A QUANTITATIVE COMPARISON OF NON-PARAMETRIC TIME-FREQUENCY REPRESENTATIONS

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ABSTRACT

In this paper we compare a variety of non-parametric time-frequency methods to determine the best time–frequency representation (TFR) for a collection of signals. These methods include quadratic time-frequency transforms, atomic decomposition and adaptive quadratic time-frequency transforms. The performance measures used to assess the TFRs include; two–dimensional correlation, IF correlation and time-frequency resolution. Synthetic signals with different time-frequency characteristics were used in the comparison to show the strengths and weaknesses of the different time-frequency methods. It was determined that adaptive quadratic time-frequency representations provide the best overall performance and should be used if no *a priori* information about the time-frequency characteristics of a signal is known.

1. INTRODUCTION

Signals with time-varying frequency content occur in many engineering applications such as telecommunications, radar, sonar, power systems, biomedicine and machine condition monitoring. Classical methods of analysis in the time domain or frequency domain cannot represent the joint time-frequency (TF) information contained in non-stationary signals. Therefore, the need for signal representation in a joint TF domain arises.

There are numerous methods for providing joint TF transformations of a signal. Quadratic time-frequency distributions (TFDs) [1], adaptive quadratic TFDs [2, 3] and atomic decomposition [4] are three non-parametric methods of time-frequency representations (TFRs).

The performance of different TFRs is varied for different signals, depending on the signals TF characteristics. However, it would be desirable to have a TFR that was optimal, or near optimal for all types of signals.

In this paper, we compare four methods for TFR to determine which method has the best performance using three signals. The signals chosen for the comparison have different TF characteristics. This will allow us to show the strengths and weaknesses of the various TF methods.

Three TF performance measures, which include, two-dimensional correlation, IF correlation and TF resolution are used to assess the performance of the TFR. The application of these performance measures also aims to determine which TFR should be used when no *a priori* information about the TF characteristics of the signal is known.

2. TIME-FREQUENCY METHODS

2.1 Quadratic TFD

The fundamental quadratic TFD from which other quadratic TFDs can be derived is the Wigner-Ville distribution (WVD). The WVD is formed by correlating a function $x(t)$ with a time and frequency translation of itself, [4], such that

$$\mathrm{WVD}_z(t,f) = \int_{-\infty}^{\infty} K_z(t,\tau) e^{-j2\pi f\tau} d\tau \qquad (1)$$

where $K_z(t,\tau) = z(t+\frac{\tau}{2})z^*(t-\frac{\tau}{2})$ is the signal kernel and $z(t)$ is the analytic associate of $x(t)$ [1]. The WVD provides high time-frequency resolution for linear FM mono-component signals and provides a positive two–dimensional Gaussian function for a Gaussian signal. However, for non-linear FM and multi-component signals, the WVD produces a number of unwanted artifacts or cross-terms [5]. By smoothing the WVD, interference from artifacts can be reduced. The Spectrogram (SPEC) and Choi-Williams distribution (CW) are two such quadratic TFDs which take the general form

$$\mathbb{E}_z(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u-t,\tau) K_z(t,\tau) e^{-j2\pi f\tau} d\tau du \qquad (2)$$

where $G(t,\tau) = w(t+\frac{\tau}{2})w(t-\frac{\tau}{2})$ for the SPEC and for the CW, $G(t,\tau) = \frac{\sqrt{\pi\sigma}}{|\tau|} e^{-\pi^2\sigma t^2/\tau^2}$ [5]. The function $w(t)$ is the analysis window of the SPEC and σ is a real positive parameter for CW.

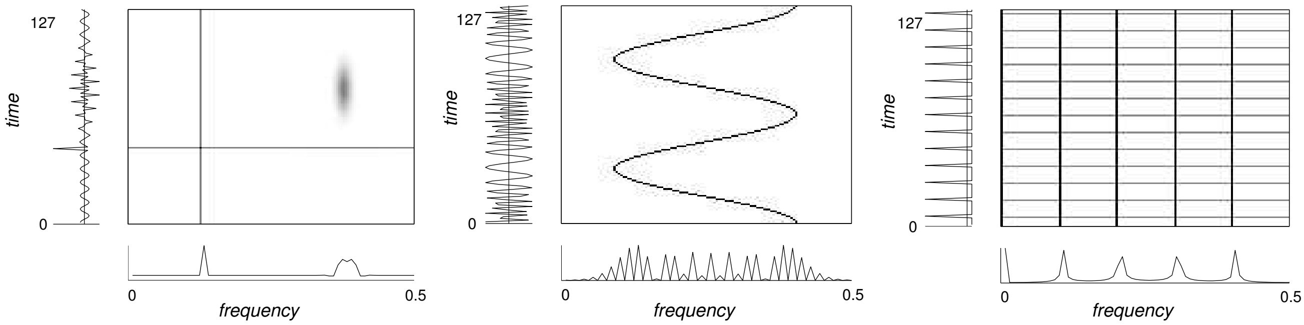
2.2 Adaptive Quadratic TFD

For the spectrogram, there is a trade off between time and frequency resolution that depends on the length of the analysis window $w(t)$. Longer window lengths provide less time localisation but better frequency resolution, whereas shorter windows can represent fast changing signal structures more accurately at the expense of poor frequency resolution. This trade off also applies to other discrete quadratic TFDs in the selection of the lag window length for the signal kernel $K_z(t,\tau)$ [6].

A number of methods for adapting window lengths have been proposed [2, 3]. The adaption criterion used in this comparison is the maximum correlation criterion [2]. Using this criterion, the optimal window scale, $p(t)$, of signal $x(t)$ at time t is given as

$$p(t) = \arg\max_{p\in P} \left| \int_{-\infty}^{\infty} x(\tau) w_p(\tau - t) e^{-j2\pi f\tau} d\tau \right| \qquad (3)$$

where P is a set of window scales and $||w_p(\tau)||_2 = 1$.



(a) “Signal One” (b) “Signal Two” (c) “Signal Three”

Figure 1: Time and frequency representations of the test signals and the ideal TFRs

2.3 Atomic Decomposition

Atomic decomposition (AD) is a method of selecting atoms ϕ_γ from a redundant set (dictionary) $\Phi = (\phi_\gamma)_{\gamma\in\Gamma}$ to provide a signal approximation by a linear superposition of the selected atoms. The subscript γ provides a parameter or list of parameters which uniquely defines each individual atom in Φ. A signal, x, can be approximated using m atoms from Φ such that

$$\hat{x} = \sum_{i=0}^{m-1} \alpha_{\gamma i}\phi_{\gamma i} \tag{4}$$

where $\alpha_{\gamma i}$ is the coefficient associated with $\phi_{\gamma i}$. An approximation error, also referred to as the residual [4], is given as $R^m x = x - \hat{x}$.

Matching pursuit (MP) is currently one of the most widely used AD techniques. It is an iterative algorithm that selects the atom which best represents the residual at each iteration. That is, at each iteration an atom is selected according to: $\phi_{\gamma i} = \arg\{\sup_{\gamma\in\Gamma}\langle R^i x, \phi_\gamma\rangle\}$. The MP decomposition of a function, x, approximating with m atoms, is given as

$$x = \sum_{i=0}^{m-1} \langle R^i x, \phi_{\gamma i}\rangle \phi_{\gamma i} + R^m x \tag{5}$$

where the inner product, $\langle R^i x, \phi_{\gamma i}\rangle$, is the coefficient value, $\alpha_{\gamma i}$, associated with atom $\phi_{\gamma i}$. Each atom in Φ is normalized such that $||\phi_\gamma||_2 = 1 \ \forall\gamma$.

Using a Gabor TF dictionary and the WVD, a TF approximation of signal $x(t)$ using the MP decomposition can be given as

$$\mathbb{E}(t,f) = \sum_{i=0}^{m-1} |\langle R^i x, \phi_{\gamma i}\rangle|^2 \mathrm{WVD}_{\phi_{\gamma i}}(t,f) \tag{6}$$

3. TEST SIGNALS

Three synthetic signals with differing TF characteristics were used in this comparison. The first signal was a multi-component signal constructed from the addition of a Dirac, sinusoid and modulated Gaussian signal. This signal is displayed on the left side of Figure 1(a), which also shows the frequency domain representation on the bottom and the ideal time-frequency representation in the middle of Figure 1(a). This signal was chosen as it has multiple components at the same time instants with differing bandwidths and durations.

The second synthetic signal, shown in Figure 1(b), was a mono-component signal with a nonlinear instantaneous frequency (IF) law. The highly varying IF requires an adaptive window length as short window lengths are desirable when the IF is changing rapidly and longer window lengths provide better resolution when the IF is changing slowly [1].

Figure 1(c) shows a periodic spike train which was the final signal used in this assessment. This signal was chosen as it has time instants with high energy concentration and a harmonic relationship between spikes. This is shown in the time and frequency plots of Figure 1(c). Both these characteristics should be reflected in the TFR.

4. PERFORMANCE MEASURES

The measures used to assess the performance of the discrete estimated TFRs were two–dimensional correlation, IF correlation and TF resolution.

The central two–dimensional correlation is defined as

$$\rho = \frac{\sum_l \sum_k \mathbb{E}_d(l,k)\mathbb{E}_e(l,k)}{\sqrt{\sum_l \sum_k \mathbb{E}_d(l,k)^2 \sum_l \sum_k \mathbb{E}_e(l,k)^2}}, \tag{7}$$

where, $\mathbb{E}_d(l,k)$ is the desired TFR, shown in Figure 1 for each of the test signals, $\mathbb{E}_e(l,k)$ is the estimated TFR and l and k are the discrete time and frequency indexes respectively. This measure was used to assess the difference, in terms of energy distribution, between the estimated TFR and the ideal version.

The IF correlation is defined as

$$\mathrm{IF} = \frac{\sum_l \sum_k \mathbb{E}_{dn}(l,k)\mathbb{E}_{en}(l,k)}{\sqrt{\sum_l \sum_k \mathbb{E}_{dn}(l,k)^2 \sum_l \sum_k \mathbb{E}_{en}(l,k)^2}}, \tag{8}$$

where the n subscript denotes a conditional normalisation of $\mathbb{E}_d$ and $\mathbb{E}_e$ such that

$$\mathbb{E}_{*n}(l,k) = \begin{cases} 1 & \text{if} \quad \mathbb{E}_{*n}(l,k) > 2E \\ 0 & \text{if} \quad \mathbb{E}_{*n}(l,k) \le 2E \end{cases}, \tag{9}$$

where,

$$E = \frac{\sum_l \sum_k \mathbb{E}_e(l,k)}{N^2} \tag{10}$$

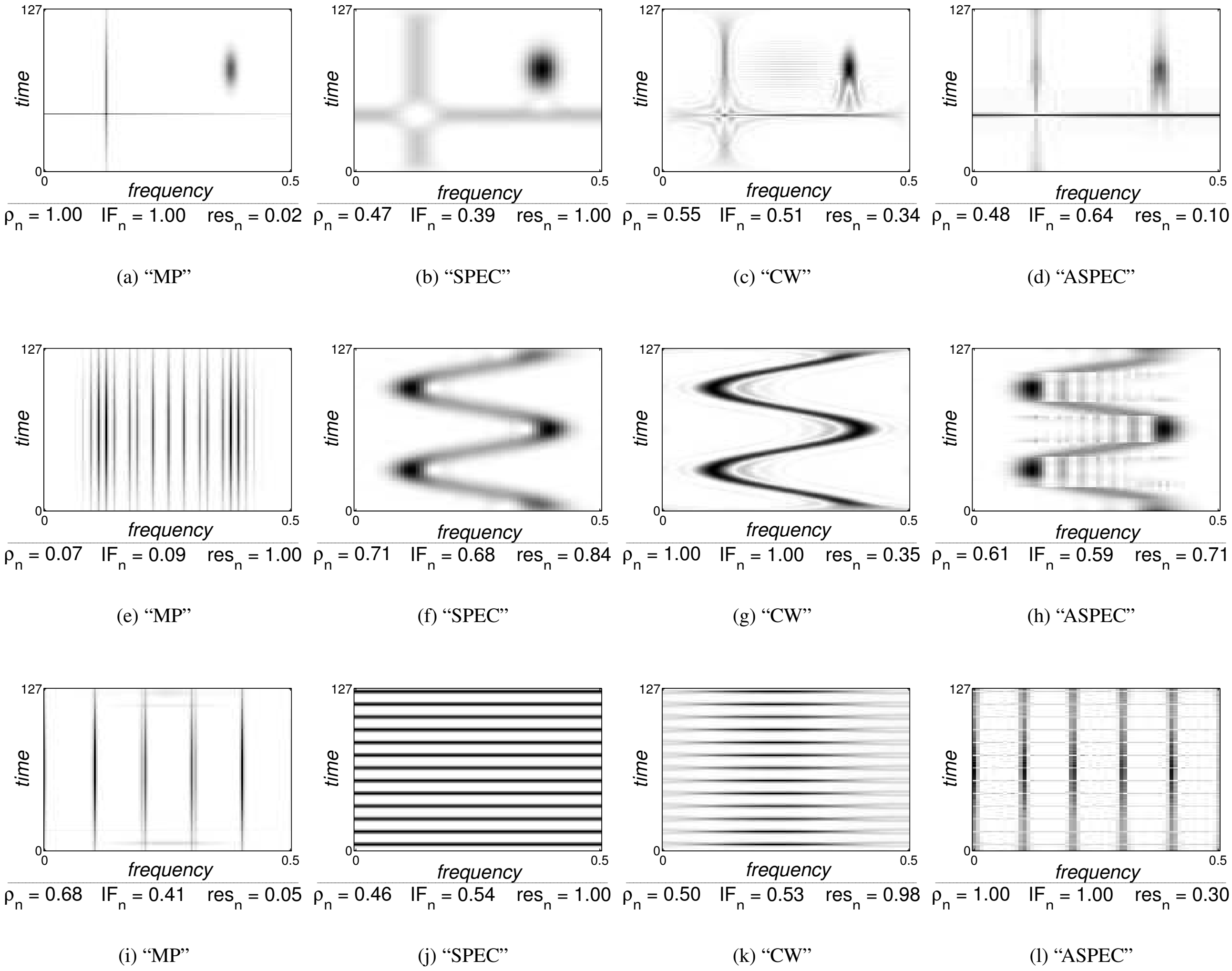


Figure 2: TFRs of test signals

and N is the number of time and frequency samples in $\mathbb{E}(l,k)$. This is a measure of the general shape of the TFR in terms of its IF law/s.

The TF resolution measure is defined as

$$\text{res} = \frac{\sum_l B(l)}{N} \cdot \frac{\sum_k T(k)}{N} \tag{11}$$

where

$$B(l) = \frac{\sum_k \mathbb{E}_e(l,k)}{\max_k \mathbb{E}_e(l,k)}, \quad T(k) = \frac{\sum_l \mathbb{E}_e(l,k)}{\max_l \mathbb{E}_e(l,k)} \tag{12}$$

The resolution measure was used to measure the energy concentration or resolution of the TFR.

5. RESULTS AND DISCUSSION

The window lengths for SPEC and CW, ($\sigma = 11$), were chosen for each test signal. The window sizes were optimized to maximise the IF correlation performance measure, as sub-optimal data window sizes may not give a true indication of the TF characteristics of the signal. Such a procedure is not required for MP and ASPEC as these are adaptive techniques.

Figures 2 (a-l) show the TFRs using MP, SPEC, CW, and the ASPEC of the three test signals. The individual performance measures are given for each TFR. However, the performance measures have been normalised to assess the relative performance between TFRs. The normalisation procedure is defined as

$$\rho_n = \frac{\rho}{\max\{\rho\}}, \quad \text{IF}_n = \frac{\text{IF}}{\max\{\text{IF}\}}, \quad \text{res}_n = \frac{\text{res}}{\max\{\text{res}\}}$$

where the normalisation occurs for each test signal.

For two–dimensional correlation and IF correlation, the value 1.00 infers the best performance and for TF resolution, the value 0.0 relates to the best performance.

For the first test signal, it is clear that MP outperforms the other TFRs according to all three performance values. This result was anticipated as the synthesised signal was created using atoms included in the time-frequency dictionary used by MP for decomposition. This demonstrated that MP can

Table 1: Average value of performance measures over the three test signals.

	ρ	IF	TF res
MP	0.58	0.50	0.36
SPEC	0.55	0.54	0.95
CW	0.68	0.68	0.56
ASPEC	0.70	0.74	0.37

provide a high resolution TFR. In addition, if *a priori* knowledge of the signals TF characteristics are known, an appropriate dictionary can be developed to provide high resolution TFRs. This has previously been shown in [7] for newborn EEG seizure signals. The spectrogram performs poorly for this type of signal, mainly as a result of the poor TF resolution it provides. For the CW, we find that ρ_n and IF_n are low. This is a result of the interfering cross terms. The ASPEC achieves high TF resolution but provides low ρ_n and IF_n, which is caused by the poor amplitude estimation of the ASPEC. Also, for the ASPEC, smearing of the Dirac and Gabor components occurs due to the sinusoid component forcing the adaption algorithm to select large window lengths for time instants where it is the dominant component. This is one problem with the chosen adaptive algorithm as the chosen window length may not be optimal for all signal components at any particular time instant. Instead it is optimised according to the dominant component at any time instant.

Figures 2(e-h) show the TFRs of the second test signal, which is a mono-component signal with a sinusoidal IF law. MP, which produced the best results for the previous signal, now provides an extremely poor TFR, not showing the IF law at all. This results in extremely poor values for the three performance measures. The SPEC follows the IF law, with relatively good amplitude estimation according to ρ_n and IF_n but has low TF resolution. The CW follows the IF law with high resolution and minimal cross terms. This is shown quantitatively with the CW providing the best results for all performance measures. The performance measures show that ASPEC provides better TF resolution than the SPEC, but due to smearing affects and poorer amplitude estimation, it is slightly worse than SPEC for ρ_n and IF_n.

The TFRs for the third test signal are shown in Figures 2(i-l). The signal is a periodic spike train. The time and frequency representations of this signal, displayed in Figure 1(c), show that the signal has a transient nature and that the spikes are harmonically related. Both these characteristics are shown in the ideal TFR given in Figure 1(c). The MP TFR shows the harmonic relationship between the spikes, but does not indicate the transient nature of the signal. However, the performance measures indicate that MP does this well with a relatively high ρ_n and a low TF resolution measure. The SPEC show the spike occurrences clearly in the TF domain, but do not show the harmonic relationship between the spikes, providing a lower ρ_n. SPEC also performs poorly with regards to the TF resolution measure. CW performs similarly to SPEC, with only slightly better performance measures. The ASPEC, however, clearly shows all time-frequency characteristics of the signal, indicating spike occurrence and showing the harmonic relationship between spikes. This is validated by the three performance measures.

The test signals used in this paper cover a wide variety of time-varying signals with TF characteristics such as components with varying duration and bandwidths, multi-component signals, components with nonlinear IF laws and harmonically related or repetitive transients.

The TFRs performed variedly, depending on the TF characteristics of the signal. However, by averaging the performance measures over the three test signals, shown in Table 1, it can be seen that ASPEC outperforms the other non-parametric TFR.

6. CONCLUSION

This paper indicates that an adaptive quadratic TFD, such as the adaptive spectrogram used in this paper, should be used for the initial TF analysis of signals for which the TF characteristics are unknown. On average, the ASPEC represents the TF energy and IF law of a range of non–stationary signals with better TF resolution. The ASPEC also provides the representation without supervised optimisation of window lengths, unlike the CW and SPEC. After analysing with ASPEC, it may then be possible to chose a more suitable representation, such as MP with a coherently designed TF dictionary.

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