Localization of Universal Problems. Local Colimits

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Abstract. The notion of the root of the category, which is a minimal (in a precise sense) weakly coreflective subcategory, is introduced in view of defining 'local' solutions of universal problems: If U is a functor from $\mathbf{C}'$ to $\mathbf{C}$ and c an object of $\mathbf{C}$, the root of the comma-category $c|U$ is called a U -universal root generated by c ; when it exists, it is unique (up to isomorphism) and determines a particular form of the locally free diagrams defined by Guitart and Lair. In this case, the analogue of an adjoint functor is an adjoint-root functor of U , taking its values in the category of pro-objects of $\mathbf{C}'$. Local colimits are obtained if U is the insertion from a category into its category of ind-objects; they generalize Diers' multicolimits. Applications to posets and Galois theory are given.

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Category Theory has been applied in a series of papers by Ehresmann and Vanbremeersch (e.g., [8]) to model complex systems such as biosystems, neural systems or social systems. For instance, in the category modelling a neural system, the internal representation of a physical object is described as the colimit of a diagram of receptors in the visual areas it activates. But there are several situations in which it would be useful to have a 'localized' notion of a colimit, for instance, to model the way an ambiguous object (which has two alternative readings) is memorized.

The locally free diagrams and locally colimit diagrams, introduced by Guitart and Lair in [12], could be used, but they are often too large and, anyway, are not uniquely determined. Particular locally free diagrams are introduced here, using the notion of the root of a category, which characterizes a minimal (in a precise sense, cf. Section 2) weakly coreflective subcategory.

If U is a functor from $\mathbf{C}'$ to $\mathbf{C}$, a U -universal root generated by an object c of $\mathbf{C}$ is defined as a root of the comma category $c|U$; when it exists, it is unique up to an isomorphism. If each object generates a U -universal root, these roots are connected by an 'adjoint-root' functor of U , which is a functor from $\mathbf{C}$ to the category $\text{Pro } \mathbf{C}'$ of pro-objects of $\mathbf{C}'$. This 'localization' of the solutions of universal problems is applied to define local colimits (which, in the discrete case, give back Diers multicolimits [4]). Applications are given to partially ordered sets and to Galois theory.

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NOTATION. A category is denoted by a boldface letter, say $\mathbf{B}$, the class of its objects by $|\mathbf{B}|$, and a generic object by a small italic letter b . We say that $\mathbf{B}$ is a poset considered as a category iff it has at most one morphism between each pair of objects; indeed, in this case, the class of its objects becomes a p(artially) o(rdered) set (in the usual sense) when it is equipped with the partial order:

$$b' \leq b \text{ iff there exists a morphism from } b' \text{ to } b;$$

this poset will be denoted by the same italic (not boldface) letter B .

Proposition $x(y)$ refers to Proposition x of Section y .

1. Category of Cylinders of a Category

The first step consists in strengthening the notion of an initial subcategory into that of a corefract. For this, the main tool will be cylinders between subcategories of a category, defined as follows:

1.1. CYLINDERS BETWEEN SUBCATEGORIES

DEFINITION. Let $\mathbf{C}$ and $\mathbf{C}'$ be two subcategories of a category $\mathbf{A}$. An $\mathbf{A}$ -cylinder from $\mathbf{C}$ to $\mathbf{C}'$ is a couple (F, f) satisfying the following conditions:

- (1) F is a map from the class $|\mathbf{C}|$ of objects of $\mathbf{C}$ to $|\mathbf{C}'|$ and f is a map associating to each object c of $\mathbf{C}$ a morphism fc from Fc to c in $\mathbf{A}$.
- (2) For each morphism $x : d \rightarrow c$ in $\mathbf{C}$, there exists at least one $x' : Fd \rightarrow Fc$ in $\mathbf{C}'$ such that $x.f d = f c.x'$.

If (F, f) is an $\mathbf{A}$ -cylinder from $\mathbf{C}$ to $\mathbf{C}'$ and (F', f') an $\mathbf{A}$ -cylinder from $\mathbf{C}'$ to $\mathbf{C}''$, their *composite* is the cylinder $(F'F, f.f'F)$ from $\mathbf{C}$ to $\mathbf{C}''$. With this composition, we get the *category of $\mathbf{A}$ -cylinders*, $\text{cyl}(\mathbf{A})$, whose objects are the subcategories of $\mathbf{A}$ and morphisms are the $\mathbf{A}$ -cylinders between them.

PROPOSITION 1. An $\mathbf{A}$ -cylinder (F, f) from $\mathbf{C}$ to $\mathbf{C}'$ is invertible in $\text{cyl}(\mathbf{A})$ iff F is a 1-1 map from $|\mathbf{C}|$ onto $|\mathbf{C}'|$ and if fc is an isomorphism of $\mathbf{A}$ for each object c of $\mathbf{C}$. Then F extends into a unique isomorphism F' from $\mathbf{C}$ to $\mathbf{C}'$.

Indeed, the inverse of (F, f) is (F^{-1}, h) where

$$hc' = (fF^{-1}c')^{-1} \quad \text{for each object } c' \text{ of } \mathbf{C}'.$$

And F extends into the isomorphism F' which sends the morphism x from d to c in $\mathbf{C}$ on the morphism $fc^{-1}.x.fd$. Then f defines an equivalence from $\text{ins } \mathbf{C}' \circ F'$ to $\text{ins } \mathbf{C}$, where $\text{ins } \mathbf{C}$ is the insertion of $\mathbf{C}$ into $\mathbf{A}$.

EXAMPLE. Let $\mathbf{A}$ be a poset considered as a category, $\mathbf{C}$ and $\mathbf{C}'$ two subposets of $\mathbf{A}$; the $\mathbf{A}$ -cylinders (R, r) from $\mathbf{C}$ to $\mathbf{C}'$ correspond to *contractions* from $\mathbf{C}$ to $\mathbf{C}'$, i.e., to order-preserving maps R such that $Rc \leq c$ for each c in $\mathbf{C}$; so there exist no invertible cylinders except identities.

1.2. COREFRACT OF A CATEGORY

DEFINITION. A subcategory $\mathbf{B}$ of $\mathbf{A}$ is called a *corefract* of $\mathbf{A}$ if there exists an $\mathbf{A}$ -cylinder (R, r) from $\mathbf{A}$ to $\mathbf{B}$ such that $rb = b$ for each object b of $\mathbf{B}$; in this case, (R, r) is called a *corefraction* from $\mathbf{A}$ to $\mathbf{B}$.

PROPOSITION 2. Let $\mathbf{B}$ be a subcategory of $\mathbf{A}$. The following conditions are equivalent:

- (1) $\mathbf{B}$ is a corefract of $\mathbf{A}$.
- (2) $\mathbf{B}$ is a full subcategory of $\mathbf{A}$, which is weakly coreflective, i.e., for each object a of $\mathbf{A}$, there exists a morphism ra from (an object of) $\mathbf{B}$ to a such that each morphism y from $\mathbf{B}$ to a has a factorization $ra.y'$ with y' in $\mathbf{B}$.

Proof. (1) Let (R, r) be a corefraction from $\mathbf{A}$ to $\mathbf{B}$, and a an object of $\mathbf{A}$. There exists at least one morphism from $\mathbf{B}$ to a , namely, $ra : Ra \rightarrow a$. If y is another morphism from an object b of $\mathbf{B}$ to a , we have $rb = b$ (by definition of a corefraction) and there exists y' in $\mathbf{B}$ such that

$$ra.y' = y.rb = y$$

(by definition of a cylinder). Hence, y factorizes through ra , so that ra is a weakly terminal object of the comma-category $\mathbf{B}|a$ over $\mathbf{A}$. And $\mathbf{B}$ is full because, if a is also in $\mathbf{B}$, then $ra = a$, which implies $y' = y$ in $\mathbf{B}$.

(2) If condition (2) is satisfied, we get a corefraction (R, r) from $\mathbf{A}$ to $\mathbf{B}$ by taking for $ra : Ra \rightarrow a$ any weakly terminal object of $\mathbf{B}|a$, if a is not in $\mathbf{B}$, and $ra = a = Ra$, if a is in $\mathbf{B}$. This defines a cylinder because, for each morphism $x : a' \rightarrow a$ in $\mathbf{A}$, there exists at least x' in $\mathbf{B}$ such that $x.ra' = ra.x'$: if a is not in $\mathbf{B}$, x' comes from the factorization of $x.ra'$ by ra , and if a is in $\mathbf{B}$, we have $ra = a$ and $x' = x.ra'$ is in the full subcategory $\mathbf{B}$. $\square$

COROLLARY. Let $\mathbf{B}$ be a poset considered as a category; it is a corefract of $\mathbf{A}$ iff it is a full coreflective subcategory; in this case, there is exactly one corefraction from $\mathbf{A}$ to $\mathbf{B}$.

Indeed, the first assertion results from the proposition, since $\mathbf{B}$, having at most one morphism between two objects, is a coreflective subcategory as soon as it is weakly coreflective. And if (R, r) and (R', r') are two refractions, ra and $r'a$ are connected by two opposite morphisms of $\mathbf{B}$, so that these morphisms are identities and $ra = r'a$.

It follows from Proposition 2 that a corefract $\mathbf{B}$ of $\mathbf{A}$ is a fortiori an initial subcategory, but in a stronger sense, namely, each comma-category $\mathbf{B}|a$ over $\mathbf{A}$ is not only connected, but has a weakly terminal object: the consequence is that the ‘zig-zag’ between two of its objects is replaced by a unique arrow or a cospan.

2. Root of a Category

The root of a category $\mathbf{A}$ characterizes a ‘minimal’ full weakly coreflective subcategory when it exists.

DEFINITION. A category $\mathbf{B}$ is called a *root* if each $\mathbf{B}$ -cylinder from $\mathbf{B}$ to $\mathbf{B}$ is invertible in $\text{cyl } \mathbf{B}$. It is called a *root of the category $\mathbf{A}$* if:

- (1) $\mathbf{B}$ is a root,
- (2) $\mathbf{B}$ is a full subcategory of $\mathbf{A}$,
- (3) There exists at least one $\mathbf{A}$ -cylinder from $\mathbf{A}$ to $\mathbf{B}$.

A category is a root iff it is a root of itself.

PROPOSITION 1. *Let $\mathbf{B}$ be a root of $\mathbf{A}$. Then:*

- (1) *If (R, r) is an $\mathbf{A}$ -cylinder from $\mathbf{A}$ to $\mathbf{B}$, the map R has a restriction which is a 1-1 map from $|\mathbf{B}|$ onto itself and rb is an isomorphism of $\mathbf{B}$ for each object b of $\mathbf{B}$.*
- (2) *$\mathbf{B}$ is a corefract of $\mathbf{A}$, minimal among all its refracts; and $\mathbf{B}$ has no nontrivial corefract.*

Proof. (1) The restrictions of R and r to $|\mathbf{B}|$ define an $\mathbf{A}$ -cylinder from $\mathbf{B}$ to itself; if $\mathbf{B}$ is full, it is also a $\mathbf{B}$ -cylinder, which must be invertible since $\mathbf{B}$ is a root. Hence, by Proposition 1(1), R has a restriction which is a 1-1 map from $|\mathbf{B}|$ onto itself and rb is an isomorphism of $\mathbf{B}$ for each object b of $\mathbf{B}$.

(2) As $\mathbf{B}$ is full, to prove that it is a corefract of $\mathbf{A}$, it suffices to prove that it is weakly coreflective in $\mathbf{A}$ (by Proposition 2(1)), i.e., that ra , for each object a of $\mathbf{A}$, is a weakly terminal object of $\mathbf{B}|a$. Indeed, let y be a morphism from b in $\mathbf{B}$ to a . By definition of the cylinder (R, r) , there exists y' in $\mathbf{B}$ such that

$$ra.y' = y.rb,$$

and, by Part 1, rb is invertible in $\mathbf{B}$. It follows that y factorizes under the form $y = ra.(y'.rb^{-1})$, with $y'.rb^{-1}$ in $\mathbf{B}$.

(3) To prove that $\mathbf{B}$ is a minimal corefract of $\mathbf{A}$, let us suppose that $\mathbf{D}$ is a corefract of $\mathbf{A}$ included in $\mathbf{B}$. As $\mathbf{B}$ is full in $\mathbf{A}$, a corefraction from $\mathbf{A}$ to $\mathbf{D}$ has a restriction (R', r') which is a corefraction from $\mathbf{B}$ to $\mathbf{D}$ and can be considered as a $\mathbf{B}$ -cylinder from $\mathbf{B}$ to $\mathbf{B}$. Such a cylinder is invertible, and thus, by Proposition 1(1), the map R' (from $|\mathbf{B}|$ to $|\mathbf{D}|$) is a 1-1 map from $|\mathbf{B}|$ onto $|\mathbf{B}|$. As $\mathbf{D}$ is full, it follows that $\mathbf{D} = \mathbf{B}$. So $\mathbf{B}$ is a minimal corefract of $\mathbf{A}$ and it has no nontrivial corefract. $\square$

COROLLARY. *A subcategory $\mathbf{B}$ of $\mathbf{A}$ is a root of $\mathbf{A}$ iff $\mathbf{B}$ is a root, which is a corefract of $\mathbf{A}$.*

PROPOSITION 2. *If $\mathbf{A}$ has two roots $\mathbf{B}$ and $\mathbf{B}'$, these roots are isomorphic both in the category of categories and in the category of $\mathbf{A}$ -cylinders.*

Proof. By definition of a root, there exist cylinders (R, r) from $\mathbf{A}$ to $\mathbf{B}$ and (R', r') from $\mathbf{A}$ to $\mathbf{B}'$. These have cylinders (R_1, r_1) from $\mathbf{B}'$ to $\mathbf{B}$ and (R'_1, r'_1) from $\mathbf{B}$ to $\mathbf{B}'$, respectively, for restrictions.

(1) The composite $(R_1, r_1).(R'_1, r'_1)$ in $\text{cyl}(\mathbf{A})$ is an $\mathbf{A}$ -cylinder (F, f) from $\mathbf{B}$ of $\mathbf{B}$ and, the root $\mathbf{B}$ being full in $\mathbf{A}$, it is also a $\mathbf{B}$ -cylinder; hence it is invertible, so that (R_1, r_1) admits $(R'_1, r'_1)(F, f)^{-1}$ as a right inverse in $\text{cyl}(\mathbf{A})$. For the same reason the composite $(R'_1, r'_1).(R_1, r_1)$ is a $\mathbf{B}'$ -cylinder from the root $\mathbf{B}'$ to itself, hence it is also invertible, and (R_1, r_1) has also a left inverse. It follows that (R_1, r_1) is invertible in $\text{cyl}(\mathbf{A})$, and the same is true for (R'_1, r'_1) .

(2) (R_1, r_1) being an invertible $\mathbf{A}$ -cylinder from $\mathbf{B}'$ to $\mathbf{B}$, Proposition 1(1) ensures that the 1-1 map R_1 from $|\mathbf{B}'|$ onto $|\mathbf{B}|$ extends into an isomorphism from the category $\mathbf{B}'$ onto the category $\mathbf{B}$. $\square$

EXAMPLES. (1) A category with an initial object admits a root formed by this object. The category $\mathbf{A}$ has a root $\mathbf{B}$ which is a discrete category iff $\mathbf{B}$ is an multi-initial subset of $\mathbf{A}$, i.e., each object a of $\mathbf{A}$ is the codomain of exactly one morphism with its domain in $\mathbf{B}$.

(2) A coproduct of roots is a root. A groupoid $\mathbf{B}$ is a root iff it is a sum of groups, because otherwise it would admit noninvertible cylinders (F, f) where F is the identity except that it sends a particular b to another object b' of the component of b .

(3) A category equivalent to a root admits a root. A groupoid $\mathbf{A}$ always has a root $\mathbf{B}$ (called its skeleton in Mac Lane [15]) constructed by selecting an object b in each component and by taking the groupoid $\mathbf{B}$ sum of the groups $\text{Hom}(b, b)$; and it is equivalent to this root.

(4) Let $\mathbf{B}$ be a root. Though it has no root other than itself, it may have a proper sub-category $\mathbf{D}$ which is a root. In particular, this will be the case, if $\mathbf{D}$ contains each morphism of $\mathbf{B}$ whose codomain is in $\mathbf{D}$; indeed, then a $\mathbf{D}$ -cylinder from $\mathbf{D}$ to $\mathbf{D}$ is extended into a $\mathbf{B}$ -cylinder (R, r) from $\mathbf{B}$ to $\mathbf{B}$ by taking $Rb = b = rb$ outside $\mathbf{D}$; as this cylinder is invertible, its restriction to the full subcategory $\mathbf{D}$ is also invertible.

3. Application to Posets and Galois Categories

Let A be a poset and B a subposet; we denote by $\mathbf{A}$ and $\mathbf{B}$ these posets considered as categories. Then $\mathbf{B}$ is a corefract of $\mathbf{A}$ (we also say that B is a corefract of A) iff, for each a in A , the set of elements of B smaller than a has a greatest element Ra , whence the unique (cf. Corollary, Proposition 2(1)) corefraction R from $\mathbf{A}$ to $\mathbf{B}$. And $\mathbf{B}$ is the root of $\mathbf{A}$ (we also say that B is the root of A) iff it is a corefract of $\mathbf{A}$ such that there exists no contraction from B to B . In particular, B is a root (of itself) iff it admits no contraction from B to B .

EXAMPLES. (1) The poset A has a discrete root iff, for each a in A , the set of elements less than a has a first element Ra , and then the root is the set of all these Ra 's.

(2) The ordered set $\mathbf{Z}$ of integers is an example of a poset which has no root.

3.1. CHARACTERIZATION OF ROOTS WHICH ARE FINITE ORDERS

PROPOSITION 1. *A finite poset B is a root iff no element of B has a predecessor in the order.*

Proof. (i) Suppose that a particular element a of B admits a predecessor a' : it means that $a' < a$ and $b \leq a'$ for each $b < a$. In this case, we define a contraction R on the poset B as follows:

$$Ra = a' \quad \text{and} \quad Rb = b \quad \text{for each } b \text{ other than } a.$$

Thus, B is not a root. (Here we have not used the finiteness condition.)

(ii) Conversely, suppose that there are no predecessors and let R be a contraction. If B is finite, it admits a set M of minimal elements m ; for such an m we must have $Rm = m$. Now the subposet defined by $B \setminus M$ is also finite, hence it admits a set M' of minimal elements. Let us prove that $Rm' = m'$ for each m' of M' . Indeed, each m strictly less than m' is in M (by definition of M') so that $Rm = m$. Now R preserving the order, we have $m = Rm \leq Rm'$. Thus, if Rm' is not equal to m' , it would be a predecessor of m' , contradicting the hypothesis. By induction we construct a finite sequence (M_n) of subsets such that M_n is the set of minimal elements of the poset obtained by subtracting from B the union of all the M_k with $k < n$. If B is finite, this sequence is finite. As done above, we prove that, if R is the identity on all the M_k with indices $k < n$, it is also the identity on M_n . Thus by induction, R is the identity on B , so that B is a root. $\square$

PROPOSITION 2. *Let $\mathbf{A}$ be a category and B a finite poset; if this poset considered as a category $\mathbf{B}$ is a subcategory of $\mathbf{A}$, then the following conditions are equivalent:*

- (i) $\mathbf{B}$ is a root of $\mathbf{A}$,

- (ii) $\mathbf{B}$ is a corefract of $\mathbf{A}$, and no element of B has a predecessor in the poset,
- (iii) $\mathbf{B}$ is a minimal full coreflective subcategory of $\mathbf{A}$.

Proof. (i) is equivalent to (ii). Indeed, Proposition 1 asserts that the second part of condition (ii) is equivalent to, say, that $\mathbf{B}$ is a root, so that it is a root of $\mathbf{A}$ iff it is a corefract of $\mathbf{A}$ (Corollary, Proposition 1(2)). In this case, it follows from Proposition 1(2) that $\mathbf{B}$ is a minimal corefract of $\mathbf{A}$, and, from its corollary, that it is a corefract iff it is a full coreflective subcategory, whence condition (iii).

It remains only to prove that (iii) implies (ii). Let $\mathbf{B}$ be a minimal full coreflective subcategory (i.e., a minimal corefract) of $\mathbf{A}$. If an element b has a predecessor b' in B , let us consider the subposet $\mathbf{C}$ determined by the elements of B other than b . We are going to show that $\mathbf{C}$ is also a corefract of $\mathbf{A}$, which will contradict that $\mathbf{B}$ is a ‘minimal’ corefract. Indeed, we define a corefraction (R', r') from $\mathbf{B}$ to $\mathbf{C}$ such that $R'b = b'$ (cf. Proposition 1) and $r'b$ is the unique morphism in $\mathbf{B}$ from b' to b . And, by composing the unique corefraction (R, r) from $\mathbf{A}$ to $\mathbf{B}$ with (R', r') , we obtain a corefraction $(R'R, r'')$ from $\mathbf{A}$ to $\mathbf{C}$. $\square$

COROLLARY. *Let A be a poset and B a subposet. If B is finite, then the following conditions are equivalent:*

- (i) B is a root of A .
- (ii) No element of B has a predecessor in B , and for each a in A the set of elements b of B smaller than a has a greatest element.
- (iii) B is a minimal corefract of A .

Indeed, this follows from the proposition since B is a corefract of A iff the set of b in B with $b \leq a$ has a maximum, for each a in A .

3.2. GALOIS CATEGORIES

The following example is important for its applications to Galois theory. Let $\mathbf{E}$ be a poset considered as a category, and F an order-preserving map from E to the lattice of subgroups of a group G . We denote by $\mathbf{B}$ the fibration associated to F (considered as a functor from $\mathbf{E}$ to $\mathbf{Cat}$): it admits E as its set of objects; the morphisms are triples (e, x, e') where $e' \leq e$ and x is in Fe ; the composition is given by:

$$(e^\circ, x', e).(e, x, e') = (e^\circ, x'.x, e').$$

PROPOSITION 3. *Let $\mathbf{B}$ be a fibration associated to an order-preserving map F from a poset E to the lattice of subgroups of a group. If F is 1-1, then $\mathbf{B}$ is a root.*

Proof. Let (R, r) be a cylinder from $\mathbf{B}$ to $\mathbf{B}$ and e an object. We have $Re \leq e$, so that the group Fre is a subgroup of Fe , and re is of the form (e, s_e, Re) for some

s_e in Fe . The definition of a cylinder implies that, for each x in Fe , there exists x' in FRe such that

$$re.(Re, x', Re) = (e, x, e).re, \quad \text{or} \quad (e, s_e.x', Re) = (e, x.s_e, Re),$$

$$\text{whence } x' = s_e^{-1}.x.s_e, \quad \text{and} \quad x = s_e.x'.s_e^{-1}.$$

In particular, for $x = s_e$ we get $x' = s_e$, so that s_e belongs to the subgroup FRe . Thus, the map sending x to x' is the internal automorphism of Fe associated to s_e , and the subgroup FRe is identical with Fe . If F is 1-1, it follows that $Re = e$ and that the cylinder is invertible. $\square$

COROLLARY. *If $\mathbf{B}$ is a Galois category, then it admits a root and is equivalent to this root.*

Indeed, let us recall (Grothendieck [10], Barr [2]) that a Galois category is a category which is equivalent to the category of continuous actions of a profinite group on a finite discrete space. Such a category admits an equivalent subcategory which is isomorphic to a fibration of the above sort (satisfying supplementary conditions) (cf. Diers [5], Lair [13]). The proposition ensures that this fibration is a root, and so is any equivalent category.

4. Universal Root Generated by an Object

Let U be a functor from $\mathbf{C}'$ to $\mathbf{C}$. The notion of the root of a category allows us to ‘localize’ the notion of a U -universal arrow while at the same time keeping its uniqueness (up to isomorphism).

For each object c of $\mathbf{C}$, let $c|U$ be the comma-category whose objects are the U -morphisms of ‘source’ c , i.e., the pairs (z, c') where c' is an object of $\mathbf{C}'$ and $z : c \rightarrow Uc'$ is a morphism in $\mathbf{C}$; a morphism from (z', e') to (z, c') is defined by $y' : e' \rightarrow c'$ in $\mathbf{C}'$ such that $Uy'.z' = z$. There exists a faithful functor Sc from $c|U$ to $\mathbf{C}'$ which sends (z, c') to c' and $y' : (z', e') \rightarrow (z, c')$ to y' .

DEFINITION. We say that $\mathbf{B}$ is a U -universal root generated by c if $\mathbf{B}$ is a root of the category $c|U$.

Thus, $\mathbf{B}$ is a U -universal root generated by c if $\mathbf{B}$ is a root (i.e., each $\mathbf{B}$ -cylinder from $\mathbf{B}$ to $\mathbf{B}$ is invertible), which is a corefract of $c|U$. By Proposition 2(1), this second condition means that $\mathbf{B}$ is a full weakly coreflective subcategory of $c|U$, i.e., that it is full and satisfies:

If c' is an object of $\mathbf{C}'$ and $z : c \rightarrow Uc'$ is a morphism in $\mathbf{C}$, then z has a weakly terminal factorization $z = Uy'.z'$, where (z', e') is a U -morphism in $\mathbf{B}$ and $y' : e' \rightarrow c'$ is a morphism in $\mathbf{C}'$; i.e., for another factorization $z = Uy''.z''$, where (z'', e'') is a U -morphism in $\mathbf{B}$ and $y'' : e'' \rightarrow c'$, there exists at least one $m : e'' \rightarrow e'$ in $\mathbf{C}'$ such that

$$y'' = y'.m \quad \text{and} \quad Um.z'' = z'.$$

From Proposition 2(3), we deduce:

PROPOSITION 1. *Let $\mathbf{B}$ be a finite poset considered as a category; $\mathbf{B}$ is a U -universal root generated by c iff $c|U$ admits $\mathbf{B}$ as a minimal full coreflective subcategory.*

PROPOSITION 2. *Two U -universal roots generated by c are isomorphic.*

Indeed, this is a particular case of Proposition 2(2).

Let us recall that Guitart and Lair [12] have refined the notion of a solution-set into that of a *locally free diagram* on c ; it can be defined as a functor from $\mathbf{D}$ to $\mathbf{C}'$ which is the composite of an initial functor from $\mathbf{D}$ to $c|U$ with Sc . But, as for solution-sets, two locally free diagrams are not isomorphic, and they can be very large. Even the locally free relatively filtered diagrams considered by Guitart [11] are unique only up to a *homotopy-equivalence* of their associated classifying spaces.

Now if $\mathbf{B}$ is a U -universal root, the restriction of Sc to $\mathbf{B}$ is a locally free diagram of a particular type, namely, such that the functor is not only initial but also has a full weak coadjoint, and it is minimal among such locally free diagrams; it is this ‘minimality’ that ensures that a U -universal root is *unique up to an isomorphism*. Lair [13, 14] has given some conditions on the form of a locally free diagram which ensure it is unique; in fact, these conditions imply that the diagram is associated with a U -universal root, and thus its uniqueness is deduced from Proposition 2 above.

EXAMPLES. (1) A set of U -morphisms is a U -universal root generated by c iff it forms a full initial subcategory of $c|U$, which means that it is a *locally free family* in the sense of Diers [4].

(2) Let $\mathbf{C}$ and $\mathbf{C}'$ be two posets considered as categories; a functor U from $\mathbf{C}'$ to $\mathbf{C}$ is then associated with an order-preserving map u from $\mathbf{C}'$ to $\mathbf{C}$. In this case, an element c of $\mathbf{C}$ generates a U -universal root (which is then unique) iff there exists a subposet B' of $\mathbf{C}'$ satisfying the following conditions:

- (i) There exists no contraction from B' to B' ;
- (ii) For each b' in B' , we have $c \leq ub'$ in the poset $\mathbf{C}$;
- (iii) For any c' in $\mathbf{C}'$ such that $c \leq uc'$, the set of elements b' in B' such that $b' \leq c'$ is not void and has a greatest element.

If B' is finite, it follows from Proposition 1(3) that condition (i) can be replaced by

- (i') No element of B' has a predecessor in the poset B' .

(3) A subgroupoid of $c|U$ is a U -universal root generated by c iff it is a sum of groups (which ensures it is a root) which is a full initial subcategory of $c|U$. In this case, the uniqueness (up to isomorphism) of the associated locally free diagram has been proved by Lair in [13]. For instance, if $\mathbf{C}$ is the category of fields and U

is the insertion of a subcategory of algebraically closed fields, each field generates a U -universal root which is the group of automorphisms of its algebraic closure.

(4) Let $\mathbf{B}$ be a subcategory of $c|U$ which is isomorphic to the fibration associated with an 1-1 order-preserving map from a poset to the lattice of subgroups of group G . By Proposition 3(3), it is a root, so that it is a U -universal root of c iff it is a corefract of $c|U$. In particular, these conditions are satisfied if the associated locally free diagram is a Galois diagram in the sense of Lair [13]. Thus, his result on the isomorphism of two Galois diagrams is a particular case of the uniqueness of the U -root (Proposition 2(4)).

(5) The last two examples have applications to Galois theory. Indeed, let us take for $\mathbf{C}$ the category whose objects are the pairs (K, P) of a commutative field K and a polynomial P in K . Example 2 of [13] proves that each (K, P) generates a universal root which is a group, with respect to the insertion of the subcategory formed by the (K, P) such that all the roots of the polynomial P are in K . And Example 3 of [13] proves that (K, P) with P separable generates a universal root which is a Galois category with respect to the insertion of the subcategory formed by the (K, P) such that at least one root of the polynomial P is in K .

5. Adjoint-Root Functors

If U is a functor from $\mathbf{C}'$ to $\mathbf{C}$ and if each object c of $\mathbf{C}$ generates a U -universal object Fc , the map F extends into an adjoint functor of U . The problem here is to generalize this result for U -universal roots. We do that by proving that such a root is a universal object with respect to the pro-object extension of the functor U .

5.1. CATEGORIES OF PRO-OBJECTS

For this, we have to consider the category $\text{Pro } \mathbf{K}$ of *pro-objects* of a category $\mathbf{K}$, which is the *free completion* of $\mathbf{K}$ (cf. Deleanu and Hilton [3], or Duskin [6] who considers only the case of filtered limits). Its objects are the functors Q from a small category to $\mathbf{K}$, and we recall from Ehresmann [7] that the morphisms from Q to Q' can be explicitly described as being the clusters between them (called ‘atlases’ in [7]) as follows:

DEFINITION. Let Q and Q' be two functors from $\mathbf{J}$ and $\mathbf{J}'$ to $\mathbf{K}$, respectively. A *cluster* from Q' to Q is a set V of triples (j', v, j) , where j' and j are objects of $\mathbf{J}'$ and $\mathbf{J}$, respectively, and v is a morphism $v : Q'j' \rightarrow Qj$ in $\mathbf{K}$ satisfying the conditions:

- (1) (Connectivity): For each j in $\mathbf{J}$, there exists at least one (j', v, j) in V , and if there exists another (j'', w, j) , both are connected by a zig-zag of $\mathbf{J}'$ from j' to j'' (in the sense that (j', v) and (j'', w) are in the same component of $Q'|Q(j)$).

- (2) (Composition by **J**): If $n : j \rightarrow j^\circ$ is a morphism in **J** and (j', v, j) is in V , then $(j', Qn.v, j^\circ)$ is also in V .
- (3) V is the largest set satisfying (1) and (2).

A set H of triples (j', v, j) satisfying conditions (1) and (2) generates a cluster obtained by adding all the (j'', w, j) connected to an element of H by a zig-zag of **J**.

The composite in $\text{Pro } \mathbf{K}$ of the cluster V with a cluster V° from Q to Q° is the cluster $V^\circ.V$ from Q' to Q° generated by the set of triples $(j', v^\circ.v, j^\circ)$ obtained by ‘composing’ (j', v, j) in V with (j, v°, j°) in V' .

We identify an object k of **K** with the insertion functor from $\{k\}$ to **K** and the morphism $u : k \rightarrow k'$ with the cluster reduced to the triple (k, u, k') . This way **K** is identified to a full subcategory of $\text{Pro } \mathbf{K}$. And a cluster from k to $Q : \mathbf{J} \rightarrow \mathbf{K}$ reduces to a set of triples (k, v_j, j) , stable by composition by **J**, such that for each object j of **J** there exists just one triple with v_j from k to Qj .

The construction of Pro is functorial: a functor F from **K** to **C** extends into a functor $\text{Pro } F$ from $\text{Pro } \mathbf{K}$ to $\text{Pro } \mathbf{C}$ mapping $Q : \mathbf{J} \rightarrow \mathbf{K}$ to the composite $F \circ Q$ and the cluster V to the cluster FV generated by the triples (j', Fv, j) if (j', v, j) is in V .

5.2. ADJOINT-ROOT FUNCTOR

Let U be a functor from **C'** to **C**, and $\text{Pro } U$ its extension to pro-categories.

PROPOSITION 1. *Let c be an object of **C** which generates a U -universal root $\mathbf{B}c$, and let Bc be the restriction of the base functor Sc to $\mathbf{B}c$. Then Bc is a $\text{Pro } U$ -universal object generated by c .*

Proof. The set of triples $(c, z, (z, e))$, where (z, e) is an object of $\mathbf{B}c$, forms a cluster Z from c to $U \circ Bc$. To prove the proposition, it suffices to prove that, if $Q : \mathbf{J} \rightarrow \mathbf{C}'$ is a functor and V is a cluster from c to $U \circ Q$, then it factorizes into a unique cluster V' from Bc to Q such that $V = U V'.Z$.

(1) Let j be an object of **J**. There exists a unique (c, v_j, j) in the cluster V . As $\mathbf{B}c$ is a root of $c|U$, there exists at least one $y : (z, e) \rightarrow (v_j, Qj)$ in $c|U$ with (z, e) in $\mathbf{B}c$, and if there is another one, both are connected by a zig-zag (in fact, a cospan) in $\mathbf{B}c$. Let V' be the set of the corresponding triples $((z, e), y, j)$, i.e., such that

$$y : e \rightarrow Qj \text{ is in } \mathbf{C}', U y.z = v_j \text{ and } (z, e) \text{ is in } \mathbf{B}c.$$

Thus, V' satisfies the connectivity condition. It is also stable by composition by **J**. Indeed, if $n : j \rightarrow j^\circ$ is a morphism of **J**, then the triple $((z, e), Qn.y, j^\circ)$ also belongs to V' . Thus V' generates a cluster.

(2) To prove that V' is a cluster, it remains to prove that each triple which is deduced from an element of V' by a zig-zag in $\mathbf{B}c$ is also in V' . Indeed, let

$((z, e), y, j)$ be in V' . Its composite with a morphism of $\mathbf{Bc}$ still defines an element of V' . So it remains to prove that, if $m : (z, e) \rightarrow (z', e')$ is a morphism of $\mathbf{Bc}$ (thus $z' = Um.z$) and if there exists $y' : e' \rightarrow e^\circ$ in $\mathbf{C}'$ such that $y'.m = y$, then V' also contains $((z', e'), y', j)$, i.e., we have $v_j = Uy'.z'$. Now it comes from the equalities

$$Uy'.z' = Uy'.Um.z = Uy.z = v_j.$$

(3) By the construction of V' , its composite with Z is the cluster V . And it is the unique cluster factorizing V , such that $V = UV'.Z$. $\square$

PROPOSITION 2. *Suppose that each object c of $\mathbf{C}$ generates a U -universal root $\mathbf{Bc}$, and let Bc be the restriction of Sc to $\mathbf{Bc}$. Then there exists a functor B from $\mathbf{C}$ to $\text{Pro } \mathbf{C}'$ sending a morphism $x : c \rightarrow c^\circ$ to the cluster Bx from Bc to Bc° formed by the triples $((z, e), y, (z^\circ, e^\circ))$, where $y : e \rightarrow e^\circ$ is a morphism of $\mathbf{C}'$ such that $Uy.z = z^\circ.x$.*

Proof. As Bc is a $\text{Pro } U$ -universal object generated by c for each c in $|\mathbf{C}|$, there is a partial adjoint functor B of $\text{Pro } U$ mapping c to Bc . It follows from the construction of Proposition 1 that this functor maps x to the cluster Bx of the form defined above. $\square$

DEFINITION. The functor B constructed in Proposition 1 is called an *adjoint-root functor of U* .

REMARK. Propositions 1 and 2 strengthen, in the case of universal roots, the result of Guitart [11] for locally free diagrams, namely, each locally free diagram generated by c is a $\text{Pro } U$ -universal object. Here we explicitly define the adjoint-root functor. Note that this result only implies that a locally free diagram is unique up to an isomorphism of $\text{Pro } \mathbf{C}'$, but this does not ensure it is unique in $\mathbf{Cat}$. The main reason for introducing universal roots has been to insure their uniqueness in $\mathbf{Cat}$ (proved in Proposition 2(4)).

6. Local Colimits

Here the results of the two preceding sections will be applied to ‘localize’ the notion of a colimit.

6.1. COLIMIT-ROOT OF A FUNCTOR

Let $\mathbf{K}$ and $\mathbf{J}$ be two categories. We take for U the canonical functor from $\mathbf{K}$ to the category $\mathbf{K}^{\mathbf{J}}$ of functors from $\mathbf{J}$ to $\mathbf{K}$. Let Q be a functor from $\mathbf{J}$ to $\mathbf{K}$.

A colimit-cone with the basis Q is a U -universal arrow generated by Q . We generalize this notion to that of a *colimit-root of Q* , defined as a U -universal root of Q , i.e., as a root of the comma-category $Q|U$.

Let us study what it means. The comma-category $Q|U$ can be identified with the *category of cones with the basis Q* , denoted by $\mathbf{C}Q$: Its objects are the (inductive) cones g with the basis Q and vertex Sg in $\mathbf{K}$, and the morphisms from g to g' are defined by the morphisms $x : Sg \rightarrow Sg'$ in $\mathbf{K}$ such that $g' = xg$, where xg denotes the cone such that $xg(j) = x.g(j)$ for each object j of $\mathbf{J}$. Let S be the faithful ‘vertex functor’ from $\mathbf{C}Q$ to $\mathbf{K}$, which maps the cone g on its vertex Sg .

The category $\mathbf{C}Q$ is void if there is no cone with the basis Q .

DEFINITION. We say that Q has a *colimit-root* $\mathbf{B}$ if $\mathbf{B}$ is a root of the category of cones $\mathbf{C}Q$; then its image by the vertex functor S is called a *local colimit* of Q .

Thus, $\mathbf{B}$ is a colimit-root of Q iff $\mathbf{B}$ is a root, which is a corefract of $\mathbf{C}Q$. More explicitly, it amounts to the three following conditions:

- (i) Each $\mathbf{B}$ -cylinder from $\mathbf{B}$ to $\mathbf{B}$ is invertible;
- (ii) $\mathbf{B}$ is a full subcategory of $\mathbf{C}Q$;
- (iii) Each cone g with the basis Q admits a weakly terminal factorization yg' through a cone g' in $\mathbf{B}$; i.e., there exists a cone g' in $\mathbf{B}$ and $y : Sg' \rightarrow Sg$ such that $g = yg'$ and, for any other factorization $g = y'g''$ through a cone g'' in $\mathbf{B}$, there is $x : Sg'' \rightarrow Sg'$ in $\mathbf{K}$ satisfying

$$y' = y.x \quad \text{and} \quad g' = xg''.$$

From Proposition 2(3), we deduce:

PROPOSITION 1. *Let $\mathbf{B}$ be a finite poset considered as a category; Q admits $\mathbf{B}$ as a colimit-root iff $\mathbf{B}$ is a minimal full coreflective subcategory of $\mathbf{C}Q$.*

The application of Proposition 2(2) to this case gives the following:

PROPOSITION 2. *Two colimit-roots of Q are isomorphic in the category of cylinders of $\mathbf{C}Q$, and two local colimits of Q are isomorphic categories.*

6.2. LOCAL COLIMITS AND LOCALLY COLIMIT DIAGRAMS

A local colimit of Q defines a *locally colimit diagram* of Q in the sense of Guitart and Lair [12]. Indeed, we recall that such a diagram can be defined as a functor from $\mathbf{D}$ to $\mathbf{K}$ which is the composite of an initial functor from $\mathbf{D}$ to $\mathbf{C}Q$ with the vertex functor. Such diagrams always exist, but they are not unique. As for the locally free diagrams, local colimits allow one to characterize minimal diagrams of a particular type, this minimality ensuring their uniqueness.

REMARK. The locally colimit diagram associated with a local colimit of Q is a *locally colimit diagram equipped with formulas* in the sense of Lair. Indeed, the condition that $\mathbf{B}$ is a root can be translated into formula of the logic of $\mathbf{B}$

without relational symbols and with only unary functional symbols. Examples of such diagrams are studied in the papers of Lair and of Ageron (e.g., [1, 13]).

EXAMPLES. (1) A multicolimit (Diers [4]) of Q is a local colimit which is discrete.

(2) A subgroupoid of $\mathbf{C}Q$ is a colimit-root of Q iff it is a sum of groups which is a full and initial subcategory of $\mathbf{C}Q$. In [13], Lair has proved that, if $\mathbf{K}$ is an accessible category with finite products, any functor to $\mathbf{K}$ admits such a local colimit.

(3) Let $\mathbf{J}$ and $\mathbf{K}$ be posets considered as categories; a functor Q from $\mathbf{J}$ to $\mathbf{K}$ is associated with an order-preserving map q from J to K . Then $\mathbf{C}Q$ is (isomorphic to) the poset $\mathbf{M}$ of upper bounds of $q(J)$, and Q admits a local colimit $\mathbf{L}$ (which is then unique) iff there exists a subposet L of M such that:

- (i) There exists no contraction from L to L ;
- (ii) For each m in M , the set of elements of L smaller than m has a greatest element.

And (by Proposition 2(3)) Q admits a *finite local colimit* iff M has a finite minimal subposet L satisfying condition (ii).

In particular, if $q(J)$ has a least upper bound b in the set m of elements smaller than m (then b is called an m -aggregate of $q(J)$ in C. Ehresmann [9]), this b is in L . It follows from Example 1(3) that Q admits a multicolimit iff $q(J)$ has an m -aggregate for each m in M , the multicolimit being the set of these subaggregates. In particular, a poset is subinductive (resp. subpreinductive) in the sense of Ehresmann [9] iff, considered as a category, it is multicomplete (resp., finitely multicomplete).

6.3. COLIMIT-ROOT FUNCTOR

We first construct a *colimit-root functor* connecting colimit-roots of functors with the same domain $\mathbf{J}$. The notation is the same as above. The following proposition is a particular case of Proposition 2(5).

PROPOSITION 3. *Suppose that each functor Q from $\mathbf{J}$ to $\mathbf{K}$ admits a colimit-root $\mathbf{B}Q$, and let GQ be the restriction to $\mathbf{B}Q$ of the vertex functor. Then there exists a functor G from $\mathbf{K}^J$ to $\text{Pro } \mathbf{K}$ sending a natural transformation T from Q to Q' to the cluster GT from GQ to GQ' formed by the triples (g, y, g') , where $y : Sg \rightarrow Sg'$ is a morphism of $\mathbf{K}$ such that $yg = g'.T$.*

This result can be extended to colimit-roots of functors with possibly different domains, using the fact that a colimit-root can also be described as a universal root with respect to the canonical functor from $\mathbf{K}$ to the category of ind-objects $\text{Ind } \mathbf{K}$ (instead of taking the ‘smaller’ functor from $\mathbf{K}$ to $\mathbf{K}^J$).

We recall that $\text{Ind } \mathbf{K}$, which represents the *free cocompletion* of the category $\mathbf{K}$, is the opposite of the category $\text{Pro}(\mathbf{K}^{\text{op}})$. More explicitly, its objects are still

the functors Q from a small category $\mathbf{J}$ to $\mathbf{K}$, and the morphisms from Q to Q' are the *coclusters* between them. Such a cocluster is a maximal set W of triples (j, w, j') , where j and j' are objects of $\mathbf{J}$ and $\mathbf{J}'$, respectively, and $w : Qj \rightarrow Q'j'$ is a morphism of $\mathbf{K}$, stable by composition by $\mathbf{J}$, and satisfying the connectivity condition: for each object j in $\mathbf{J}$ there are triples in W with source j and two of them are connected by a zig-zag in $\mathbf{J}'$.

A functor with as its domain the category $\mathbf{1}$ is identified with the object k of $\mathbf{K}$ it determines, so that $\mathbf{K}$ is identified with a full subcategory of $\text{Ind } \mathbf{K}$; we denote by Y the insertion. A cocluster from Q to k then reduces to a cone with the basis Q and vertex k . It follows that the comma-category $Q|Y$ becomes the category of cones $\mathbf{C}Q$, so that a colimit-root of Q can also be defined as a Y -universal root of Q . We can then apply Proposition 2(5) to Y to get the following

PROPOSITION 4. *Let $\mathbf{M}$ be a full subcategory of $\text{Ind } \mathbf{K}$ whose objects Q have a colimit-root $\mathbf{B}Q$; let GQ be the functor from $\mathbf{B}Q$ to $\mathbf{K}$ restriction of the vertex functor. Then there exists a colimit-root functor from $\mathbf{M}$ to $\text{Pro } \mathbf{K}$ mapping Q on GQ and mapping a cocluster W from Q to Q' to the cluster from GQ to GQ' formed by the triples (g, y, g') , where $y : Sg \rightarrow Sg'$ is a morphism of $\mathbf{K}$ such that $yg = g'.W$.*

REMARK. All the results of this article can be transposed to the opposite categories to generalize U -couniversal arrows, where U is a functor, into U -couniversal coroots, and limit-cones into local limits of a functor.

References

1. Ageron, P.: Catégories accessibles à produits fibrés, *Diagrammes* **36** (1996).
2. Barr, M.: Abstract Galois theory, *J. Pure Appl. Algebra* **19** (1980), 21–42.
3. Deleanu, A. and Hilton, P.: Borsuk shape and Grothendieck categories of pro-objects, *Math. Proc. Cambridge* **79** (1976), 473–482.
4. Diers, Y.: Catégories localisables, Thèse Université Paris VI, 1971.
5. Diers, Y.: Catégories multialgébriques galoisiennes, *Cahiers Topologie Géom. Différentielle Catégoriques* **XXXIII**(1) (1992), 55–69.
6. Duskin, J.: Pro-objects (d'après Verdier), *Séminaire Heidelberg–Strasbourg* 1966–67, Exposé 6.
7. Ehresmann, A. C.: Comments on part IV-1. In: *Charles Ehresmann: Oeuvres complètes et commentées* IV-1, Amiens, 1981, p. 370.
8. Ehresmann, A. C. and Vanbreemersch, J.-P.: Hierarchical evolutive systems: A mathematical model for complex systems, *Bull. Math. Biol.* **49**(1) (1987), 13–50. And Multiplicity principle and emergence in memory evolutive systems, *SAMS* **26** (1997), 81–117.
9. Ehresmann, C.: Groupoïdes sous-inductifs, *Ann. Inst. Fourier* (Grenoble) **13**(2) (1963), 1–60; reprinted in *Charles Ehresmann: Oeuvres complètes et commentées*, partie II-1, Amiens, 1982, p. 257.
10. Grothendieck, A.: *Revêtements étales et groupe fondamental*, *S.G.A. 1*, Lecture Notes in Math. 224, Springer, 1971.

11. Guitart, R.: On the geometry of computations I and II, *Cahiers Topologie Géom. Différentielle Catégorique* **XXVII**(4) (1988), 107–136; **XXIX**(4) (1990), 297–326.
12. Guitart, R. and Lair, C.: Calcul syntaxique des modèles et calcul des formules internes, *Diagrammes* **4** (1980).
13. Lair, C.: Diagrammes localement libres. Extensions de corps et théorie de Galois, *Diagrammes* **10** (1983).
14. Lair, C.: Sur le profil d'esquissabilité des catégories modelables possédant les noyaux, *Diagrammes* **36** (1996).
15. Mac Lane, S.: *Categories for the Working Mathematician*, Springer, 1971.