16th European Signal Processing Conference (EUSIPCO 2008), Lausanne, Switzerland, August 25-29, 2008, copyright by EURASIP

DESIGN OF HIGHER DENSITY DUAL-TREE DISCRETE WAVELET TRANSFORM WITH FEW DEGREES OF FREEDOM

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ABSTRACT

We propose a method for the design of FIR filters that define higher density discrete wavelet transforms forming a dual tree. The design is based on a parameterization, using positive trigonometric polynomials, of the convex space of parameters characterizing the filters. Using only one or two degrees of freedom, we are able to design, via a search procedure, complex wavelets that are nearly analytic and have high regularity.

1. INTRODUCTION

The higher density discrete wavelet transform (HD-DWT), introduced in [6], is an overcomplete transform using the three-channel filter bank shown in Fig. 1. The signal in the first channel is processed recursively with an identical filter bank, giving a redundancy factor of three for the complete tree. Like other expansive transforms, the HD-DWT can perform better than the critically sampled DWT in diverse applications.

The dual-tree complex wavelet transform [7] employs two real DWTs that operate in parallel on the input signal, emulating a complex transform. This construction is possible also with HD-DWTs [2, 9]. In this paper, our aim is to obtain optimized dual-tree HD-DWTs, built with FIR filters. We consider only self-Hilbertian trees, for which the filters used in the second filter bank (identical in structure with the first, shown in Fig. 1) are

$$\begin{aligned} G_0(z) &= z^{-N} H_0(z^{-1}), \\ G_1(z) &= z^{-N} H_1(z^{-1}), \\ G_2(z) &= z^{-N} H_2(z^{-1}), \end{aligned} \tag{1}$$

i.e. their impulse responses are the reversed of those from the first filter bank. Thus, the filters will be almost linear phase and the resulting complex wavelets will be almost symmetric.

Let $\psi_{h,i}(t)$, $\psi_{g,i}(t)$, $i = 1,2$, be the wavelets generated by the filters $H_0(z)$, $H_i(z)$ and $G_0(z)$, $G_i(z)$, respectively. The two HD-DWTs form a dual-tree if $\psi_{g,i}(t)$ is approximately equal to the Hilbert transform of $\psi_{h,i}(t)$. Equivalently, the complex wavelet $\psi_i(t) = \psi_{h,i}(t) + j\psi_{g,i}(t)$ is approximately analytic and so its spectrum $\Psi_i(\Omega)$ is approximately zero for negative frequencies. For quantifying the analyticity property, we use an ∞-norm measure

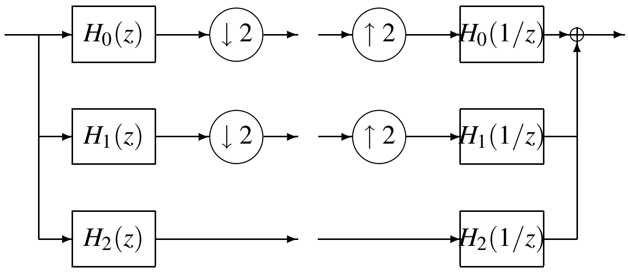

Figure 1: Filter bank used for the implementation of the HD-DWT.

$$E_{1,i} = \frac{\max_{\Omega<0} |\Psi_i(\Omega)|}{\max_{\Omega>0} |\Psi_i(\Omega)|} \tag{2}$$

or a least-squares measure

$$E_{2,i} = \frac{\int_{-\infty}^{0} |\Psi_i(\Omega)|^2 d\Omega}{\int_{0}^{\infty} |\Psi_i(\Omega)|^2 d\Omega}. \tag{3}$$

We optimize (2) and (3) for HD-DWTs that have few degrees of freedom (left by the perfect reconstruction and regularity constraints), using a characterization of the convex set of free parameters. A similar but simpler procedure has been used for critically sampled DWTs [3]. We are aware of only two other methods for designing dual-tree HD-DWTs. In [2], the case of two degrees of freedom is treated using a convex combination of minimal length product filters; by its nature, the search is only partial (a line in a 2D space). In [9], the filters of the second tree are not defined by (1) and the design involves no explicit optimization.

The content of this paper is as follows. In section 2, we review the relationship among the filters that generate a HD-DWT. In section 3, we present our design algorithm, that exploits the convexity of the admissible set of parameters that define the filters. Section 4 gives the experimental results (better than those from [2]) and examples of optimal filters.

2. PROPERTIES OF HD-DWT FILTERS

We consider filters $H_0(z), H_1(z), H_2(z)$ that are FIR of degree N and have the following properties. The filter $H_0(z)$ has K_0 zeros at $z = -1$, i.e.

$$H_0(z) = \left(\frac{1+z^{-1}}{2}\right)^{K_0} A(z), \tag{4}$$

This work is supported by the Academy of Finland, project No. 213462, Finnish Centre of Excellence program (2006 - 2011). B.Dumitrescu is on leave from Department of Automatic Control and Computers, "Politehnica" University of Bucharest.

the degree of $A(z)$ being $M = N - K_0$. The polynomial

$$R(z) = A(z)A(z^{-1}) = \sum_{k=-M}^{M} r_k z^{-k} \quad (5)$$

is symmetric and nonnegative on the unit circle.

The filters $H_1(z)$ and $H_2(z)$ have K_2 zeros at $z = 1$.

To satisfy perfect reconstruction (PR) constraints, the filter $H_1(z)$ is taken as

$$H_1(z) = z^{-\alpha}\left(\frac{1+z^{-1}}{2}\right)^{K_1}\left(\frac{1-z^{-1}}{2}\right)^{K_2} A(-z^{-1})(-z)^M, \quad (6)$$

where α is an integer such that $\alpha + K_2 + M$ is odd and $K_1 = K_0 - K_2 > 0$.

The only remaining PR constraint is

$$H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) + H_2(z)H_2(z^{-1}) = 2. \quad (7)$$

Hence, using (4) and (6), it results that the nonnegative polynomial

$$S(z) = H_2(z)H_2(z^{-1}) \quad (8)$$

must satisfy the condition

$$\begin{aligned} 2S(z) &= 2 - \left(\frac{z+2+z^{-1}}{4}\right)^{K_0} R(z) \quad (9) \\ &- \left(\frac{z+2+z^{-1}}{4}\right)^{K_1}\left(\frac{-z+2-z^{-1}}{4}\right)^{K_2} R(-z). \end{aligned}$$

The highest degree coefficient of $S(z)$ is

$$-\frac{1}{4^{K_0}}\left[1 + (-1)^{M+K_2}\right] r_M. \quad (10)$$

It results that the degree of $S(z)$ is N if $M + K_2$ is even and $N - 1$ if $M + K_2$ is odd.

Since $S(z)$ has K_2 zeros at $z = 1$, we can write

$$S(z) = \left(\frac{-z+2-z^{-1}}{4}\right)^{K_2} P(z), \quad (11)$$

where $P(z)$ is a symmetric polynomial positive on the unit circle of degree

$$N_p = \begin{cases} N - K_2, & \text{if } M + K_2 \text{ is even,} \\ N - K_2 - 1, & \text{otherwise.} \end{cases} \quad (12)$$

The above material is mostly drawn from [6]. What follows is a new view of PR relations. It is clear that the PR condition (9) is linear in the coefficients of $R(z)$ and $P(z)$. Denoting

$$\mathbf{r} = [r_0\ r_1\ \ldots\ r_M]^T,\ \ \mathbf{p} = [p_0\ p_1\ \ldots\ p_{N_p}]^T \quad (13)$$

the vectors of distinct coefficients of $R(z)$ and $P(z)$, the PR condition (9) can be written in the compact form

$$\mathbf{C}_1\mathbf{r} + \mathbf{C}_2\mathbf{p} = \mathbf{e} \Leftrightarrow [\mathbf{C}_1\ \mathbf{C}_2]\begin{bmatrix}\mathbf{r}\\ \mathbf{p}\end{bmatrix} = \mathbf{e} \Leftrightarrow \mathbf{C}\mathbf{x} = \mathbf{e}, \quad (14)$$

where the matrices $\mathbf{C}_1$, $\mathbf{C}_2$ can be built easily from the convolution matrices that appear in the polynomial multiplications from (9) and $\mathbf{e} = [2\ 0\ \ldots\ 0]^T$, $\mathbf{x} = [\mathbf{r}^T\ \mathbf{p}^T]^T$. The system (14) has $N_p + K_2 + 1$ equations (the number of distinct coefficients of $S(z)$) and $N_p + M + 2$ unknowns (the number of distinct coefficients of $R(z)$ and $P(z)$). Let us denote

$$L = M - K_2 + 1 = N - K_0 - K_2 + 1 \quad (15)$$

the number of degrees of freedom in the system (14), i.e. the difference between the number of variables and the number of equations in (14).

As argued in [6], we always take $K_1 = 1$ (otherwise the filter $H_1(z)$ has a very small norm and so its channel will contain only an insignificant amount of the energy of the input signal) and so we have $K_2 = K_0 - 1$. The values of K_0 and L determine the degrees of all filters that appear in this section.

As a side note, since $M + K_2$ and $M - K_2$ have the same parity, it follows from (15) that $M + K_2$ and L have opposite parities, and so (12) can be written

$$N_p = \begin{cases} N - K_0 + 1, & \text{if } L \text{ odd,} \\ N - K_0, & \text{if } L \text{ even.} \end{cases} \quad (16)$$

The filters designed in [6] have minimum length, i.e. $N = K_0 + K_2 - 1$, $L = 0$. In this case, the system (14) has a single solution and, moreover, the resulting polynomials $R(z)$ and $P(z)$ are nonnegative. (Note that in [6], this solution is presented in a different way.) If $L > 0$, then there are many (an infinite number of) solutions to the system (14) and only some of them correspond to nonnegative polynomials. We will see in the next section how to select appropriate solutions.

3. DESIGN IDEA AND ALGORITHM

An essential premise for the optimization of dual-tree HDDWT is the ability to characterize the set of nonnegative solutions $R(z)$, $P(z)$ to the system (14), for given K_0 and L. A first remark is that this set is convex. This follows from the fact that the set of polynomials that are nonnegative on the unit circle is convex and from the linearity of the constraint (14). Moreover, the set is bounded; this is a consequence of the PR constraint (7), which implies that $|H_0(e^{j\omega})|^2 + |H_2(e^{j\omega})|^2 \leq 2$, $\forall\omega$, which in turn puts bounds on $R(e^{j\omega})$ and $P(e^{j\omega})$.

We split the system $\mathbf{C}\mathbf{x} = \mathbf{e}$ (14) such as to isolate the free variables, for example by taking them as the first L elements of $\mathbf{x}$

$$[\mathbf{A}\ \tilde{\mathbf{C}}]\begin{bmatrix}\tilde{\mathbf{r}}\\ \tilde{\mathbf{x}}\end{bmatrix} = \mathbf{e}, \quad (17)$$

where $\tilde{\mathbf{r}} \in \mathbb{R}^L$ contains the free variables and $\tilde{\mathbf{x}} \in \mathbb{R}^{N_p+K_0}$ contains the dependent variables; the square matrix $\tilde{\mathbf{C}}$ is nonsingular. Given the first L elements of $\mathbf{x}$ (they actually belong to $\mathbf{r}$), the dependent variables can be computed by

$$\tilde{\mathbf{x}} = \tilde{\mathbf{C}}^{-1}(\mathbf{e} - \mathbf{A}\tilde{\mathbf{r}}). \quad (18)$$

Now, the problem to be solved is what values can take the elements of $\tilde{\mathbf{r}}$. Although they belong to a convex set, the optimization criteria (2) and (3) are not convex functions and so we cannot formulate a convex optimization problem. However, we can devise a search procedure, based on the following remark.

Let us assume that, for a given nonnegative integer ℓ, the values of the variables r_0, r_1, ..., $r_{\ell-1}$ are known. (We have reduced the problem to a section through the set of all admissible $\tilde{\mathbf{r}}$.) Then, we can solve the following optimization problem

$$\begin{aligned} r_{\ell,\min}(r_0,\ldots,r_{\ell-1}) = \min_{\tilde{\mathbf{r}}} \ & r_\ell \\ \text{s.t.} \ & \mathbf{C}_1\mathbf{r} + \mathbf{C}_2\mathbf{p} = \mathbf{e} \\ & r_0,\ldots,r_{\ell-1} = \text{given values} \\ & R(\omega) \geq 0,\ P(\omega) \geq 0,\ \forall\omega \end{aligned} \quad (19)$$

A similar maximization problem gives $r_{\ell,\max}(r_0,\ldots,r_{\ell-1})$ and thus, for given $r_0,\ldots,r_{\ell-1}$, all admissible values of r_ℓ belong to the interval $[r_{\ell,\min}, r_{\ell,\max}]$. The key point here is that, using the trace parameterization of nonnegative trigonometric polynomials [1, 4, 5], the problem (19) can be expressed as an SDP problem and solved efficiently.

We propose the following general approach to the optimization of dual-tree HD-DWT.

Step 0. Initial data: integers K_0 and L.

Step 1. Find $r_{0,\min}$ and $r_{0,\max}$ by solving (19) for $\ell = 0$. Take some values $r_0 \in [r_{0,\min}, r_{0,\max}]$.

Step 2. For each r_0 from the previous step, find $r_{1,\min}(r_0)$ and $r_{1,\max}(r_0)$ by solving (19) for $\ell = 1$; take values $r_1 \in [r_{1,\min}(r_0), r_{1,\max}(r_0)]$.

⋮

Step L. For each set of values r_0, r_1, ..., r_{L-2} generated in the previous steps, find $r_{L-1,\min}(r_0,\ldots,r_{L-2})$ and $r_{L-1,\max}(r_0,\ldots,r_{L-2})$ by solving (19) for $\ell = L-1$; take values $r_{L-1} \in [r_{L-1,\min}(r_0,\ldots,r_{L-2}), r_{L-1,\max}(r_0,\ldots,r_{L-2})]$.

Step L + 1. For each set of values r_0, ..., r_{L-1} generated in the previous steps, compute $\tilde{\mathbf{x}}$ from (18) and thus all the coefficients of $R(z)$, $P(z)$ are available.

Compute all spectral factors $A(z)$ of $R(z)$ and compute $H_0(z)$ and $H_1(z)$ from (4) and (6), respectively.

Compute all spectral factors $B(z)$ of $P(z)$ and put

$$H_2(z) = \left(\frac{1+z^{-1}}{2}\right)^{K_2} B(z). \quad (20)$$

For each combination of filters $H_0(z)$, $H_1(z)$, $H_2(z)$ compute the wavelets $\psi_{h,i}(t)$, $\psi_{g,i}(t)$, $i = 1,2$, and the analyticity measures (2) and (3).

Output: the combination of filters that gives the minimum values of $\max(E_{1,1}, E_{1,2})$ (for the ∞-norm measure) and $\max(E_{2,1}, E_{2,2})$ (for the least-squares measure).

In the actual implementation for $L = 1$ and $L = 2$, we have used uniform grids covering the intervals $[r_{0,\min}, r_{0,\max}]$ (typically few hundred points) and $[r_{1,\min}(r_0), r_{1,\max}(r_0)]$ (typically one hundred points). A second run of the algorithm, on a fine grid around the best values of r_0 (for $L = 1$) or r_0 and r_1 (for $L = 2$), was used to compute good approximations of the optimal filters. To speed up the algorithm, we have computed the spectral factors of $P(z)$ (and thus $H_2(z)$) only if the values of the analyticity measures for the first complex wavelet (generated by $H_0(z)$ and $H_1(z)$) were good, i.e. less than a predefined threshold.

A different approach (that we have not yet experimented) would be to generate only one set r_0, ..., r_{L-1} at a time. This approach is somewhat slower, since there are more problems (19) to solve, but not significantly so, since the computation time for (19) is relatively small. This can open the way to other optimization procedures rather than the brute force search we have used. For example, randomized search or a genetic algorithm search are possible ways to deal with the case $L \geq 3$, for which the complexity of the grid search can become too high.

Finally, here are some considerations regarding the accuracy of the algorithm. A cause of inaccuracy may be the spectral factorization, especially if there are roots on the unit circle. In our algorithm, the polynomials to be factorized are $R(z)$ and $P(z)$, which typically have no roots on the unit circle. Another potential numerical trouble source may be the positivity itself of the computed $R(z)$ and $P(z)$, whose lack prevents the existence of a spectral factorization. It appears that, for the filter degrees we have experimented ($K_0 \leq 8$), the system (14) is sufficiently well-conditioned and the SDP problem (19) can be accurately solved, hence the computed polynomials are indeed nonnegative for practically all admissible values of the parameters.

K_0	r_0	$E_1(\%)$	r_0	$E_2(\%)$
2	2.4769	38.8	2.4218	13.8
3	3.8450	17.7	3.9631	2.31
4	12.264	42.7	10.608	10.5
5	35.704	18.9	34.237	1.97
6	124.80	10.0	124.30	1.29
7	267.46	14.6	281.82	2.02
8	1469.5	9.57	1382.4	0.543

Table 1: Optimal values of the analyticity measures for $L = 1$ degree of freedom.

K_0	r_0	r_1	$E_1(\%)$	r_0	r_1	$E_2(\%)$
2	1.981	0.0795	16.5	1.928	0.0926	2.77
3	4.665	−1.204	12.4	4.385	−0.9896	1.18
4	7.249	−1.419	5.55	7.249	−1.414	0.275
5	31.29	−18.79	5.70	30.45	−18.13	0.418
6	62.04	−36.47	3.22	61.22	−35.75	0.097
7	122.4	−66.68	6.35	124.7	−68.33	0.448
8	1381	−1143	6.65	1377	−1139	0.377

Table 2: Optimal values of the analyticity measures for $L = 2$ degrees of freedom.

4. EXPERIMENTAL RESULTS

We have implemented the design algorithm from the previous section in Matlab and run it on a PC. The SDP problem (19) was solved using SeDuMi [8]. The accuracy of the solver is set to 10^{-12}. We have performed our search for one and two degrees of freedom and values of K_0 from 2 to 8. Even for the most time consuming search ($L = 2$, $K_0 = 8$), an overnight calculation was sufficient (less than 8 hours, actually).

For one degree of freedom ($L = 1$), the degree of the filters is $N = 2K_0 - 1$ and the only parameter in the design is r_0. (Note that $\alpha = 1$ in (6), hence $H_1(z)$ is delayed with one sample with respect to $H_0(z)$.) The optimal results are given in Table 1. The optimal values of the E_1 criterion are relatively bad. However, for the E_2 analyticity measure, some of

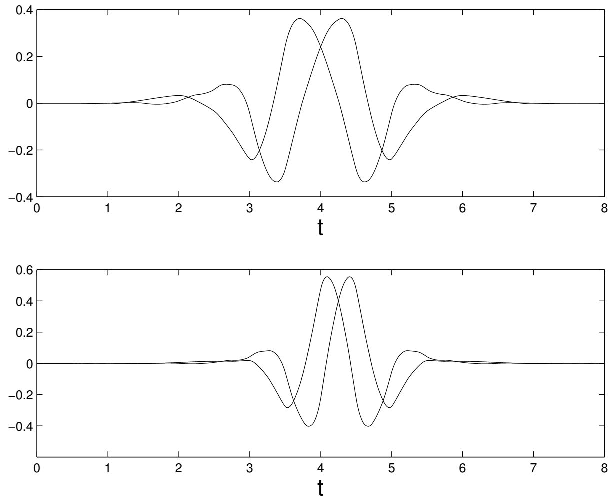

Figure 2: Wavelets $\psi_{h,1}(t)$, $\psi_{g,1}(t)$ (up) and $\psi_{h,2}(t)$, $\psi_{g,2}(t)$ (down), for the E_2-optimal filters with $L = 2$, $K_0 = 4$.

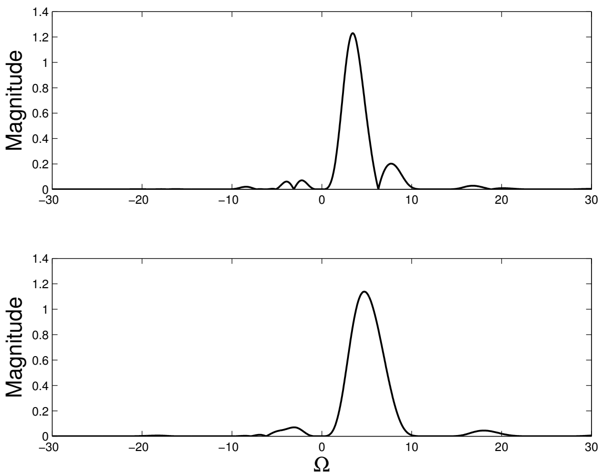

Figure 3: Spectrum of the complex wavelets $\psi_1(t)$ (up) and $\psi_2(t)$(down), for the E_2-optimal filters with $L = 2$, $K_0 = 4$.

the values are satisfactory, especially for $K_0 = 8$ and $K_0 = 6$.

For two degrees of freedom ($L = 2$), the degree of the filters is $N = 2K_0$ and the parameters are r_0 and r_1. (The degree of $H_2(z)$ is $N-1$, due to (16).) The optimal results are displayed in Table 2. All the optimal values of E_2 are better than the corresponding values reported in [2] (where E_1 is not considered). For $K_0 \geq 4$, the values are less than 0.5%. The coefficients of the E_2-optimal filters for $K_0 = 4$ are given in Table 3. The corresponding wavelets are given in Fig. 2, and the spectra of the complex wavelets in Fig. 3. In both figures, the upper part corresponds to the wavelets generated by the filters from the first and second channel of the filter bank from Fig. 1, and the lower part to wavelets generated by the filters from the first and third channel. One can remark the good approximation of the analyticity property. For $K_0 = 6$, the same information as above is given in Table 4 and Figs. 4 and 5.

5. CONCLUSION

We have presented a method for designing dual-tree higher density discrete wavelet transform. Allowing one or two free parameters (the other variables being determined by perfect reconstruction and regularity constraints), we have been able to find HD-DWT filters that, together with their reversed versions, form complex wavelets that are approximately analytic. The method is efficient due to the parameterization of the convex set of admissible coefficients.

Further work will be dedicated to the design of dual-tree HD-DWT with three or more degrees of freedom, where improved search methods may be necessary.

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H_0	H_1	H_2
−0.053607650180478	0.018685262199982	0.007890466250877
−0.071333162453333	0.019599595188741	0.009813252147803
0.287927268940389	−0.076756822228470	−0.024763820696285
0.735917577096619	−0.180093747766496	−0.197831653510008
0.532687388432642	0.327842203334109	0.489824880660473
0.024751437332038	0.124612014670132	−0.347745087443976
−0.078585488205987	−0.323378293486099	0.027048473785034
0.017770929211224	0.035882137907623	0.035763488806082
0.018685262199982	0.053607650180478	

Table 3: Coefficients of E_2-optimal filters, $L = 2$, $K_0 = 4$.

H_0	H_1	H_2
0.011687955709184	−0.003568879741754	0.000866103658883
0.009216241237504	−0.002373104825526	0.000214011171626
−0.090423159123521	0.026768968989524	−0.009082914397474
−0.101058956717152	0.024185846627524	−0.007372170013552
0.313978563762228	−0.070352598117225	0.018933326455294
0.743179629297713	−0.171504481850303	0.169515860776227
0.518632784649025	0.313554322291325	−0.452548260888904
0.033573980507397	0.089007477642637	0.387079528423482
−0.067577903225681	−0.340193587892848	−0.068601260428724
0.026960541519068	0.074843932586531	−0.055389072843445
0.024377419157068	0.085479730180161	0.010433005600948
−0.004764654657982	−0.014159670180864	0.005951842485639
−0.003568879741754	−0.011687955709184	

Table 4: Coefficients of E_2-optimal filters, $L = 2$, $K_0 = 6$.

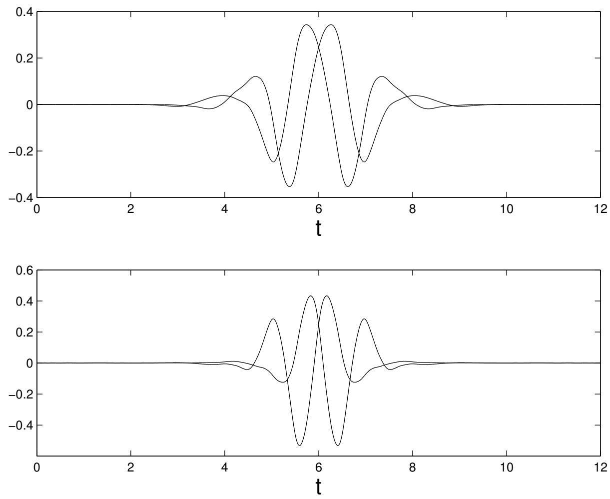

Figure 4: Wavelets $\psi_{h,1}(t)$, $\psi_{g,1}(t)$ (up) and $\psi_{h,2}(t)$, $\psi_{g,2}(t)$ (down), for the E_2-optimal filters with $L = 2$, $K_0 = 6$.

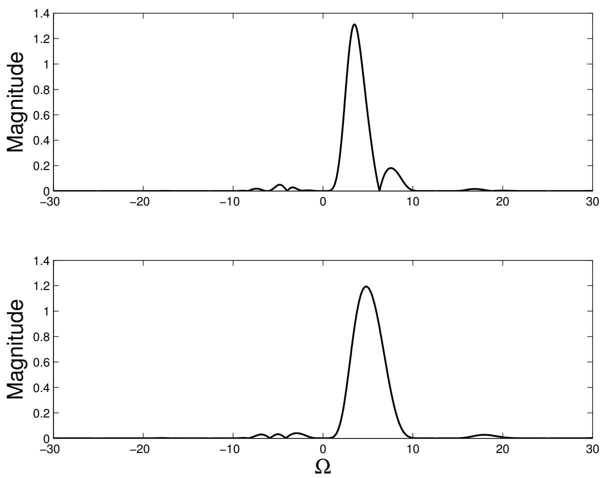

Figure 5: Spectrum of the complex wavelets $\psi_1(t)$ (up) and $\psi_2(t)$(down), for the E_2-optimal filters with $L = 2$, $K_0 = 6$.