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Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p-Laplacian

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Abstract. We study the asymptotic behaviour of nonnegative solutions to: $u_t = \Delta_p u^m$ using an entropy estimate based on a sub-family of the Gagliardo-Nirenberg inequalities – or, in the limit case $m = (p-1)^{-1}$, on a logarithmic Sobolev inequality in $W^{1,p}$ – for which optimal functions are known.

Diffusions nonlinéaires et constantes optimales dans des inégalités de type Sobolev : comportement asymptotique d'équations faisant intervenir le p-Laplacien

Résumé. *Nous étudions le comportement asymptotique des solutions positives ou nulles de : $u_t = \Delta_p u^m$ à l'aide d'une estimation d'entropie qui repose sur l'utilisation d'une sous-famille des inégalités de Gagliardo-Nirenberg – ou, dans le cas limite $m = (p-1)^{-1}$, d'une inégalité de Sobolev logarithmique dans $W^{1,p}$ – pour laquelle on connait des fonctions optimales.*

Version française abrégée

Nous considérons dans $\mathbb{R}^d$ des solutions positives ou nulles de

$$u_t = \Delta_p u^m \ , \tag{0}$$

où Δ_p désigne le p-Laplacien. Le premier résultat de cette note concerne le cas $m = 1$.

THÉORÈME 0.1. – *Supposons que $m = 1$, $d \geq 2$, $\frac{2d+1}{d+1} \leq p < d$. Soit u une solution de (0) avec donnée initiale u_0 dans $L^1 \cap L^\infty$ telle que $u_0^q \in L^1$, où $q = \frac{2p-3}{p-1}$ (si $p < 2$) et $\int |x|^{\frac{p}{p-1}} u_0 \, dx < +\infty$. Alors pour tout $s > 1$, il existe une constante $K > 0$ telle que, pour tout $t > 0$,*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2}\,\frac{q}{s} + d(1-\frac{1}{s}))} \quad si\ p \geq 2\ , \quad s \geq q\ ,$$

$$\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2qs} + d(q-\frac{1}{s}))} \quad si\ p \leq 2\ , \quad s \geq \tfrac{1}{q}\ ,$$

où $\alpha = (1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = R(t) = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (d+1)p - 2d$, $u_\infty(t,x) = \frac{1}{R^d}\, v_\infty(\log R, \frac{x}{R})$ avec, pour tout $x \in \mathbb{R}^d$, $v_\infty(x) = (C - \frac{p-2}{p}\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ si $p \neq 2$ et $v_\infty(x) = C\, e^{-|x|^2/2}$ si $p = 2$.

Ici on note $\|v\|_c = \left(\int |v|^c\, dx\right)^{1/c}$ pour tout $c > 0$. La preuve consiste à montrer que la fonction v définie par $u(t,x) = R^{-d}\, v(\log R, \frac{x}{R})$, où R est donné dans le Théorème 0.1, est telle que son *entropie* $\Sigma = \int [\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)]\ dx$ avec $\sigma(s) = \frac{s^q-1}{q-1}$ si $q \neq 1$, $\sigma(s) = s\log s$ si $q = 1$ ($p = 2$), a une décroissance exponentielle. Dans le cas $p = 2$, il suffit d'utiliser l'inégalité logarithmique de Sobolev : voir par exemple [13]. Dans les autres cas, on utilise une sous-famille des inégalités de Gagliardo-Nirenberg avec constantes optimales (voir [6]).

THÉORÈME 0.2. – *Soit $1 < p < d$, $1 < a \leq \frac{p(d-1)}{d-p}$, $b = p\,\frac{a-1}{p-1}$. Il existe une constante strictement positive $\mathcal{S}$ telle que pour toute fonction $w \in W^{1,p}_{\rm loc}$ vérifiant $\|w\|_a + \|w\|_b < +\infty$,*

$$\begin{cases} \|w\|_b \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_a^{1-\theta} & \text{avec } \theta = \frac{(a-p)d}{(a-1)(dp-(d-p)a)} \quad \text{si } a > p \\ \|w\|_a \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_b^{1-\theta} & \text{avec } \theta = \frac{(p-a)d}{a(d(p-a)+p(a-1))} \quad \text{si } a < p \end{cases}$$

avec, à une translation près, égalité pour toute fonction de la forme $\overline{w}(x) = A\,(1 + B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$, où $(A,\ B) \in \mathbb{R}^2$ est tel que B a le même signe que $a - p$.

On en déduit, par un argument de changement d'échelle, une inégalité inhomogène : $\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$, où $\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p\ dx - \frac{1}{q}\,(\frac{d}{1-\kappa_p} + \frac{p}{p-2}) \int v^q\ dx$, avec $\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$, qui montre qu'on a, pour tout $t > 0$, $\Sigma(t) \leq e^{-\alpha\, t}\, \Sigma(0)$. La fin de la preuve repose sur une inégalité de type Csiszár-Kullback selon laquelle, *si f et g sont deux fonctions de L^q avec $q \in (1,2]$, positives ou nulles, alors*

$$\int [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g} - 1)]\, g^q\ dx \geq \frac{q}{2}\, \min\left(\|f\|_q^{q-2}, \|g\|_q^{q-2}\right)\, \|f-g\|_q^2\ .$$

On s'intéresse ensuite au cas $m \neq 1$, d'un point de vue formel, car à part pour $p = 2$ [9] ou pour $m = 1$ [10], il n'y a apparemment pas de résultat de convergence dans L^∞, ni même d'existence globale. Soit $q = 1 + m - \frac{1}{p-1}$. Selon que q est plus grand ou plus petit que 1, on retrouve encore deux régimes différents avec un cas limite correspondant à $q = 1$ ($m = (p-1)^{-1}$) que l'on traite grâce à une généralisation à $W^{1,p}$ de l'inégalité logarithmique de Sobolev : voir le Théorème 0.4 ci-dessous. On supposera donc que
(H) u est une solution de donnée initiale $u_0 \in L^1 \cap L^\infty$ telle que $u_0^q \in L^1$ (si $m < \frac{1}{p-1}$) et $\int |x|^{\frac{p}{p-1}} u_0 dx < +\infty$, qui est bien définie pour tout $t > 0$, appartient à $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d, (1 + |x|^{\frac{p}{p-1}})\, dx) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, et telle que u^q et $t \mapsto \int u^{-\frac{1}{p-1}}\, |\nabla u|^p\ dx$ appartiennent respectivement à $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d)$ et à $L^1_{\rm loc}(\mathbb{R}^+)$.

THÉORÈME 0.3. – *Supposons que $d \geq 2$, $1 < p < d$, $\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{p}{p-1}$ et $q = 1 + m - \frac{1}{p-1}$. Soit u une solution de (0) vérifiant (H). Alors il existe une constante $K > 0$ telle que, pour tout $t > 0$,*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2} + d(1-\frac{1}{q}))} \quad si\ \tfrac{1}{p-1} \leq m \leq \tfrac{p}{p-1}\ ,$$

$$\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}} \qquad si\ \tfrac{d-(p-1)}{d(p-1)} \leq m \leq \tfrac{1}{p-1}\ ,$$

où $\alpha = (1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$ et $R = R(t) = (1 + \gamma\, t)^{1/\gamma}$, $\gamma = (md+1)(p-1) - (d-1)$, $u_\infty(t,x) = \frac{1}{R^d}\, v_\infty(\log R, \frac{x}{R})$ avec, pour tout $x \in \mathbb{R}^d$, $v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ si $m \neq (p-1)^{-1}$ et $v_\infty(x) = C\, e^{-(p-1)^2 |x|^{p/(p-1)}/p}$ si $m = (p-1)^{-1}$.

La preuve est similaire à celle du cas $m = 1$, à l'exception du cas $q = 1$ (qui correspond à $m = (p-1)^{-1}$), et pour lequel on utilise l'inégalité logarithmique de Sobolev dans $W^{1,p}$ suivante (voir [6] pour une preuve dans le cas de la forme invariante par changement d'échelle) :

THÉORÈME 0.4. – *Soit $1 < p < d$. Pour toute fonction $w \in W^{1,p}$, $w \neq 0$, on a*

$$\int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right)\ dx + \frac{d}{p^2}\, \|w\|_p^p \left(1 - \log \mathcal{L}_p - \log(\frac{d}{p^2 \lambda})\right) \leq \lambda\, \|\nabla w\|_p^p$$

avec $\mathcal{L}_p = \frac{p}{d}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\Gamma(\frac{d}{2}+1)/\Gamma(d\,\frac{p-1}{p}+1)\right]^{\frac{p}{d}}$*. Pour* $\lambda = (p-1)\,p^{p-1}$*, cette inégalité est de plus optimale avec égalité si et seulement si* $w = v^{1/p}$ *est égale, à une translation et à une multiplication par une constante près, à* $v_\infty^{1/p}$*.*

Une preuve détaillée du Théorème 0.1 sera donnée dans [7].

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1. Introduction and main result

The long time behaviour of solutions to nonlinear diffusion equations has been extensively studied, but most of the results are concerned with nonlinearities involving the function itself rather than its derivatives, like in the case of the porous medium equation. Although it is well known that equations involving the p-Laplacian have similar properties and can be studied using the same type of tools [10], much less is known for such type of diffusions.

Recently, an approach based on an *entropy-entropy production method* [1, 3, 11, 2] has been developed, but it has apparently not been sufficient to catch the *intermediate asymptotics, i.e.* to determine in an appropriate L^s norm the decay rate of the difference of the solution with the Barenblatt-Prattle type self-similar solution, in the case of the p-Laplacian (which is already known for a long time, by different methods, for $s = \infty$: see [9]). Independently of the *entropy-entropy production method,* the rate of decay of the entropy functionals in connection with optimal constants in Sobolev type inequalities has been investigated in [4, 5] for $p = 2$, and new optimal constants were found in [6] (see Theorem 2.2 for results on Gagliardo-Nirenberg inequalities and Theorem 3.2 for a new and optimal logarithmic Sobolev inequality in $W^{1,p}$). These results are a consequence of a recent uniqueness result by Serrin & Tang [12] for ground states involving the p-Laplacian. The main purpose of this note is to present the application of these optimality results to the asymptotic behaviour of evolution equations involving the p-Laplacian.

THEOREM 1.1. – *Assume that* $d \geq 2$, $\frac{2d+1}{d+1} \leq p < d$ *and let* u *be a solution in* $\mathbb{R}^d$ *to*

$$u_t = \Delta_p u \tag{1}$$

corresponding to a nonnegative initial data u_0 *in* $L^1 \cap L^\infty$ *such that* $u_0^q \in L^1$ *with* $q = \frac{2p-3}{p-1}$ *(in case* $p < 2$*) and* $\int |x|^{\frac{p}{p-1}} u_0\,dx < +\infty$*. For any* $s > 1$*, there exists a constant* $K > 0$ *such that, for any* $t > 0$*,*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_s \leq K\,R^{-(\frac{\alpha}{2}\,\frac{q}{s}+d(1-\frac{1}{s}))} \quad \textit{if } p \geq 2\,, \quad s \geq q\,,$$

$$\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_s \leq K\,R^{-(\frac{\alpha}{2qs}+d(q-\frac{1}{s}))} \quad \textit{if } p \leq 2\,, \quad s \geq \tfrac{1}{q}\,,$$

where $\alpha = (1-\frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = R(t) = (1+\gamma\,t)^{1/\gamma}$, $\gamma = (d+1)p-2d$, $u_\infty(t,x) = \frac{1}{R^d}\,v_\infty(\log R, \frac{x}{R})$ *with, for any* $x \in \mathbb{R}^d$, $v_\infty(x) = (C - \frac{p-2}{p}\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $p \neq 2$ *and* $v_\infty(x) = C\,e^{-|x|^2/2}$ *if* $p = 2$.

Here Δ_p is the p-Laplace operator or p-Laplacian, for some $p > 1$: $\Delta_p w = \nabla\cdot(|\nabla w|^{p-2}\nabla w)$. All integrals are taken over $\mathbb{R}^d$ and we shall use the notation $\|v\|_c = \left(\int |v|^c\,dx\right)^{1/c}$ for any $c > 0$. Unless it is explicitely specified, functions spaces are concerned with functions defined on $\mathbb{R}^d$ and the measure is Lebesgue's measure.

The rest of this note is devoted to a sketch of the main steps of the proof of this result (Section 2.) and to a natural extension (Theorem 3.1) to more general diffusion, but only at a formal level (Section 3.) since uniform convergence for large time or even existence results are apparently not available in the literature. For standard results on nonlinear diffusions, we refer to [8]. A detailed proof will be given in a forthcoming paper [7].

2. Entropy and optimal constants in Sobolev type inequalities

In this section we are giving a sketch of the proof of Theorem 1.1. Let v be such that $u(t,x) = R^{-d}\, v(\log R, \frac{x}{R})$. If u is a solution of (1), v is a solution of

$$v_t = \Delta_p v + \nabla\cdot(x\,v)\ , \tag{2}$$

with initial data u_0, and v_∞ is a stationary, nonnegative radial and nonincreasing solution of (2). Notice that the constant C is uniquely determined by the condition $M = \|v_\infty\|_1$ and can be explicitely computed with the help of the Γ function. For $q>0$, define the *entropy* by

$$\Sigma = \Sigma[v] = \int \left[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\right]\ dx$$

where σ is defined on $(0,+\infty)$ by $\sigma(s) = \frac{s^q-1}{q-1}$ if $q\neq 1$, $\sigma(s) = s\log s$ if $q=1$ ($p=2$). Then $\frac{d\Sigma}{dt} = -q\,(I_1+I_2+I_3+I_4)$ where

$$I_1 = \int v^{-\frac{1}{p-1}}\,|\nabla v|^p\ dx \qquad I_3 = -\frac{d}{q}\int v^q\ dx$$
$$I_2 = \int |x|^{\frac{p}{p-1}} v\ dx \qquad I_4 = \int |\nabla v|^{p-2}\nabla v\cdot |x|^{\frac{1}{p-1}-1} x\ dx$$

The discussion for the heat equation $p=2$ relies on the logarithmic Sobolev inequality: see for instance [13]. From now on, we assume $p\neq 2$. Using Hölder's inequality, we can estimate I_4 in terms of I_1 and I_2:

LEMMA 2.1. – *Let $\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$. With the above notations and under the assumptions of Theorem 1.1, any solution v of (2) with initial data u_0 satisfies: $\frac{1}{q}\frac{d\Sigma}{dt} \leq -(1-\kappa_p)(I_1+I_2) - I_3$.*

Next, consider a subfamily of the Gagliardo-Nirenberg inequalities for which the optimal constants and optimal functions are known. We refer to [6] for a proof.

THEOREM 2.2. – *Let $1<p<d$, $1<a\leq \frac{p(d-1)}{d-p}$, and $b = p\,\frac{a-1}{p-1}$. There exists a positive constant $\mathcal{S}$ such that for any function $w\in W^{1,p}_{\rm loc}$ with $\|w\|_a + \|w\|_b < +\infty$,*

$$\begin{cases} \|w\|_b \leq \mathcal{S}\,\|\nabla w\|_p^\theta\ \|w\|_a^{1-\theta} & \text{with } \theta = \frac{(a-p)d}{(a-1)(dp-(d-p)a)} \quad \text{if } a>p \\ \|w\|_a \leq \mathcal{S}\,\|\nabla w\|_p^\theta\ \|w\|_b^{1-\theta} & \text{with } \theta = \frac{(p-a)d}{a(d(p-a)+p(a-1))} \quad \text{if } a<p \end{cases}$$

and equality holds for any function taking, up to a translation, the form $\overline{w}(x) = A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$ for some $(A,\ B)\in \mathbb{R}^2$, where B has the sign of $a-p$.

With the special choice $b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, we can rephrase Theorem 2.2 into a single non homogeneous inequality for $v = w^b$. For $p\neq 2$, let $\mathcal{F}[v] = \int v^{-\frac{1}{p-1}}|\nabla v|^p\ dx - \frac{1}{q}\left(\frac{d}{1-\kappa_p} + \frac{p}{p-2}\right)\int v^q\ dx$, with κ_p defined as in Lemma 2.1.

COROLLARY 2.3. – *Let $d\geq 2$ be an integer and assume that $\frac{2d+1}{d+1}\leq p<d$. Then for any nonnegative measurable function v for which all integrals involved in the definition of $\mathcal{F}$ are well defined, $\mathcal{F}[v]\geq \mathcal{F}[v_\infty]$ where v_∞ is defined as in Theorem 1.1 in order that $\|v\|_{L^1} = \|v_\infty\|_{L^1}$.*

Proof. – Notice first that $\frac{d}{1-\kappa_p} + \frac{p}{p-2}$ is positive for $p>2$ and negative for $p<2$. One can perform a scaling leaving $\|v\|_1$ invariant in the first case, $\|v\|_q$ in the second case. This reduces the inequality for v to one of the cases of Theorem 2.2. The result follows by identification of a minimizer. □

As a straightforward consequence, $\Sigma \leq \frac{q}{1-\kappa_p}\ \frac{p-1}{p}\ [(1-\kappa_p)(I_1+I_2)+I_3]$, which shows the exponential decay of the entropy. Notice that this inequality is autonomous (the moment in $|x|^{\frac{p}{p-1}}$ is the same on both sides of the inequality).

COROLLARY 2.4. – *If v is a solution of (2) corresponding to an initial data satisfying the conditions of Theorem 1.1, then, with $\alpha = (1-\kappa_p)\,\frac{p}{p-1}$, for any $t>0$, $\Sigma(t)\leq e^{-\alpha\,t}\,\Sigma(0)$.*

The conclusion of the proof of Theorem 1.1 then holds with the help of the following lemma which is one of the variants of the Csiszár-Kullback inequality: see [7] for more details. In order to treat the case $q < 1$ ($p < 2$), one has to notice first that $\frac{s^q-1}{q-1} - \frac{q}{q-1}(s-1)$ can be written as $(\frac{1}{q}-1)^{-1}[(s^q)^{\frac{1}{q}} - 1 - \frac{1}{q}((s^q)-1)]$ for any $s > 0$.

LEMMA 2.5. – *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in $\mathbb{R}^d$. Assume that $q \in (1,2]$. Then $\int_\Omega [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g}-1)]\, g^q\, dx \geq \frac{q}{2}\, \min(\|f\|^{q-2}_{L^q(\Omega)}, \|g\|^{q-2}_{L^q(\Omega)})\, \|f-g\|^2_{L^q(\Omega)}$.*

3. Extension to other nonlinear diffusions involving the p-Laplacian

This section is devoted to the extension of the above results to the equation

$$u_t = \Delta_p u^m \tag{3}$$

for some positive exponent m. The result stated below is formal in the sense that apparently a complete existence theory for such an equation is not yet available, except in the special cases $p = 2$ or $m = 1$ [9, 10, 8]. Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$ (depending if p is bigger or smaller than 2). The case $q = 1$ ($m = (p-1)^{-1}$) is a limiting case, which involves a generalized logarithmic Sobolev inequality: see Theorem 3.2. We shall assume that:

(H) the solution u corresponding to a given nonnegative initial data u_0 in $L^1 \cap L^\infty$, such that $u_0^q \in L^1$ (in case $m < \frac{1}{p-1}$) and $\int |x|^{\frac{p}{p-1}} u_0\, dx < +\infty$, is well defined for any $t > 0$, belongs to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d, (1+|x|^{\frac{p}{p-1}})\, dx) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, such that u^q and $t \mapsto \int u^{-\frac{1}{p-1}}\, |\nabla u|^p\, dx$ belong to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d)$ and $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ respectively.

THEOREM 3.1. – *Assume that $d \geq 2$, $1 < p < d$, $\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{p}{p-1}$ and $q = 1 + m - \frac{1}{p-1}$. Let u be a solution of (3) satisfying (H). Then there exists a constant $K > 0$ such that for any $t > 0$,*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2} + d(1-\frac{1}{q}))} \quad \text{if } \frac{1}{p-1} \leq m \leq \frac{p}{p-1}\ ,$$

$$\|u^q(t,\cdot) - u^q_\infty(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}} \qquad \text{if } \frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{1}{p-1}\ ,$$

where $\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\, \frac{p}{p-1}$ and $R = R(t) = (1+\gamma t)^{1/\gamma}$, $\gamma = (md+1)(p-1) - (d-1)$, $u_\infty(t,x) = \frac{1}{R^d}\, v_\infty(\log R, \frac{x}{R})$ with, for any $x \in \mathbb{R}^d$, $v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ if $m \neq (p-1)^{-1}$ and $v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ if $m = (p-1)^{-1}$.

The rescaled function v defined as in Section 2., with R now given as above, satisfies the rescaled equation $v_t = \Delta_p v^m + \nabla \cdot (x\, v)$. The main steps of the proof are the same as in the case $m = 1$, if $q \neq 1$. With the same definition of Σ as in Section 2., Lemma 2.1 applies without modification. With $w = v^{(mp+q-(m+1))/p} = v^{1/b}$ and $a = b\, q = p\, \frac{m(p-1)+p-2}{mp(p-1)-1}$, Theorem 2.2 also applies and provides a non homogeneous inequality like the one of Corollary 2.3, thus giving the same result as in Corollary 2.4.

The case $q = 1$ (which corresponds to $m = (p-1)^{-1}$) is a limiting case, for which we can use the generalization to $W^{1,p}$ of the logarithmic Sobolev inequality.

THEOREM 3.2. – *Let $1 < p < d$. Then for any $w \in W^{1,p}$, $w \neq 0$,*

$$\int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right)\, dx + \frac{d}{p^2}\, \|w\|_p^p \left(1 - \log \mathcal{L}_p - \log(\frac{d}{p^2 \lambda})\right) \leq \lambda\, \|\nabla w\|_p^p$$

with $\mathcal{L}_p = \frac{p}{d}\, \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\Gamma(\frac{d}{2}+1)/\Gamma(d\,\frac{p-1}{p}+1)\right]^{\frac{p}{d}}$. For $\lambda = (p-1)\, p^{p-1}$, Inequality (3.2) is morevover optimal with equality if and only if $w = v^{1/p}$ is equal, up to a translation and a multiplication by a constant, to $v_\infty^{1/p}$.

For convenience one uses here a non invariant under scaling inequality, which is equivalent to the following optimal form (see [6]): $\int |w|^p \log |w| \; dx \leq \frac{d}{p^2} \log \left[\mathcal{L}_p \|\nabla w\|_p^p \right]$ for any $w \in W^{1,p}(\mathbb{R}^d)$ such that $\|w\|_p = 1$.

As a conclusion, notice that the convergence in L^s for any $s \in [q, +\infty)$ or $[1/q, +\infty)$, depending whether $q \geq 1$ or $q \leq 1$, would hold as soon as a uniform convergence result of v to v_∞ is true. This is apparently known only in the case $m = 1$ [10] or $p = 2$ [9].

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