FRACTIONAL ANALYTIC SIGNALS

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Abstract

The fractional Hilbert transform (FHT) is a linear combination of a signal and its Hilbert transform, and invites extensions of the classical analytic signal. Mathematical properties of the FHT and three possible versions of fractional analyticity are explored in detail.

0. Introduction

Gabor introduced the Hilbert transform into signal theory in 1946 [1], by defining a complex extension of a real signal $S(t)$ as

$$Z(t) = S(t) + iHS(t)$$

$$H = \text{Hilbert transform}, \quad i = \sqrt{-1}$$

A signal is provably orthogonal to its Hilbert transform (HT), and Gabor's extension provides ready estimates of instantaneous frequency and amplitude. These extended (or 'analytic') signals have non-negative Fourier frequencies, an unique property inherited directly from the HT; they are also eigenfunctions of the HT. The recently introduced fractional Hilbert transform (FHT) is a linear combination of a signal and its HT; Gabor's recipe, therefore, invites exploration of fractional analytic versions of the classical concept.

1. Preliminaries

The Hilbert transform is the quintessential singular integral operator [2]

$$HS(t) = (1/\pi) \int S(x)/(t-x)dx \quad (1.0)$$

where $S(x)$ is a real signal of finite energy and the integral is understood in the Cauchy Principal Value sense

$$HS(t) = (1/\pi) \lim_{\varepsilon \rightarrow 0} \int_{|t-x|>\varepsilon} S(x)/(t-x)dx$$

Formally, the Hilbert transform amounts to convolution with a singular kernel

$$HS = S * 1/\pi x$$

hence the fundamental property that the Fourier multiplier $m_H(\omega)$ associated with the operator H is [2]

$$m_H(\omega) = -i \operatorname{sgn} \omega \quad (1.1)$$

in the sense that

$$FHS(\omega) = m_H(\omega)FS(\omega)$$

where the Fourier transform is taken to be

$$FS(\omega) = \hat{S}(\omega) = (2\pi)^{-1/2} \int S(t)\exp(-i\omega t)dt$$

The notion of a multiplier may also be introduced in a relative sense $m_{X,Y}(\omega)$ is a multiplier of $X(t)$ with respect to $Y(t)$ whenever

$$\hat{X}(\omega) = m_{X,Y}(\omega)\hat{Y}(\omega)$$

It follows from (1.1) that the analytic signal $Z(t)$ has a real-valued multiplier $m_{Z,S}(\omega)$ relative to the original signal $S(t)$

$$\hat{Z}(\omega) = m_{Z,S}(\omega)\hat{S}(\omega)$$

where

$$m_{Z,S}(\omega) = 1 + im_H(\omega) = 1 + \text{sgn}\omega \quad (1.2)$$

Clearly, it is the vanishing of $m_{Z,S}(\omega)$ for negative frequencies that forces this spectral feature onto the analytic signal itself. Property (1.1) also opens the door to the following additional properties [3], [4]

$$H^2 = -I \quad (I = \text{Identity operator}) \quad (1.3)$$

showing H to be an operator equivalent of the imaginary unit, so that

$$H^{-1} = -H \quad (1.4)$$

$$(HS_1(t), S_2(t)) = (S_1(t), H^{-1}S_2(t)) \quad (1.5)$$

or equivalently

$$(HS_1(t), S_2(t)) + (S_1(t), HS_2(t)) = 0 \quad (1.6)$$

where $(\cdot, \cdot)$ is the L^2 inner product,

$$\int (HS(t))^2 dt = \int S(t)^2 dt \quad (\text{energy conservation}) \quad (1.7)$$

and more generally

$$(HS_1(t), HS_2(t)) = (S_1(t), S_2(t)) \quad (1.8)$$

$$(S(t), HS(t)) = 0 \quad (\text{orthogonality}) \quad (1.9)$$

or what is the same

$$(\text{Re}Z(t), \text{Im}Z(t)) = 0 \quad (1.10)$$

$$\int HS(t)dt = 0 \quad (\text{cancellation}) \quad (1.11)$$

$$HZ(t) = -iZ(t) \quad (\text{eigenfunction property}) \quad (1.12)$$

Estimates of (square) instantaneous amplitude and frequency are given as

$$A(t)^2 = Z(t)Z(t)^* = S(t)^2 + (HS(t))^2 \quad (1.13)$$

$$f(t) = D_t(\text{atan}(ImZ(t)/ReZ(t)))$$

$$= (S(t)D_t(HS(t)) - HS(t)D_tS(t))/A(t)^2, \quad D_t = d/dt \quad (1.14)$$

by analogy with complex numbers. Note also that

$$\int A(t)^2 dt = 2 \int S(t)^2 dt \quad (1.15)$$

so the analytic signal provides about 3dB of total energy boost. As an aside, property (1.4) allows a rewriting of the instantaneous amplitude as

$$A(t)^2 = (HS(t))^2 - S(t)H(HS)(t) \quad (1.13)$$

and formal substitution of the operator D_t for the Hilbert transform produces Kaiser's version of instantaneous energy [5]

$$\Psi(t) = (D_t S(t))^2 - S(t)D_t^2 S(t), \quad D_t^2 = d^2/dt^2 \quad (1.14)$$

The substitution is certainly appropriate when restricted to single-frequency sinusoids.

2. The FHT

Because the Hilbert transform retains signal domain, the orthogonality property (1.9) leads easily to a fractional Hilbert transform [6], [7] of order α (see Fig. 1):

$$H_\alpha S(t) = S(t)\cos\alpha + HS(t)\sin\alpha \quad (2.0)$$

In operator terms, $H_0 = I$ and $H_{\pi/2} = H$

How many properties of the Hilbert transform remain intact? Direct computation yields the following list

$$FH_\alpha S(\omega) = m_{H_\alpha}(\omega) \hat{S}(\omega)$$

where

$$m_{H_\alpha}(\omega) = \cos\alpha - i\operatorname{sgn}\omega\sin\alpha \quad (2.1)$$

is the Fourier multiplier associated with the FHT, satisfying

$$\begin{aligned} m_{H_\alpha}(\omega) &= \exp(i\alpha) & \omega < 0 \\ &= \cos\alpha & \omega = 0 \\ &= \exp(-i\alpha) & \omega > 0 \end{aligned}$$

and in terms of the Fourier spectrum

$$\begin{aligned} |FH_\alpha S(\omega)| &= |\hat{S}(\omega)| & \omega \neq 0 \\ &= |\cos\alpha| |\hat{S}(0)| & \text{at } \omega = 0 \end{aligned}$$

$$H_\alpha H_\beta = H_{\alpha+\beta} \quad (\text{semi-group property}) \quad (2.2)$$

so that now

$$H_\alpha^{-1} = H_{-\alpha} \quad (2.3)$$

$$(H_\alpha S_1(t), S_2(t)) = (S_1(t), H_{-\alpha} S_2(t)) \quad (2.4)$$

implying that

$$(H_\alpha S_1(t), H_\beta S_2(t)) = (H_{\alpha-\beta} S_1(t), S_2(t)) \quad (2.5)$$

$$(H_\alpha S_1(t), H_\alpha S_2(t)) = (S_1(t), S_2(t)) \quad (2.6)$$

and in particular

$$\int (H_\alpha S(t))^2 dt = \int S(t)^2 dt \quad (\text{energy conservation}) \quad (2.7)$$

Also,

$$(H_\alpha S_1(t), S_2(t)) + (S_1(t), H_\alpha S_2(t)) = 2(S_1(t), S_2(t))\cos\alpha \quad (2.8)$$

$$(H_\alpha S(t), H_\beta S(t)) = (S(t), S(t))\cos(\alpha - \beta) \quad (2.9)$$

$$\int H_\alpha S(t) dt = \cos\alpha \int S(t) dt \quad (2.10)$$

$$H_\alpha Z(t) = \exp(-i\alpha)Z(t) \quad (\text{eigenfunction property}) \quad (2.11)$$

To see (2.2), for example,

$$\begin{aligned} H_\beta H_\alpha S(t) &= \cos\beta H_\alpha S(t) + \sin\beta H(H_\alpha S)(t) \\ &= \cos\beta(S(t)\cos\alpha + HS(t)\sin\alpha) + \sin\beta(HS(t)\cos\alpha - S(t)\sin\alpha) \end{aligned}$$

$$\begin{aligned}
&= S(t)(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + HS(t)(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= S(t)\cos(\alpha + \beta) + HS(t)\sin(\alpha + \beta) \\
&= H_{\alpha+\beta}S(t)
\end{aligned}$$

Some differential properties for signals in the range of the FHT are listed next; for simplicity, let $W_\alpha(t) = H_\alpha S(t)$

Then

$$\partial_\alpha W_\alpha(t) = HW_\alpha(t) = W_{\alpha+\pi/2}(t) \quad , \quad \partial_\alpha = \partial/\partial\alpha \quad (2.12)$$

implying that

$$H\partial_\alpha W_\alpha(t) = \partial_\alpha HW_\alpha(t) = -W_\alpha(t) \quad (2.13)$$

and also

$$(W_\alpha(t), \partial_\alpha W_\alpha(t)) = 0 \quad (2.14)$$

Thus the range of the FHT imitates the behavior of sines and cosines, except, of course, that here the differentiator and Hilbert transformer operate with respect to different labels. And now, on to the matter of extending the analytic signal.

3. Fractional analytic signals

The introduction of the FHT actually suggests three possible definitions of a fractional analytic signal

$$Z1_\alpha(t) = H_\alpha Z(t) \quad , \quad Z1_0(t) = Z(t) \quad (3.0a)$$

$$Z2_\alpha(t) = S(t) + iH_\alpha S(t) \quad , \quad Z2_{\pi/2}(t) = Z(t) \quad (3.0b)$$

$$Z3_\alpha(t) = S(t)\cos\alpha + iHS(t)\sin\alpha \quad , \quad Z3_{\pi/4}(t) = 2^{-1/2}Z(t) \quad (3.0c)$$

The first option is drawn from (2.11) and has geometric appeal; the second option replaces the HT by the FHT, while the third option is a complexification of the FHT. Frequency, amplitude, eigensolution and differential properties of these options are explored next.

Option a) $Z1_\alpha(t) = H_\alpha Z(t)$

From the definition of the FHT and the eigenfunction property (1.12)

$$Z1_\alpha(t) = Z(t)\cos\alpha - iZ(t)\sin\alpha = \exp(-i\alpha)Z(t) \quad (3.1a)$$

hence fractional order in $Z1_\alpha(t)$ is induced by rotation in the complex plane; computation of the Fourier spectrum is immediate

$$|\hat{Z}1_\alpha(\omega)| = |\hat{Z}(\omega)| \quad (3.2a)$$

so the spectrum of $Z1_\alpha(t)$ is identical to $Z(t)$ independently of fractional order, a predictable outcome. In addition,

$$\hat{Z}1_\alpha(\omega) = \exp(-i\alpha)\hat{Z}(\omega) = \exp(-i\alpha)(1 + \text{sgn}\omega)\hat{S}(\omega)$$

showing that the Fourier multiplier of $Z_{1\alpha}(t)$ with respect to $S(t)$ is

$$m_{Z_{1\alpha}, S}(\omega) = \exp(-i\alpha)(1 + \operatorname{sgn}\omega) \quad (3.3a)$$

This multiplier satisfies

$$\begin{aligned} m_{Z_{1\alpha}, S}(\omega) &= 0 & \omega < 0 \\ &= \exp(-i\alpha) & \omega = 0 \\ &= 2\exp(-i\alpha) & \omega > 0 \end{aligned}$$

Estimation of the instantaneous fractional amplitude is equally routine

$$A_{1\alpha}(t)^2 = Z_{1\alpha}(t)Z_{1\alpha}(t)^* = Z(t)Z(t)^* = A(t)^2 \quad (3.4a)$$

As for fractional instantaneous frequency, (3.1a) and the polar form of the analytic signal make matters readily apparent

$$Z_{1\alpha}(t) = \exp(-i\alpha)A(t)\exp(i(\Phi(t))) = A(t)\exp(i(\Phi(t) - \alpha))$$

thus the instantaneous phase of $Z_{1\alpha}(t)$ is merely altered by the constant order α , so that

$$f_{1\alpha}(t) = \partial_t(\Phi(t) - \alpha) = f(t) \quad (3.5a)$$

Next, since $Z_{1\alpha}(t)$ is the FHT image of $Z(t)$, differential property (2.12) also holds here

$$\partial_\alpha Z_{1\alpha}(t) = HZ_{1\alpha}(t) = Z_{1\alpha+\pi/2}(t) \quad (3.6a)$$

which, together with (3.1a) yields

$$HZ_{1\alpha}(t) = \exp(-i\pi/2)\exp(-i\alpha)Z(t) = -iZ_{1\alpha}(t) \quad (3.7a)$$

so $Z_{1\alpha}(t)$ is also an eigenfunction of the Hilbert transform with the same eigenvalue as $Z(t)$.

In addition

$$H\partial_\alpha Z_{1\alpha}(t) = \partial_\alpha HZ_{1\alpha}(t) = -Z_{1\alpha}(t) \quad (3.8a)$$

Option b) $Z_{2\alpha}(t) = S(t) + iH_\alpha S(t)$

The multiplier $m_{Z_{2\alpha}, S}(\omega)$ of $Z_{2\alpha}(t)$ with respect to $S(t)$ follows from (2.1) and does show

negative frequencies

$$\begin{aligned} m_{Z_{2\alpha}, S}(\omega) &= 1 + \operatorname{sgn}\omega \sin\alpha + i\cos\alpha & (3.1b) \\ &= 1 + i\exp(i\alpha) & \omega < 0 \\ &= 1 + i\cos\alpha & \omega = 0 \\ &= 1 + i\exp(-i\alpha) & \omega > 0 \end{aligned}$$

thus

$$\begin{aligned} |\hat{Z}_{2\alpha}(\omega)|^2 &= 2(1 + \operatorname{sgn}\omega \sin\alpha)|\hat{S}(\omega)|^2 & \omega \neq 0 \\ &= (1 + \cos^2\alpha)|\hat{S}(0)|^2 & \text{at } \omega = 0 \end{aligned} \quad (3.2b)$$

The corresponding estimates of instantaneous fractional amplitude and frequency mimic the conventional expressions

$$A2_{\alpha}(t)^2 = S(t)^2 + (H_{\alpha}S(t))^2 \quad (3.3b)$$

$$f2_{\alpha}(t) = (S(t)\partial_t H_{\alpha}S(t) - (H_{\alpha}S(t))\partial_t S(t))/A2_{\alpha}(t)^2 \quad (3.4b)$$

and these connect immediately with the conventional amplitude/frequency estimates

$$A2_{\alpha}(t)^2 = A(t)^2 + (S(t)^2 - (HS(t))^2)\cos^2\alpha + S(t)HS(t)\sin 2\alpha \quad (3.5b)$$

$$f2_{\alpha}(t) = (A(t)/A2_{\alpha}(t))^2 f(t)\sin\alpha \quad (3.6b)$$

The instantaneous frequency estimate in this fractional domain is therefore a non-linearly scaled version of the original. Note also the appearance of cross terms involving the signal and its Hilbert transform in the instantaneous amplitude; nevertheless, (1.7) and (1.9) imply that

$$\int A2_{\alpha}(t)^2 dt = \int A(t)^2 dt \quad (3.7b)$$

so in this fractional setting the total instantaneous amplitude energy matches that of the analytic signal. As for differential features, the following relations are evident from (2.12)

$$(H - \partial_{\alpha})Z2_{\alpha}(t) = HS(t) \quad (3.8b)$$

$$\partial_{\alpha}HZ2_{\alpha}(t) = H\partial_{\alpha}Z2_{\alpha}(t) \quad (3.9b)$$

Even though their actions differ here, the operators ∂_{α} and H still commute.

No transparent eigensolution property appears available for $Z2_{\alpha}(t)$. However, (2.2) yields that

$$H_{-\alpha}Z2_{\alpha}(t) = H_{-\alpha}S(t) + iS(t) = i(S(t) - iH_{-\alpha}S(t)) = iZ2_{-\alpha}(t)^* \quad (3.10b)$$

Option c) $Z3_{\alpha}(t) = S(t)\cos\alpha + iHS(t)\sin\alpha$

The Fourier multiplier $m_{Z3_{\alpha},S}(\omega)$ of $Z3_{\alpha}(t)$ relative to $S(t)$ follows from (1.1), turns out to

be real-valued, and again shows negative frequencies

$$\begin{aligned} m_{Z3_{\alpha},S}(\omega) &= \cos\alpha + \operatorname{sgn}\omega\sin\alpha \\ &= \cos\alpha - \sin\alpha & \omega < 0 \\ &= \cos\alpha & \omega = 0 \\ &= \cos\alpha + \sin\alpha & \omega > 0 \end{aligned} \quad (3.1c)$$

The spectrum of $Z3_{\alpha}(t)$ satisfies

$$\begin{aligned} |Z3_{\alpha}(\omega)|^2 &= (1 + \operatorname{sgn}\omega\sin 2\alpha)|S(\omega)|^2 & \omega \neq 0 \\ &= \cos^2\alpha |S(0)|^2 & \text{at } \omega = 0 \end{aligned} \quad (3.2c)$$

Estimates of instantaneous fractional amplitude and frequency in this setting are

$$A3_{\alpha}(t)^2 = S(t)^2\cos^2\alpha + (HS(t))^2\sin^2\alpha = A(t)^2\sin^2\alpha + S(t)^2\cos 2\alpha \quad (3.3c)$$

$$f3_{\alpha}(t) = (A(t)/A3_{\alpha}(t))^2 f(t)\sin\alpha\cos\alpha \quad (3.4c)$$

Note that this time

$$\int A^2_{3\alpha}(t) dt = \int S^2(t) dt = 1/2 \int A^2(t) dt \quad (3.5c)$$

Differential and eigensolution properties are immediate

$$\partial_{\alpha} Z_{3\alpha}(t) = -Z_{3\alpha-\pi/2}(t) \quad (3.6c)$$

$$H Z_{3\alpha}(t) = -i Z_{3\alpha-\pi/2}(t)^* \quad (3.7c)$$

The operators ∂_{α} and H commute

$$\partial_{\alpha} H Z_{3\alpha}(t) = H \partial_{\alpha} Z_{3\alpha}(t) = -i Z_{3\alpha}(t)^* \quad (3.8c)$$

4. Relative merits

It is perhaps less than satisfying that more than one version of fractional analyticity is available; however, the following observations may help narrow the choice in practical applications

(4.1) Only $Z_{1\alpha}(t)$ and $Z_{3\alpha}(t)$ are in keeping with the spirit of the analytic signal, in that the respective real and imaginary parts are orthogonal independently of α .

(4.2) Only $Z_{1\alpha}(t)$ matches in spectral amplitude the conventional analytic signal at all orders regardless of frequency.

(4.3) Only the total instantaneous amplitude energy of $Z_{2\alpha}(t)$ (and, trivially, $Z_{1\alpha}(t)$) equals that of $Z(t)$.

(4.4) Only $Z_{3\alpha}(t)$ has a real-valued multiplier relative to the original signal $S(t)$ for any order α , retaining that property of the analytic signal.

(4.5) DC components (i.e., at $\omega = 0$) have the following spectral ordering

$$|\hat{Z}_{3\alpha}(0)| \leq |\hat{Z}_{1\alpha}(0)| = |\hat{Z}(0)| \leq |\hat{Z}_{2\alpha}(0)|$$

Much like the FHT itself, $Z_{3\alpha}(t)$ offers variable reduction of DC contributions down to zero; on the other hand, $Z_{2\alpha}(t)$ allows boosting of DC components relative to $Z(t)$.

(4.6) Comparison of non-zero frequency behavior is less clear-cut. Obviously

$$|\hat{Z}_{3\alpha}(\omega)| < |\hat{Z}_{1\alpha}(\omega)| \text{ and } |\hat{Z}_{2\alpha}(\omega)| \leq |\hat{Z}_{1\alpha}(\omega)|$$

From (3.2b) and (3.2c), comparing spectra among $Z_{2\alpha}(t)$ and $Z_{3\alpha}(t)$ requires examination of the

functions $M_2(\alpha, \omega) = 2(1 + \text{sgn}\omega \sin\alpha)$ and $M_3(\alpha, \omega) = 1 + \text{sgn}\omega \sin 2\alpha$. Let

$$P = \{0 \leq \alpha \leq 2\pi \mid M_2(\alpha, \omega) = M_3(\alpha, \omega), \omega > 0\}$$

$$N = \{0 \leq \alpha \leq 2\pi \mid M_2(\alpha, \omega) = M_3(\alpha, \omega), \omega < 0\}$$

The MATLAB 'fzero' function conveniently provides the following numerical estimates (in radians)

$$P = \{3.39858591533875, 5.17072446854282\} \equiv \{p_1, p_2\}$$

$$N = \{1.11246083863676, 2.88459939184084\} \equiv \{n_1, n_2\}$$

Thus for $\omega > 0$

$$|\hat{Z}_{3\alpha}(\omega)| < |\hat{Z}_{2\alpha}(\omega)| \quad \text{when} \quad 0 \leq \alpha < p_1 \quad \text{or} \quad p_2 < \alpha \leq 2\pi$$

while

$$|\hat{Z}_{2\alpha}(\omega)| < |\hat{Z}_{3\alpha}(\omega)| \quad \text{when} \quad p_1 < \alpha < p_2$$

Similarly, for $\omega < 0$

$$|\hat{Z}_{3\alpha}(\omega)| < |\hat{Z}_{2\alpha}(\omega)| \quad \text{when} \quad 0 \leq \alpha < n_1 \quad \text{or} \quad n_2 < \alpha \leq 2\pi$$

while

$$|\hat{Z}_{2\alpha}(\omega)| < |\hat{Z}_{3\alpha}(\omega)| \quad \text{when} \quad n_1 < \alpha < n_2$$

5. Conclusion

Mathematical properties of the recently-introduced fractional Hilbert transform have been presented here in greater detail than available elsewhere. In particular, the important semi-group property has been shown to hold; this is not the case for another fractional variant proposed by Zayed [8], which borrows from the fractional Fourier transform formulation. Three possible extensions of the classical analytic signal have been proposed and explored. One extension produced a rotation of the analytic signal in the complex plane, a second replaced the Hilbert transform by its fractional version as the imaginary component, while a third option was defined through a complexification of the FHT. It remains for practical applications to determine a preference among these three options.

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