Superlink derivations

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Contents

1 Glossary

Variable	Description
$a_{ik}, b_{ik}, c_{ik}, P_{ik}$	Momentum coefficients
a_h, q_q	Horizontal and vertical axis scaling factors for elliptical pump curve
A_o	Maximum area of flow of orifice o
A_{ik}	Cross-sectional area of flow at link ik
A_{dk}	Cross-sectional area of flow at downstream end of superlink k
$A_{s,Ik}$	Area of water surface at junction Ik
A_{sj}	Area of water surface at superjunction j
A_{uk}	Cross-sectional area of flow at upstream end of superlink k
B_{ik}	Top width of flow at link ik
C_o	Coefficient of discharge of orifice o
C_{dk}	Coefficient of discharge at downstream end of superlink k
C_{uk}	Coefficient of discharge at upstream end of superlink k
C_{wR}	Rectangular weir discharge coefficient for weir w
C_{wT}	Triangular weir discharge coefficient for weir w
D_{Ik}, E_{Ik}	Continuity coefficients
g	Acceleration due to gravity
h_{dk}	Water depth at downstream of superlink k
h_{Ik}	Water depth at junction Ik
h_{uk}	Water depth at upstream of superlink k
H_j	Head at junction j (invert elevation + water depth)
H_{jdk}	Head at junction downstream of superlink k (invert elevation + water depth)
H_{juk}	Head at junction upstream of superlink k (invert elevation + water depth)
H_{juo}	Head at junction upstream of orifice o (invert elevation + water depth)
H_{jdo}	Head at junction downstream of orifice o (invert elevation + water depth)
ΔH_{dk}	Head difference at downstream end of superlink k
ΔH_{uk}	Head difference at upstream end of superlink k
L_w	Length of transverse weir w
$NBDj$	Number of superlinks with downstream end attached to superjunction j
$NBUj$	Number of superlinks with upstream end attached to superjunction j
Q_{ik}	Discharge in link ik
Q_{Ik}	Discharge at junction Ik
$Q_{o,Ik}$	External flow input at junction Ik
$Q_{o,j}$	External flow input at superjunction j
Q_o	Discharge from orifice o
Q_{dk}	Discharge at downstream end of superlink k
Q_{uk}	Discharge at upstream end of superlink k

Variable	Description
s_w	Side slope (run/rise) for triangular portion of weir w
$S_{o,ik}$	Channel bottom slope at link ik
$S_{f,ik}$	Friction head loss slope at link ik
$S_{L,ik}$	Local head loss slope at link ik
Δt	Time step
u_{Ik}	Velocity of flow at junction Ik
U_{Ik}, V_{Ik}, W_{Ik}	Forward recurrence relation coefficients
X_{Ik}, Y_{Ik}, Z_{Ik}	Backward recurrence relation coefficients
Δx_{ik}	Length of link ik
$y_{max,o}$	Maximum height of orifice o
$y_{max,w}$	Maximum height of weir w
$z_{inv,j}$	Invert elevation of superjunction j
$z_{inv,dk}$	Invert elevation at downstream end of superlink k
$z_{inv,uk}$	Invert elevation at upstream end of superlink k
$z_{inv,jdo}$	Invert elevation of superjunction at downstream end of orifice o
$z_{inv,juo}$	Invert elevation of superjunction at upstream end of orifice o
$z_{inv,jdp}$	Invert elevation of superjunction at downstream end of pump p
$z_{inv,jup}$	Invert elevation of superjunction at upstream end of pump p
$z_{inv,jdw}$	Invert elevation of superjunction at downstream end of weir w
$z_{inv,juw}$	Invert elevation of superjunction at upstream end of weir w
z_o	Offset elevation of orifice o
z_p	Offset elevation of pump p
z_w	Offset elevation of weir w

2 Basic equations

The two governing equations for SUPERLINK are continuity and conservation of momentum.

Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{1}$$

Conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{2}$$

3 Discretization of momentum

Starting with the equation for conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{3}$$

The following discretization scheme can be applied to link ik:

$$\begin{aligned}(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + u_{I+1k}Q_{I+1k}^{t+\Delta t} - u_{Ik}Q_{Ik}^{t+\Delta t} \\ +gA(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) - gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x = 0\end{aligned} \tag{4}$$

This equation can be written in terms of the following coefficient equation:

$$\boxed{a_{ik}Q_{i-1k}^{t+\Delta t} + b_{ik}Q_{ik}^{t+\Delta t} + c_{ik}Q_{i+1k}^{t+\Delta t} = P_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})} \tag{5}$$

Where:

$$\boxed{a_{ik} = -\max(u_{Ik}, 0)} \tag{6}$$

$$\boxed{c_{ik} = -\max(-u_{I+1k}, 0)} \tag{7}$$

$$\boxed{b_{ik} = \frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - a_{ik} - c_{ik}} \tag{8}$$

$$\boxed{P_{ik} = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik} \Delta x_{ik}} \tag{9}$$

This coefficient equation can be verified by substituting the expressions for the coefficients:

$$\begin{aligned} -\max(u_{Ik},0)Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} + \max(u_{Ik},0) + \max(-u_{I+1k},0)\right)Q_{ik}^{t+\Delta t} \\ - \max(-u_{I+1k},0)Q_{i+1k}^{t+\Delta t} \\ = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik}\Delta x_{ik} + g A_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{10}$$

Assuming $u_{ik} > 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$\begin{aligned} -u_{Ik}Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} + u_{Ik}\right)Q_{ik}^{t+\Delta t} \\ = Q_{ik}^t\frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik}\Delta x_{ik} + g A_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{11}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} \\ + g A_{ik}\left(\frac{n_{ik}^2 |Q_{ik}^t| Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t| Q_{ik}^{t+\Delta t}}{g C_{ik}^2 A_{cik}^2 \Delta x_{ik}}\right)\Delta x_{ik} \\ = g A_{ik} S_{o,ik}\Delta x_{ik} + g A_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{12}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} + g A_{ik}(S_{f,ik} + S_{L,ik})\Delta x_{ik} \\ = g A_{ik} S_{o,ik}\Delta x_{ik} + g A_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{13}$$

Which further simplifies to the original combined mass and momentum balance:

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} \\ + g A_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + g A_{ik}(S_{f,ik} - g A_{ik} S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik} \end{aligned} \tag{14}$$

Alternatively, assuming $u_{ik} < 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$\begin{aligned} u_{I+1k} Q_{i+1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik} |Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - u_{I+1k} \right) Q_{ik}^{t+\Delta t} \\ = Q_{ik}^t \frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{15}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t}) u_{I+1k} \\ + g A_{ik} \left(\frac{n_{ik}^2 |Q_{ik}^t| Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t| Q_{ik}^{t+\Delta t}}{g C_{ik}^2 A_{cik}^2 \Delta x_{ik}} \right) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{16}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t}) u_{I+1k} + g A_{ik} (S_{f,ik} + S_{L,ik}) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{17}$$

Which simplifies to the original combined mass and momentum balance:

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t}) u_{I+1k} \\ + g A_{ik} (h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + g A_{ik} (S_{f,ik} - g A_{ik} S_{o,ik} \Delta x_{ik} + S_{L,ik}) \Delta x_{ik} \end{aligned} \tag{18}$$

4 Discretization of continuity

Starting with the continuity equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{19}$$

The following discretization scheme can be applied to junction Ik:

$$Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t} + \left(\frac{B_{ik} \Delta x_{ik}}{2} + \frac{B_{i-1k} \Delta x_{i-1k}}{2} + A_{s,Ik} \right) \cdot \frac{h_{Ik}^{t+\Delta t} - h_{Ik}^t}{\Delta t} = Q_{0,Ik} \tag{20}$$

Through substitution, the discretized continuity equation can be represented as follows:

$$\boxed{Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t} + E_{Ik} h_{Ik}^{t+\Delta t} = D_{Ik}} \tag{21}$$

Where:

$$\boxed{D_{Ik} = Q_{0,Ik} + \frac{h_{Ik}^{t}}{\Delta t} \left(\frac{B_{ik} \Delta x_{ik}}{2} + \frac{B_{i-1k} \Delta x_{i-1k}}{2} + A_{s,Ik} \right)} \tag{22}$$

$$\boxed{E_{Ik} = \frac{1}{\Delta t} \left(\frac{B_{ik} \Delta x_{ik}}{2} + \frac{B_{i-1k} \Delta x_{i-1k}}{2} + A_{s,Ik} \right)} \tag{23}$$

5 Recurrence relationships

The SUPERLINK algorithm uses a series of recurrence relationships to embed channel dynamics into the solution matrix. In this section, the forward and backward recurrence relations for each superlink are derived.

5.1 Forward recurrence

Starting at the upstream end of superlink k, the continuity and momentum equations can be written as:

$$Q_{2k}^{t+\Delta t} - Q_{1k}^{t+\Delta t} + E_{2k} h_{2k}^{t+\Delta t} = D_{2k} \tag{24}$$

$$a_{1k} Q_{0k}^{t+\Delta t} + b_{1k} Q_{1k}^{t+\Delta t} + c_{1k} Q_{2k}^{t+\Delta t} = P_{1k} + g A_{1k} (h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{25}$$

Assuming $Q_{0k}^{t+\Delta t} = Q_{1k}^{t+\Delta t}$, the continuity and momentum equations can be combined as follows:

$$a_{1k} Q_{1k}^{t+\Delta t} + b_{1k} Q_{1k}^{t+\Delta t} + c_{1k} (Q_{1k}^{t+\Delta t} - E_{2k} h_{2k}^{t+\Delta t} + D_{2k}) = P_{1k} + g A_{1k} (h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{26}$$

$$(a_{1k} + b_{1k} + c_{1k}) Q_{1k}^{t+\Delta t} = E_{2k} c_{2k} h_{2k}^{t+\Delta t} + (P_{1k} + c_{1k} D_{2k}) + g A_{1k} (h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{27}$$

$$(a_{1k} + b_{1k} + c_{1k}) Q_{1k}^{t+\Delta t} = (E_{2k} c_{1k} - g A_{1k}) h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k} c_{1k}) + g A_{1k} h_{1k}^{t+\Delta t} \tag{28}$$

$$Q_{1k}^{t+\Delta t} = \frac{(E_{2k}c_{1k} - gA_{1k})h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k}c_{1k}) + gA_{1k}h_{1k}^{t+\Delta t}}{a_{1k} + b_{1k} + c_{1k}} \tag{29}$$

Thus for the upstream end of superlink k, the following equation holds:

$$\boxed{Q_{1k}^{t+\Delta t} = U_{1k}h_{2k}^{t+\Delta t} + Y_{1k} + Z_{1k}h_{1k}^{t+\Delta t}} \tag{30}$$

Where:

$$\boxed{T_{1k} = a_{1k} + b_{1k} + c_{1k}} \tag{31}$$

$$\boxed{U_{1k} = \frac{E_{2k}c_{1k} - gA_{1k}}{T_{1k}}} \tag{32}$$

$$\boxed{V_{1k} = \frac{P_{1k} - D_{2k}c_{1k}}{T_{1k}}} \tag{33}$$

$$\boxed{W_{1k} = \frac{gA_{1k}}{T_{1k}}} \tag{34}$$

For the next element downstream, the continuity and momentum equations can be written:

$$Q_{3k}^{t+\Delta t} - Q_{2k}^{t+\Delta t} + E_{3k}h_{3k}^{t+\Delta t} = D_{3k} \tag{35}$$

$$a_{2k}Q_{1k}^{t+\Delta t} + b_{2k}Q_{2k}^{t+\Delta t} + c_{2k}Q_{3k}^{t+\Delta t} = P_{2k} + gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) \tag{36}$$

Combining the two equations together and simplifying yields the following equation:

$$\begin{split} a_{2k}(Q_{2k}^{t+\Delta t} + E_{2k}h_{2k}^{t+\Delta t} - D_{2k}) + (b_{2k})Q_{2k}^{t+\Delta t} + c_{2k}(Q_{2k} - E_{3k}h_{3k}^{t+\Delta t} + D_{3k}) \\ -P_{2k} - gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) = 0 \end{split} \tag{37}$$

$$\begin{split} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} + (E_{2k}a_{2k} - gA_{2k})h_{2k}^{t+\Delta t} \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-D_{2k}a_{2k} + D_{3k}c_{2k} - P_{2k}) = 0 \end{split} \tag{38}$$

Multiplying $h_{2k}^{t+\Delta t}$ by $(U_{1k}-E_{2k})/(U_{1k}-E_{2k})$ and rearranging yields:

$$\begin{aligned}(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t}+\frac{(E_{2k}a_{2k}-gA_{2k})(U_{1k}-E_{2k})}{(U_{1k}-E_{2k})}h_{2k}^{t+\Delta t}\\+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k})=0\end{aligned}\tag{39}$$

Note that:

$$U_{1k}h_{2k}^{t+\Delta t}=(Q_{1k}^{t+\Delta t}-V_{1k}-W_{1k}h_{1k}^{t+\Delta t})\tag{40}$$

$$E_{2k}h_{2k}^{t+\Delta t}=(D_{2k}-Q_{2k}^{t+\Delta t}+Q_{1k}^{t+\Delta t})\tag{41}$$

Thus:

$$\begin{aligned}(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t}\\+\frac{(E_{2k}a_{2k}-gA_{2k})}{(U_{1k}-E_{2k})}[(Q_{1k}^{t+\Delta t}-V_{1k}-W_{1k}h_{1k}^{t+\Delta t})-(D_{2k}-Q_{2k}^{t+\Delta t}+Q_{1k}^{t+\Delta t})]\\+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k})=0\end{aligned}\tag{42}$$

Allowing $Q_{1k}^{t+\Delta t}$ to be eliminated:

$$\begin{aligned}(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t}+\frac{(E_{2k}a_{2k}-gA_{2k})}{U_{1k}-E_{2k}}Q_{2k}^{t+\Delta t}\\+\frac{(E_{2k}a_{2k}-gA_{2k})(-W_{1k})}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}\\+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(E_{2k}a_{2k}-gA_{2k})\frac{(-V_{1k}-D_{2k})}{(U_{1k}-E_{2k})})=0\end{aligned}\tag{43}$$

Rearranging:

$$\begin{aligned}\left(a_{2k}+b_{2k}+c_{2k}-\frac{gA_{2k}-E_{2k}a_{2k}}{U_{1k}-E_{2k}}\right)Q_{2k}^{t+\Delta t}\\+\frac{(gA_{2k}-E_{2k}a_{2k})W_{1k}}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}\\+\left(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(gA_{2k}-E_{2k}a_{2k})\frac{V_{1k}+D_{2k}}{U_{1k}-E_{2k}}\right)=0\end{aligned}\tag{44}$$

$$\left(a_{2k} + b_{2k} + c_{2k} - \frac{gA_{2k} - E_{2k}a_{2k}}{U_{1k} - E_{2k}}\right)Q_{2k}^{t+\Delta t}$$
$$= (E_{3k}c_{2k} - gA_{2k})h_{3k}^{t+\Delta t}$$
$$+\left(P_{2k} + D_{2k}a_{2k} - D_{3k}c_{2k} - (gA_{2k} - E_{2k}a_{2k})\frac{V_{1k} + D_{2k}}{(U_{1k} - E_{2k})}\right) \tag{45}$$
$$-\frac{(gA_{2k} - E_{2k}a_{2k})W_{1k}}{U_{1k} - E_{2k}}h_{1k}^{t+\Delta}$$

Generalizing for $i = 2$, $I = 2$:

$$\left(a_{ik} + b_{ik} + c_{ik} - \frac{gA_{ik} - E_{Ik}a_{ik}}{U_{I-1k} - E_{Ik}}\right)Q_{ik}^{t+\Delta t}$$
$$= (E_{I+1k}c_{ik} - gA_{ik})h_{I+1k}^{t+\Delta t}$$
$$+\left(P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{ik}a_{ik})\frac{V_{I-1k} + D_{Ik}}{U_{I-1k} - E_{Ik}}\right) \tag{46}$$
$$-\frac{(gA_{ik} - E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k} - E_{Ik}}h_{1k}^{t+\Delta}$$

Condensing in terms of the coefficients yields the following recurrence relation for the mass and momentum balance in the forward direction:

$$\boxed{Q_{ik}^{t+\Delta t} = U_{Ik}h_{I+1k}^{t+\Delta t} + V_{Ik} + W_{Ik}h_{1k}^{t+\Delta t}} \tag{47}$$

Where:

$$\boxed{U_{Ik} = \frac{E_{I+1k}c_{ik} - gA_{ik}}{T_{ik}}} \tag{48}$$

$$\boxed{V_{Ik} = \frac{P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{Ik}a_{ik})\frac{V_{I-1k}+D_{Ik}}{U_{I-1k}-E_{Ik}}}{T_{ik}}} \tag{49}$$

$$\boxed{W_{Ik} = -\frac{(gA_{ik} - E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k} - E_{Ik}}} \tag{50}$$

$$\boxed{T_{ik} = \left(a_{ik} + b_{ik} + c_{ik} - \frac{gA_{ik} - E_{Ik}a_{ik}}{U_{I-1k} - E_{Ik}}\right)} \tag{51}$$

5.2 Backward recurrence

Starting at the downstream end of superlink k, the continuity and momentum equations can be written as:

$$Q_{nk}^{t+\Delta t} - Q_{nk-1}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} = D_{Nk} \tag{52}$$

$$a_{nk} Q_{nk-1}^{t+\Delta t} + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk+1}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{53}$$

Assuming $Q_{nk}^{t+\Delta t} = Q_{nk+1}^{t+\Delta t}$, the continuity and momentum equations can be combined and simplified as follows:

$$a_{nk} (Q_{nk}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} - D_{Nk}) + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{54}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = -E_{Nk} a_{nk} h_{Nk}^{t+\Delta t} + (P_{nk} + a_{nk} D_{Nk}) + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{55}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = (g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t} \tag{56}$$

$$Q_{nk}^{t+\Delta t} = \frac{(g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t}}{(a_{nk} + b_{nk} + c_{nk})} \tag{57}$$

Thus for the downstream end of superlink k, the mass and momentum balance can be represented by the following equation:

$$\boxed{Q_{nk}^{t+\Delta t} = X_{Nk} h_{Nk}^{t+\Delta t} + Y_{Nk} + Z_{Nk} h_{Nk+1}^{t+\Delta t}} \tag{58}$$

Where:

$$\boxed{O_{nk} = a_{nk} + b_{nk} + c_{nk}} \tag{59}$$

$$\boxed{X_{Nk} = \frac{(g A_{nk} - E_{Nk} a_{nk})}{O_{nk}}} \tag{60}$$

$$\boxed{Y_{Nk} = \frac{P_{nk} + D_{Nk}a_{nk}}{O_{nk}}} \tag{61}$$

$$\boxed{Z_{Nk} = -\frac{gA_{nk}}{O_{nk}}} \tag{62}$$

For the next element upstream, the continuity and momentum equations can be written:

$$Q_{nk-1}^{t+\Delta t} - Q_{nk-2}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} = D_{Nk-1} \tag{63}$$

$$a_{nk-1}Q_{nk-2}^{t+\Delta t} + b_{nk-1}Q_{nk-1}^{t+\Delta t} + c_{nk-1}Q_{nk}^{t+\Delta t} = P_{nk-1} + gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) \tag{64}$$

$$\begin{aligned} a_{nk-1}(Q_{nk-1}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} - D_{Nk-1}) + (b_{nk-1})Q_{nk-1}^{t+\Delta t} + c_{nk-1}(Q_{nk-1} - E_{Nk}h_{Nk}^{t+\Delta t} + D_{Nk}) \\ -P_{nk-1} - gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) = 0 \end{aligned} \tag{65}$$

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + (-E_{Nk}c_{nk-1} + gA_{nk-1})h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} - P_{nk-1}) = 0 \end{aligned} \tag{66}$$

Multiplying $h_{Nk}^{t+\Delta t}$ by $(X_{Nk} + E_{Nk})/(X_{Nk} + E_{Nk})$ and rearranging:

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(X_{Nk} + E_{Nk})}{(X_{Nk} + E_{Nk})}h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1}) = 0 \end{aligned} \tag{67}$$

Note that:

$$X_{Nk}h_{Nk}^{t+\Delta t} = (Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk}h_{Nk+1}^{t+\Delta t}) \tag{68}$$

$$E_{Nk}h_{Nk}^{t+\Delta t} = (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t}) \tag{69}$$

Thus:

$$
\begin{aligned}
&(a_{nk-1}+b_{nk-1}+c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}[(Q_{nk}^{t+\Delta t}-Y_{Nk}-Z_{Nk}h_{Nk+1}^{t+\Delta t})+(D_{Nk}-Q_{nk}^{t+\Delta t}+Q_{nk-1}^{t+\Delta t})] \\
&+(E_{Nk-1}a_{nk-1}-gA_{nk-1})h_{Nk-1}^{t+\Delta t}+(-P_{nk-1}-D_{Nk-1}a_{nk-1}+D_{Nk}c_{nk-1})=0
\end{aligned}
\tag{70}
$$

Allowing $Q_{nk}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
&(a_{nk-1}+b_{nk-1}+c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}Q_{nk-1}^{t+\Delta t}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})(-Z_{Nk})}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1}-gA_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(-P_{nk-1}-D_{Nk-1}a_{nk-1}+D_{Nk}c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})(-Y_{Nk}+D_{Nk})}{(X_{Nk}+E_{Nk})}\right)=0
\end{aligned}
\tag{71}
$$

Rearranging:

$$
\begin{aligned}
&\left(a_{nk-1}+b_{nk-1}+c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1}-gA_{nk-1})h_{Nk-1}^{t+\Delta t}-\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+\left(-P_{nk-1}-D_{Nk-1}a_{nk-1}+D_{Nk}c_{nk-1}+(gA_{nk-1}-E_{Nk}c_{nk-1})\frac{(D_{Nk}-Y_{Nk})}{(X_{Nk}+E_{Nk})}\right)=0
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
&\left(a_{nk-1}+b_{nk-1}+c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&=(gA_{nk-1}-E_{Nk-1}a_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(P_{nk-1}+D_{Nk-1}a_{nk-1}-D_{Nk}c_{nk-1}-(gA_{nk-1}-E_{Nk}c_{nk-1})\frac{(D_{Nk}-Y_{Nk})}{(X_{Nk}+E_{Nk})}\right) \\
&+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{73}
$$

Generalizing for $i=nk-1$, $I=Nk-1$:

$$
\begin{aligned}
\left(a_{ik} + b_{ik} + c_{ik} + \frac{(gA_{ik} - E_{I+1k}c_{ik})}{(X_{I+1k} + E_{I+1k})} \right) Q_{ik}^{t+\Delta t} \\
= (gA_{ik} - E_{Ik}a_{ik}) h_{Ik}^{t+\Delta t} \\
+ \left(P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{I+1k}c_{ik}) \frac{(D_{I+1k} - Y_{I+1k})}{(X_{I+1k} + E_{I+1k})} \right) \\
+ \frac{(gA_{ik} - E_{I+1k}c_{ik}) Z_{I+1k}}{(X_{I+1k} + E_{I+1k})} h_{Nk+1}^{t+\Delta t}
\end{aligned} \tag{74}
$$

Condensing in terms of the coefficients yields the following recurrence relation for the mass and momentum balance in the backwards direction:

$$
\boxed{Q_{ik}^{t+\Delta t} = X_{ik} h_{Ik}^{t+\Delta t} + Y_{Ik} + Z_{Ik} h_{Nk+1}^{t+\Delta t}} \tag{75}
$$

Where:

$$
\boxed{X_{Ik} = \frac{gA_{ik} - E_{Ik}a_{ik}}{O_{ik}}} \tag{76}
$$

$$
\boxed{Y_{Ik} = \frac{P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{I+1k}c_{ik}) \frac{(D_{I+1k} - Y_{I+1k})}{(X_{I+1k} + E_{I+1k})}}{O_{ik}}} \tag{77}
$$

$$
\boxed{Z_{Ik} = \frac{(gA_{ik} - E_{I+1k}c_{ik}) Z_{I+1k}}{(X_{I+1k} + E_{I+1k}) O_{ik}}} \tag{78}
$$

$$
\boxed{O_{ik} = \left(a_{ik} + b_{ik} + c_{ik} + \frac{gA_{ik} - E_{I+1k}c_{ik}}{X_{I+1k} + E_{I+1k}} \right)} \tag{79}
$$

6 Inlet hydraulics

6.1 Depth at upstream end of superlink

The discharge at the upstream end of a superlink is given by:

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \tag{80}$$

Where:

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \tag{81}$$

Squaring and rearranging provides the depth boundary condition at the upstream end:

$$Q_{uk}^2 = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{82}$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{83}$$

$$\boxed{h_{uk} = -\frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{2 C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk}} \tag{84}$$

6.2 Depth at downstream end of superlink

The discharge at the downstream end of a superlink is given by:

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \tag{85}$$

Where:

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \tag{86}$$

Squaring and rearranging provides the depth boundary condition at the downstream end:

$$Q_{dk}^2 = 2 C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \tag{87}$$

$$|Q_{dk}^t|Q_{dk}^{t+\Delta t} = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{88}$$

$$\boxed{h_{dk} = \frac{|Q_{dk}^t|Q_{dk}^{t+\Delta t}}{2C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk}} \tag{89}$$

6.3 Superlink boundary conditions

From the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k} h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k} h_{dk}^{t+\Delta t} \tag{90}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk} h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk} h_{uk}^{t+\Delta t} \tag{91}$$

From the depth boundary conditions at the ends of each superlink:

$$h_{uk} = \gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \tag{92}$$

$$h_{dk} = \gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \tag{93}$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{94}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{95}$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \tag{96}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \tag{97}$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk}Q_{uk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} - X_{1k}z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} - Z_{1k}z_{inv,dk} \quad (98)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk}Q_{dk}^{t+\Delta t} + U_{Nk}H_{jdk} - U_{Nk}z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + W_{Nk}H_{juk} - W_{Nk}z_{inv,uk} \quad (99)$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1)Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} + \pi_1 \quad (100)$$

$$0 = W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1)Q_{dk}^{t+\Delta t} + W_{Nk}H_{juk} + U_{Nk}H_{jdk} + \pi_2 \quad (101)$$

Where:

$$\pi_1 = Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk} \quad (102)$$

$$\pi_2 = V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk} \quad (103)$$

$$(104)$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (105)$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (106)$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \quad (107)$$

Expanding:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1)(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \tag{108}$$

Arranging in terms of the unknown heads:

$$\begin{aligned} Q_{uk}^{t+\Delta t} = [(U_{Nk}\gamma_{dk} - 1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ [(U_{Nk}\gamma_{dk} - 1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ [(U_{Nk}\gamma_{dk} - 1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)] \end{aligned} \tag{109}$$

$$\begin{aligned} Q_{dk}^{t+\Delta t} = [(-W_{Nk}\gamma_{uk})(-X_{1k}) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk} - 1)(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk} - 1)(-\pi_2)] \end{aligned} \tag{110}$$

Finally, the flow rates at the upstream and downstream ends of superlink k can be expressed as:

$$\boxed{Q_{uk}^{t+\Delta t} = \alpha_{uk}H_{juk}^{t+\Delta t} + \beta_{uk}H_{jdk}^{t+\Delta t} + \chi_{uk}} \tag{111}$$

$$\boxed{Q_{dk}^{t+\Delta t} = \alpha_{dk}H_{juk}^{t+\Delta t} + \beta_{dk}H_{jdk}^{t+\Delta t} + \chi_{dk}} \tag{112}$$

Where:

$$\boxed{\alpha_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*}} \tag{113}$$

$$\boxed{\beta_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*}} \tag{114}$$

$$\boxed{\chi_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*}} \tag{115}$$

$$\boxed{\alpha_{dk} = \frac{(1 - X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*}} \tag{116}$$

$$\boxed{\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*}} \tag{117}$$

$$\boxed{\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*}} \tag{118}$$

$$\boxed{D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk})} \tag{119}$$

$$\boxed{\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g}} \tag{120}$$

$$\boxed{\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g}} \tag{121}$$

7 Forming the solution matrix

The equations for the flows at the ends of each superlink are given by:

$$\sum_{l=1}^{NBDj} Q_{dk_l}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{uk_m}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j)}{\Delta t} \tag{122}$$

Substituting the linear expressions for the upstream and downstream flows:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} &= \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_{jdk_l}^{t+\Delta t} + \chi_{dk_l}) \\ &- \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_{juk_m}^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{123}$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$\frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_j^{t+\Delta t} + \chi_{dk_l}) \\ - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_j^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \tag{124}$$

Rearranging:

$$\left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{uk_m} - \sum_{l=1}^{NBDj} \beta_{dk_l}\right) H_j^{t+\Delta t} - \sum_{l=1}^{NBDj} \alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{uk_m} H_{jdk_m}^{t+\Delta t} \\ = \frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dk_l} - \sum_{m=1}^{NBUj} \chi_{uk_m} + Q_{o,j} \tag{125}$$

The continuity equation for each superjunction can thus be redefined in terms of the following coefficients.

$$\boxed{F_{j,j} H_j^{t+\Delta t} + \sum_{\ell=1}^{NBDj} \Phi_{j,juk_\ell} H_{juk_\ell}^{t+\Delta t} + \sum_{m=1}^{NBUj} \Psi_{j,jdk_m} H_{jdk_m}^{t+\Delta t} = G_j} \tag{126}$$

Where:

$$\boxed{F_{j,j} = \frac{A_{sj}}{\Delta t} - \sum_{\ell=1}^{NBDj} \beta_{dk_\ell} + \sum_{m=1}^{NBUj} \alpha_{uk_m}} \tag{127}$$

$$\boxed{\Phi_{j,juk_\ell} = -\alpha_{dk_\ell}} \tag{128}$$

$$\boxed{\Psi_{j,jdk_m} = \beta_{uk_m}} \tag{129}$$

$$\boxed{G_j = \frac{A_{sj}}{\Delta t} H_j^t + Q_{0,j} - \sum_{\ell=1}^{NBDj} \chi_{uk_\ell} + \sum_{m=1}^{NBUj} \chi_{dk_m}} \tag{130}$$

Solution matrix equation for example network

For the example network in Ji (1998), the sparse matrix equation is given as:

$$Ax = b \tag{131}$$

$$\begin{bmatrix} F_{1,1} & \Psi_{1,2} & 0 & 0 & 0 & 0 \\ \Phi_{2,1} & \Psi_{2,2} & \Psi_{2,3} & 0 & \Psi_{2,5} & 0 \\ 0 & \Phi 3,2 & F_{3,3} & \Psi_{3,4} & \Phi_{3,5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \Phi_{5,2} & \Psi_{5,3} & 0 & F_{5,5} & \Psi_{5,6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} H_1^{t+\Delta t} \\ H_2^{t+\Delta t} \\ H_3^{t+\Delta t} \\ H_4^{t+\Delta t} \\ H_5^{t+\Delta t} \\ H_6^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{bmatrix} \tag{132}$$

Expanding the coefficients:

$$A = \begin{bmatrix} (\frac{A_{s1}}{\Delta t} + \alpha_{u1}) & \beta_{u1} & 0 & 0 & 0 & 0 \\ -\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} + \alpha_{u4} - \beta_{d1}) & \beta_{u2} & 0 & \beta_{u4} & 0 \\ 0 & -\alpha_{d2} & (\frac{A_{s3}}{\Delta t} + \alpha_{u3} - \beta_{d2} - \beta_{d6}) & \beta_{u3} & -\alpha_{d6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha_{d4} & \beta_{u6} & 0 & (\frac{A_{s5}}{\Delta t} + \alpha_{u5} + \alpha_{u6} - \beta_{d4}) & \beta_{u5} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{133}$$

$$b = \begin{bmatrix} \frac{A_{s1} H_1^t}{\Delta t} - \chi_{u1} + Q_{0,1} \\ \frac{A_{s2} H_2^t}{\Delta t} + \chi_{d1} - (\chi_{u2} + \chi_{u4}) + Q_{0,2} \\ \frac{A_{s3} H_3^t}{\Delta t} + (\chi_{d2} + \chi_{d6}) - \chi_{u3} + Q_{0,3} \\ H_{4,bc} \\ \frac{A_{s5} H_5^t}{\Delta t} + \chi_{d4} - (\chi_{u5} + \chi_{u6}) + Q_{0,5} \\ H_{6,bc} \end{bmatrix} \tag{134}$$

8 Representing orifices

For orifices, six different flow cases are possible:

- Side-mounted orifice with both sides submerged
- Side-mounted orifice with one side submerged

- Side-mounted orifice with weir-like flow
- Bottom-mounted orifice with both sides submerged
- Bottom-mounted orifice with one side submerged
- No-flow condition

8.1 Governing equations for orifices

The governing equations for each condition are presented here:

Side-mounted orifice with both sides submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o + \omega y_{max,o}$
- $\min(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o + \frac{\omega y_{max,o}}{2}$

The effective head is computed as:

$$H_{e,o} = |H_{juo} - H_{jdo}| \tag{135}$$

And the flow is computed as:

$$Q_o = \mathrm{sgn}(H_{juo} - H_{jdo}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{136}$$

Side-mounted orifice with one side submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o + \omega y_{max,o}$
- $\min(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) < z_o + \frac{\omega y_{max,o}}{2}$

The effective head is computed as:

$$H_{e,o} = \cdot\big[\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,jdo}) - (z_o + \frac{\omega y_{max,o}}{2})\big] \tag{137}$$

And the flow is computed as:

$$Q_o = \text{sgn}(H_{juo} - H_{jdo}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{138}$$

Side-mounted orifice with weir-like flow

This flow regime occurs when both of the following conditions are met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o$
- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) < z_o + \omega y_{max,o}$

The effective head is computed as:

$$H_{e,o} = \max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) - z_o \tag{139}$$

And the flow is computed as:

$$Q_o = \frac{C_o A_o \sqrt{g}}{\omega y_{max,o}} \sqrt{H_{e,o}} \tag{140}$$

Bottom-mounted orifice with both sides submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o$
- $\min(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o$

The effective head is computed as:

$$H_{e,o} = |H_{juo} - H_{jdo}| \tag{141}$$

And the flow is computed as:

$$Q_o = \text{sgn}(H_{juo} - H_{jdo}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{142}$$

Bottom-mounted orifice with one side submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) > z_o$
- $\min(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) < z_o$

The effective head is computed as:

$$H_{e,o} = \cdot \big[\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,do}) - z_o\big] \tag{143}$$

And the flow is computed as:

$$Q_o = \text{sgn}(H_{juo} - H_{jdo}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{144}$$

No-flow condition

This flow regime occurs when the following condition is met:

- $\max(H_{juo} - z_{inv,juo}, H_{jdo} - z_{inv,juo}) \leq z_o$

In this case, the effective head and flow are both zero:

$$H_{e,o} = 0 \tag{145}$$

$$Q_o = 0 \tag{146}$$

8.2 Representing orifice equations in the solution matrix

Orifices can be represented in the solution matrix as follows.

Define the following indicator functions:

$$\Omega(H_{juo}, H_{jdo}) = \begin{cases} 1, & H_{juo} \geq H_{jdo} \\ 0, & o/w \end{cases} \tag{147}$$

$$\tau(o) = \begin{cases} 1, & \text{orifice } o \text{ is side-mounted} \\ 0, & \text{orifice } o \text{ is bottom-mounted} \end{cases} \tag{148}$$

Similarly, define boolean-valued functions to represent the following flow conditions:

Submerged on high-head side

$$\Theta_{o,1} = \begin{cases} 1, & \Omega H_{juo} + (1-\Omega) H_{jdo} > z_o + z_{inv,juo} + \tau \omega y_{max,o} \\ 0, & o/w \end{cases} \tag{149}$$

Submerged on low-head side

$$\Theta_{o,2} = \begin{cases} 1, & (1-\Omega) H_{juo} + \Omega H_{jdo} > z_o + z_{inv,juo} + \frac{\tau \omega y_{max,o}}{2} \\ 0, & o/w \end{cases} \tag{150}$$

Above bottom rim on high-head side

$$\Theta_{o,3} = \begin{cases} 1, & \Omega H_{juo} + (1-\Omega) H_{jdo} > z_o + z_{inv,juo} \\ 0, & o/w \end{cases} \tag{151}$$

The flow through an orifice can now be represented using the following linearized coefficient equation:

$$\boxed{Q_o^{t+\Delta t} = \alpha_o H_{juo}^{t+\Delta t} + \beta_o H_{jdo}^{t+\Delta t} + \chi_o} \tag{152}$$

Where:

$$\boxed{\alpha_o = \begin{cases} \gamma_o \omega^2, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o \Omega (-1)^{1-\Omega} \omega^2, & \Theta_{o,1} \wedge \neg\Theta_{o,2} \\ \frac{\gamma_o}{2 y_{max,o}^2} \Omega (-1)^{1-\Omega}, & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases}} \tag{153}$$

$$\boxed{\beta_o = \begin{cases} -\gamma_o \omega^2, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o (1-\Omega)(-1)^{1-\Omega}\omega^2, & \Theta_{o,1} \wedge \neg\Theta_{o,2} \\ \frac{\gamma_o}{2y_{max,o}^2}(1-\Omega)(-1)^{1-\Omega}, & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases}} \tag{154}$$

$$\boxed{\chi_o = \begin{cases} 0, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o (-1)^{1-\Omega}(-z_{inv,uo} - z_o - \frac{\tau \omega y_{max,o}}{2}), & \Theta_{o,1} \wedge \neg\Theta_{o,2} u^2 \\ \frac{\gamma_o}{2y_{max,o}^2}(-z_{inv,uo} - z_o), & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases}} \tag{155}$$

$$\boxed{\gamma_o = \frac{2gC_o^2 A_o^2}{|Q_o^t|}} \tag{156}$$

These equations can be added to the solution matrix in much the same way as the linearized superlink coefficients ($\alpha_{uk}, \beta_{uk}, \chi_{uk}, \alpha_{dk}, \beta_{dk}, \chi_{dk}$). The system of equations is now represented by the following matrix equation:

$$(A + O)x = b + b_o \tag{157}$$

With the A matrix and b vector representing the original SUPERLINK system. The O matrix and b_o vector represent the orifice equations. The elements of O are defined by the following coefficients:

$$\boxed{F_{j,j}^o = -\sum_{\ell=1}^{NBDj} \beta_{o_\ell} + \sum_{m=1}^{NBUj} \alpha_{o_m}} \tag{158}$$

$$\boxed{\Phi_{j,juo_\ell}^o = -\alpha_{o_\ell}} \tag{159}$$

$$\boxed{\Psi_{j,jdo_m}^o = \beta_{o_m}} \tag{160}$$

And the elements of b_o defined by:

$$\boxed{G_j^o = -\sum_{\ell=1}^{NBDj} \chi_{o_\ell} + \sum_{m=1}^{NBUj} \chi_{o_m}} \tag{161}$$

9 Representing weirs

This section discusses the governing equations for weirs, and explains how weirs can be incorporated into the solution matrix. Only transverse weirs will be considered.

9.1 Governing equations for weirs

First, without loss of generality, assume all weirs can be represented as trapezoidal weirs (given that both rectangular and triangular weirs are special cases of the trapezoidal weir).

The effective head on a weir can be defined as:

$$H_{e,w} = \max\left(\max\left(H_{juw}, H_{jdw}\right) - (z_w + z_{inv,juw} + (1-\omega)y_{max,w}), 0\right) \tag{162}$$

The flow through a trapezoidal weir is the sum of the flow through the rectangular and triangular sections:

$$Q_w = C_{wR} L_w H_{e,w}^{3/2} + C_{wT} s_w H_{e,w}^{5/2} \tag{163}$$

The flow at the next time step can thus be estimated as:

$$Q_w^{t+\Delta t} = \frac{C_{wR} L_w H_{e,w}^t + C_{wT} s_w (H_{e,w}^t)^2}{|Q_w^t|} H_{e,w}^{t+\Delta t} \tag{164}$$

9.2 Representing weirs in the solution matrix

Define the following indicator function:

$$\Omega(H_{juw}, H_{jdw}) = \begin{cases} 1, & H_{juw} \geq H_{jdw} \\ 0, & o/w \end{cases} \tag{165}$$

Similarly, define boolean-valued functions to represent the following flow conditions:

Submerged on high-head side

$$\Theta_{w,1} = \begin{cases} 1, & \Omega H_{juw} + (1-\Omega)H_{jdw} > z_w + z_{inv,juw} + (1-\omega)y_{max,w} \\ 0, & o/w \end{cases} \tag{166}$$

Submerged on low-head side

$$\Theta_{w,2} = \begin{cases} 1, & (1-\Omega)H_{juw} + \Omega H_{jdw} > z_o + z_{inv,juw} + (1-\omega)y_{max,w} \\ 0, & o/w \end{cases} \tag{167}$$

The flow through a weir can now be represented using the following linearized coefficient equation:

$$\boxed{Q_w^{t+\Delta t} = \alpha_w H_{juw}^{t+\Delta t} + \beta_w H_{jdw}^{t+\Delta t} + \chi_w} \tag{168}$$

Where:

$$\boxed{\alpha_w = \begin{cases} \gamma_w, & \Theta_{w,1} \land \Theta_{w,2} \\ \gamma_w \Omega(-1)^{1-\Omega}, & \Theta_{w,1} \land \neg\Theta_{w,2} \\ 0, & \neg\Theta_{w,1} \end{cases}} \tag{169}$$

$$\boxed{\beta_w = \begin{cases} -\gamma_w, & \Theta_{w,1} \land \Theta_{w,2} \\ \gamma_w (1-\Omega)(-1)^{1-\Omega}, & \Theta_{w,1} \land \neg\Theta_{w,2} \\ 0, & \neg\Theta_{w,1} \end{cases}} \tag{170}$$

$$\boxed{\chi_w = \begin{cases} 0, & \Theta_{w,1} \land \Theta_{w,2} \\ \gamma_w (-1)^{1-\Omega}[-z_{inv,juw} - z_w - (1-\omega)y_{max,w}], & \Theta_{w,1} \land \neg\Theta_{w,2} \\ 0, & \neg\Theta_{w,3} \end{cases}} \tag{171}$$

$$\boxed{\gamma_w = \frac{C_{wR} L_w H_{e,w}^t + C_{wT} s_w (H_{e,w}^t)^2}{|Q_w^t|}} \tag{172}$$

Thus, a system with weirs can be represented as:

$$(A + W)x = b + b_w \tag{173}$$

With the A matrix and b vector representing the original SUPERLINK system. The W matrix and b_w vector represent the weir equations. The elements of W are defined by the following coefficients:

$$\boxed{F^w_{j,j} = -\sum_{\ell=1}^{NBDj} \beta_{w_\ell} + \sum_{m=1}^{NBUj} \alpha_{w_m}} \tag{174}$$

$$\boxed{\Phi^w_{j,juw_\ell} = -\alpha_{w_\ell}} \tag{175}$$

$$\boxed{\Psi^w_{j,jdw_m} = \beta_{w_m}} \tag{176}$$

And the elements of b_w defined by:

$$\boxed{G^w_j = -\sum_{\ell=1}^{NBDj} \chi_{w_\ell} + \sum_{m=1}^{NBUj} \chi_{w_m}} \tag{177}$$

10 Representing pumps

10.1 Governing equations for pumps

The relationship between flow and head in a pump is usually defined by a pump curve. For this implementation, we assume that the flow/head relationship can be approximated by an ellipse centered at the origin defined over the support $[H_{min,p}, H_{max,p}]$.

First, define the effective head for the pump as follows:

$$H_{e,p} = \begin{cases} H_{max,p}, & H_{jdp} - H_{jup} > H_{max,p} \\ H_{jdp} - H_{jup}, & H_{min,p} < H_{jdp} - H_{jup} < H_{max,p} \\ H_{min,p}, & H_{jdp} - H_{jup} < H_{min,p} \end{cases} \tag{178}$$

Then, using the elliptical approximation, the flow through the pump can be represented as:

$$Q_p = \omega \sqrt{a_q^2 (1 - \frac{H_{e,p}^2}{a_h^2})} \tag{179}$$

10.2 Representing pumps in the solution matrix

Define boolean-valued functions to represent the following flow conditions:

Submerged inlet

$$\Theta_{p,1} = \begin{cases} 1, & H_{jup} \geq z_{inv,jup} + z_p \\ 0, & o/w \end{cases} \tag{180}$$

Head in pump curve range

$$\Theta_{p,2} = \begin{cases} 1, & H_{min,p} < H_{jdp} - H_{jup} < H_{max,p} \\ 0, & o/w \end{cases} \tag{181}$$

The flow through a pump can now be represented using the following linearized coefficient equation:

$$\boxed{Q_p^{t+\Delta t} = \alpha_p H_{jup}^{t+\Delta t} + \beta_p H_{jdp}^{t+\Delta t} + \chi_p} \tag{182}$$

$$\boxed{\alpha_p = \begin{cases} \gamma_p \omega^2, & \Theta_{p,1} \wedge \Theta_{p,2} \\ 0, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases}} \tag{183}$$

$$\boxed{\beta_p = \begin{cases} -\gamma_p \omega^2, & \Theta_{p,1} \wedge \Theta_{p,2} \\ 0, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases}} \tag{184}$$

$$\boxed{\chi_p = \begin{cases} \frac{a_q^2}{|Q_p^t|}, & \Theta_{p,1} \wedge \Theta_{p,2} \\ \omega\sqrt{a_q^2\left(1 - \frac{(H_{e,p}^t)^2}{a_h^2}\right)}, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases}} \tag{185}$$

$$\boxed{\gamma_p = \frac{a_q^2 |H_{dp}^t - H_{up}^t|}{a_h^2 |Q_p^t|}} \tag{186}$$

Thus, a system with pumps can be represented as:

$$(A + P)x = b + b_p \tag{187}$$

With the A matrix and b vector representing the original SUPERLINK system. The P matrix and b_p vector represent the weir equations. The elements of P are defined by the following coefficients:

$$\boxed{F^p_{j,j} = -\sum_{\ell=1}^{NBDj} \beta_{p_\ell} + \sum_{m=1}^{NBUj} \alpha_{p_m}} \tag{188}$$

$$\boxed{\Phi^p_{j,jup_\ell} = -\alpha_{p_\ell}} \tag{189}$$

$$\boxed{\Psi^p_{j,jdp_m} = \beta_{p_m}} \tag{190}$$

And the elements of b_p defined by:

$$\boxed{G^p_j = -\sum_{\ell=1}^{NBDj} \chi_{p_\ell} + \sum_{m=1}^{NBUj} \chi_{p_m}} \tag{191}$$