{\rho_i}{(2\pi\theta)^{\frac{3}{2}}} \exp\Big\{ -\frac{e_i\psi(x)}{m_i\theta} - \frac{v^2}{2\theta} \Big\}, \tag{133}$$

and

$$-\triangle_x \psi = \sum_{i=1}^{n} \frac{e_i}{\epsilon_0} \rho_i \exp\Big\{ -\frac{e_i\psi(x)}{m_i\theta} \Big\}. \tag{134}$$

Moreover, if we denote by M_i the initial mass of the species i, and by E_i the initial energy, the conservations of these quantities ensure that

$$M_i = m_i\, \rho_i \int_{x\in\Omega} \exp\Big\{ -\frac{e_i\psi(x)}{m_i\theta} \Big\}\, dx, \tag{135}$$

and

$$E = \sum_{i=1}^{n} 3\, m_i\, \rho_i\, \theta \int_{x\in\Omega} \exp\Big\{ -\frac{e_i\psi(x)}{m_i\theta} \Big\}\, dx + \epsilon_0 \int_{x\in\Omega} |\nabla_x\psi|^2 dx. \tag{136}$$

We make the change of variable

$$V(x) = \frac{\psi(x)}{\theta}, \tag{137}$$

and denote

$$\lambda_i = \frac{e_i}{m_i}, \tag{138}$$

$$M = \sum_{i=1}^{n} M_i, \tag{139}$$

$$H(t) = \frac{2E}{3M + \sqrt{9M^2 + 4E\epsilon_0 t^2}}. \tag{140}$$

Equations (135) and (136) become

$$\rho_i = \frac{\mu_i}{\int_{x\in\Omega} e^{-\lambda_i V(x)} dx}, \tag{141}$$

and

$$\epsilon_0\, \theta^2 \int_{x\in\Omega} \|\nabla_x V\|^2\, dx + 3\,\theta M - E = 0. \tag{142}$$

Finally,

$$\theta = \frac{2E}{3M + \sqrt{9M^2 + 4E\,\epsilon_0 \int_{x\in\Omega} |\nabla_x V|^2 dx}}. \tag{143}$$

Therefore,

$$g_i(t,x,v) = \frac{\mu_i}{\int_{x\in\Omega} e^{-\lambda_i V(x)} dx} \Big(2\pi H(||\nabla_x V||_{L^2(\Omega)})\Big)^{-\frac{3}{2}}$$

$$\exp\left(-\lambda_i V(x) - \frac{|v|^2}{2H(||\nabla_x V||_{L^2(\Omega)})}\right), \tag{144}$$

and

$$\psi(x) = 2\,E\,H(||\nabla_x V||_{L^2(\Omega)}), \tag{145}$$

where V satisfies:

$$-\frac{3}{2}\frac{\chi^{'}\,\triangle_x V}{1 + \sqrt{1 + \chi^{'} \int_{x\in\Omega} |\nabla_x V|^2\,dx}} = \sum_{i=1}^{n} \lambda_i \frac{e^{-\lambda_i V(x)}}{\int_{x\in\Omega} e^{-\lambda_i V(x)}\,dx}, \tag{146}$$

with

$$\chi^{'} = \frac{4E\epsilon_0}{9M^2}, \tag{147}$$

and this equation can be solved exactly as before.

Appendix

In this section, we give a proof of the following lemma:

Lemma 2 : *Let Ω be a regular (of class C^2) open set of $I\!R^3$ and p belong to $]1,+\infty[$. There exists a continuous function F defined on $[0,+\infty[$ such that*

$$\forall\,U \in W_0^{1,p}(\Omega), \quad \log\left(\int_{x\in\Omega} e^{-U}\,dx\right) \geq F\Big(\|\nabla_x U\|_{L^p(\Omega)}\Big), \tag{148}$$

with

$$F(t) = -\frac{p}{p-1}\log t + a + o(1) \quad \textit{when} \quad t \to +\infty, \tag{149}$$

and

$$a = \frac{p}{p-1} \times \left(\log\Big(\frac{p}{p-1}\,\frac{|\partial\Omega|}{K(\Omega)}\Big) - 1\right), \tag{150}$$

where $K(\Omega)$ is the constant appearing in Hardy's inequality (168).

Proof of lemma 2: The proof is divided in two steps.
First, we prove that

$$\forall U \in W_0^{1,p}(\Omega), \quad \log\left(\int_{x\in\Omega} e^{-U}\, dx\right) \geq F_1\Big(\|\nabla_x U\|_{L^p(\Omega)}\Big), \tag{151}$$

with

$$F_1(t) = A - B\, t, \tag{152}$$

where A is a real constant and B a strictly positive real constant. This step is necessary to prove that F is continuous and bounded in a neighborhood of $t = 0_+$.
In a second step, we prove that

$$\forall U \in W_0^{1,p}(\Omega), \quad \log\left(\int_{x\in\Omega} e^{-U}\, dx\right) \geq F_2\Big(\|\nabla_x U\|_{L^p(\Omega)}\Big), \tag{153}$$

where F_2 is a continuous function on $]0, +\infty[$ such that

$$F_2(t) = -\frac{p}{p-1}\log t + a + o(1) \quad \textit{when} \quad t \to +\infty. \tag{154}$$

The lemma is therefore proved with $F = \max(F_1, F_2)$.

First step of the proof: Let U belong to $W_0^{1,p}(\Omega)$. According to Jensen's inequality, we have

$$\int_{x\in\Omega} e^{-U}\, dx \quad \geq \quad |\Omega|\ \exp\left\{-\left(\frac{\int_{x\in\Omega} U\, dx}{|\Omega|}\right)\right\}. \tag{155}$$

Hölder's inequality ensures that

$$\int_{x\in\Omega} U\, dx \leq |\Omega|^{1-1/p}\ \|U\|_{L^p(\Omega)}, \tag{156}$$

and, according to Poincaré's inequality, we get

$$\|U\|_{L^p(\Omega)} \leq C(\Omega)\ \|\nabla_x U\|_{L^p(\Omega)}, \tag{157}$$

where $C(\Omega)$ is a strictly positive constant, and

$$\int_{x\in\Omega} U\, dx \quad \leq \quad C(\Omega)\ |\Omega|^{1-1/p}\, \|\nabla_x U\|_{L^p(\Omega)}. \tag{158}$$

Therefore,

$$\log\left(\int_{x\in\Omega} e^{-U}\ dx\right) \geq F_1\Big(\|\nabla_x U\|_{L^p(\Omega)}\Big), \tag{159}$$

with

$$F_1(t) = \ln|\Omega| - \frac{C(\Omega)}{|\Omega|^{1/p}}\ t. \tag{160}$$

Second step of the proof: Let us define the following quantities for $x \in \Omega$ and $\lambda > 0$,

$$d(x) = d(x, \partial\Omega), \tag{161}$$

and

$$\Omega_\lambda = \{\ x \in \Omega\ /\quad d(x) \leq 1/\lambda\ \}. \tag{162}$$

Let U belong to $W_0^{1,p}(\Omega)$ and define

$$V = \frac{1}{\lambda}\ U^+. \tag{163}$$

Then

$$\begin{aligned}
\int_{x\in\Omega} e^{-U}\ dx &\geq \int_{x\in\Omega} e^{-U^+}\ dx \\
&\geq \int_{x\in\Omega} e^{-\lambda V}\ dx \\
&\geq \int_{x\in\Omega_\lambda} e^{-\lambda V}\ dx \\
&\geq \int_{x\in\Omega_\lambda} e^{-\frac{V}{d}\ \lambda d}\ dx \\
&\geq \int_{x\in\Omega_\lambda} e^{-\frac{V}{d}}\ dx.
\end{aligned} \tag{164}$$

Using Jensen's inequality, we get

$$\int_{x\in\Omega_\lambda} e^{-\frac{V}{d}}\ dx \quad\geq\quad |\Omega_\lambda| \quad \exp\left\{ -\left(\frac{\int_{x\in\Omega_\lambda} \frac{V}{d}\ dx}{|\Omega_\lambda|}\right)\right\}. \tag{165}$$

Then, Hölder's inequality ensures that

$$\int_{x\in\Omega_\lambda} \frac{V}{d}\ dx \quad\leq\quad |\Omega_\lambda|^{1-1/p}\ \|\frac{V}{d}\|_{L^p(\Omega_\lambda)}. \tag{166}$$

Moreover

$$\int_{x\in\Omega_\lambda} \frac{V}{d}\, dx \quad \leq \quad |\Omega_\lambda|^{1-1/p}\, \|\frac{V}{d}\|_{L^p(\Omega)}, \tag{167}$$

since $\Omega_\lambda \subset \Omega$. According to Hardy's inequality, we get

$$\|\frac{V}{d}\|_{L^p(\Omega)} \quad \leq \quad K(\Omega)\, \|\nabla_x V\|_{L^p(\Omega)}, \tag{168}$$

where $K(\Omega)$ is a strictly positive constant. Therefore,

$$\int_{x\in\Omega} e^{-U}\, dx \quad \geq \quad |\Omega_\lambda| \quad \exp\left\{ -\left(\frac{K(\Omega)}{|\Omega_\lambda|^{1/p}} \quad \|\nabla_x V\|_{L^p(\Omega)} \right) \right\}. \tag{169}$$

Let us define the functions,

$$\lambda(t) = \left(\frac{p-1}{p}\, \frac{K(\Omega)}{|\partial\Omega|^{1/p}}\, t \right)^{\frac{p}{p-1}}, \tag{170}$$

and

$$\eta(t) = \frac{|\partial\Omega|}{\lambda(t)\, |\Omega_{\lambda(t)}|} - 1. \tag{171}$$

The choice of the form of $\lambda(t)$ has been made in order to maximise the value of the constant a which appears in (149). It is clear that η is positive, continuous on $]0,+\infty[$ and that

$$\lim_{t\to+\infty} \eta(t) = 0. \tag{172}$$

Let us assume that

$$t = \|\nabla_x U\|_{L^p(\Omega)}, \quad \lambda = \lambda(t). \tag{173}$$

We get

$$\|\nabla_x V\|_{L^p(\Omega)} = \frac{1}{\lambda(t)}, \quad \|\nabla_x U\|_{L^p(\Omega)} = \frac{t}{\lambda(t)}. \tag{174}$$

But

$$|\Omega_{\lambda(t)}| = \frac{|\partial\Omega|}{\lambda(t)} \quad \Big(1+\eta(t)\Big)^{-1}, \tag{175}$$

therefore

$$\frac{\|\nabla_x V\|_{L^p(\Omega)}}{|\Omega_{\lambda(t)}|^{1/p}} = \frac{\Big(1+\eta(t)\Big)^{1/p}\, t}{|\partial\Omega|^{1/p}\, \lambda(t)^{1-1/p}}, \tag{176}$$

and
$$\frac{K(\Omega)\,\|\nabla_x V\|_{L^p(\Omega)}}{|\Omega_{\lambda(t)}|^{1/p}} = \frac{p}{p-1}\Big(1+\eta(t)\Big)^{1/p}. \tag{177}$$

Finally
$$\log\left(\int_{x\in\Omega} e^{-U}\,dx\right) \quad \geq \quad F_2\Big(\|\nabla_x U\|_{L^p(\Omega)}\Big), \tag{178}$$

with
$$F_2(t) = a - \frac{p}{p-1}\ln t - \ln\Big(1+\eta(t)\Big) + \frac{p}{p-1}\left(1-\Big(1+\eta(t)\Big)^{1/p}\right). \tag{179}$$

The function F_2 is therefore continuous, and
$$F_2(t) = -\frac{p}{p-1}\ln t + a + o(1) \quad \text{when } t \to +\infty, \tag{180}$$

which ends the second step of the proof.

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