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High order asymptotic dynamics of a nonlinearly coupled electromechanical system

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Abstract

A nonlinearly coupled mathematical model of an electro-magneto-mechanical system is studied via the multiple scale approach in order to investigate its weakly nonlinear dynamics and analytically predict its salient features. The obtained amplitude modulation equations up to the third order perturbation allow to analytically describe the mechanical and electrical responses in terms of frequency-response curves and stability scenarios. A critical threshold of Hopf bifurcation is detected and analyzed as a function of the main system parameters. The subsequent extension of the asymptotic scheme up to the fifth order proves to grasp also the post-critical behavior, providing with the accurate identification of the amplitude of the quasiperiodic responses characterizing the unstable region of the ODEs system.

Keywords: Electromechanical system, Multiple Scale Method, Stability Analysis, Post-critical Behavior

1. Introduction

The nonlinear dynamics of an electromechanical system is considered in this study. The system consists of a linear oscillator nonlinearly coupled through an electromagnet to a linear electric circuit. As it is known, electromechanical systems are characterized by the interaction among inertial, electric, and magnetic circuits, and they are nowadays widely used for devices that monitor and control machine and structural systems. Depending on the specific application, electromechanical devices range from macro to micro and nano scales. Irrespective of the scale, they can be modeled by lumped physical elements, such as mass or inductance, for which basic dynamic principles apply, leading to coupled ordinary differential equations (ODEs)[1]. The conversion between electrical and mechanical energy may take place in electromechanical motion devices through energy transfer due to coupling electromagnetic field.

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15 In electromagnetic circuits, the dynamical equations take the form of conservation of charge and the Faraday–Henry law of flux change. If the system has a mechanical state variable, such as displacement, then the inductance may be a function of it. In such a case the magnetomotive force, induced by a current exciting a coil, acting on a ferromagnetic movable member depends on its displacement thereby triggering oscillations. By exploiting this principle, a number
20 of devices have been proposed so far.

Magnetic solenoid actuators are electromagnetic devices that convert electrical energy into a mechanical push/pull force or motion finding a number of applications across different industries. Solenoids are variable reluctance translational machines integrating movable (plunger) and stationary (fixed) members made from high permeability ferromagnetic materials; in these actuators
25 the inductance and the reluctance force generated both vary nonlinearly with position [2–4]. Synchronous reluctance motors are rotational electromechanical devices in which the magnetizing inductance is a function of the rotor angular displacement [3]. From a modeling perspective, in electrodynamic loudspeakers
30 it has been recently proved that the nonlinearity of inductance as function of displacement is a significant source of distortion; a linear approximation of this voice-coil inductance dependence was considered in [5].

The nonlinear dynamics stemming from the electromagnetic coupling characterized by displacement dependent induction was also addressed by a number
35 of studies related to classical problems, such as the Bethenod’s pendulum. In this system a pendulum with a ferromagnetic bob experiences growing and sustained oscillations when it is made to vary the inductance of a coil excited by high-frequency current [6]. In this work, a quadratic law for the inductance dependence on the pendulum oscillation angle was used to approximate the
40 experimental bell shaped profile. A similar electromechanical system was considered in [7]. The electromechanical system consists of a linear electrical circuit whose inductance varies linearly with the position of a bar. The bar, which is made of magnetic material forming part of the inductance core, represents the mass of a damped, linear, mechanical oscillator and is forced by the ponderomotive action of the electromagnetic field of the inductance. The nonlinear
45 amplitude frequency response, merely of the electrical circuit, and a region of instability were analytically obtained using a modified Ritz averaging technique. Later, the same two degrees of freedom system was considered in [8, 9] as an example of self-oscillatory system. Combination frequencies arise due to the
50 nonlinear interaction between the dynamic variables, so that the oscillations become quasiperiodic. As a result of the interaction of these oscillations with the harmonic source, energy transfer from the high-frequency (electric) source to the low-frequency (mechanical) oscillations occurs. An asymptotic approach assuming a quasiperiodic electrical response lead to obtain the slowly varying
55 functions describing amplitude and phase of the mechanical oscillator.

From the analytical point of view it is worth mentioning the similarity between the nonlinearly coupled behavior in liquid-shell systems and the electromechanical system at hand. The analogy in the governing differential equations is obtained by adopting a simplified liquid-shell system in which only one

60 liquid mode is considered [7, 10].

In [11, 12] the same electromechanical system here tackled is numerically thoroughly investigated. The occurrence of significant bifurcations marking the transition from regular to non regular responses are detected. Moreover, a wide unstable region with high-amplitude quasiperiodic and chaotic responses characterizes the system's nonlinear dynamics. Aiming at predicting and describing this peculiar behavior while completing the outcomes presented in a companion paper, a purely analytical asymptotic approach is here proposed. This analysis turns out to be effective due to the persistence of the unstable region also for rather low values of the nonlinear coupling. In developing the asymptotic procedure, useful suggestions come from [11], where the analytical approach performed though a slow-fast time decomposition analysis needed to reach high orders to catch the dynamical phenomena. In the light of this, a high order Multiple Scale Method [13–15] is applied to the model. The paper is organized as follows. After the introductory section, and a brief description of the physical model under analysis in Section 2, the multiple scale formulation is presented in Sections 3 and 4; it is shown that the asymptotic approach up to the third order enables to identify the frequency response curve around the primary resonance and to analyze its stability. By extending the asymptotic approach up to the fifth order, the post-critical regime is then addressed in Section 5; the concluding remarks are eventually reported.

2. Modeling

The physical model herein considered is composed of a linear oscillator nonlinearly coupled to a linear electric circuit by means of an electromagnet [11]. Under the hypothesis of linear dependence of the inductance L on the mechanical displacement x , i.e. $L(x) = L_0 + L_1x$, the equations of motion for the electro-mechanical system are:

$$\begin{aligned} \ddot{x}(t) + 2\zeta_m\omega_m\dot{x}(t) + \omega_m^2x(t) &= \frac{\xi\alpha\omega_m^4\dot{q}(t)^2}{\omega_e^4} \\ (1 + \alpha x(t))\ddot{q}(t) + (2\zeta_e\omega_e + \alpha\dot{x}(t))\dot{q}(t) + \omega_e^2q(t) &= \beta\omega_e^2\sin(\Omega t) \end{aligned} \quad (1)$$

where $x(t)$ and $q(t)$ are the mechanical and electrical (charge) displacements, respectively, and

$$\omega_m^2 = \frac{k}{m}, \omega_e^2 = \frac{1}{L_0C}, \zeta_m = \frac{c}{2\sqrt{km}}, \zeta_e = \frac{R\sqrt{C}}{2\sqrt{L_0}}, \xi = \frac{L_0}{2m}, \alpha = \frac{L_1}{L_0}$$

m, c, k are the mass, viscous dissipation and stiffness parameters of the mechanical oscillator, C, R are capacitance and resistance of the electrical circuit, and β and Ω are amplitude and frequency of the harmonic voltage, respectively.

85 The coupling terms in (1), which represent also the sole source of nonlinearity into the model, are governed by the α parameter, describing the inductance coupling, therefore it is of particular interest to study the effect of such parameter on the dynamical response. Figure 1(a) displays three frequency-response

(a)

(b)

Figure 1: 3D representation of the frequency-response curves at different values of the non-linear coupling α , at $\mu = 0.2$, $\beta = 1.0$ (a) and relevant stability chart in the Ω - α plane (b). In (a), Black lines: stable solutions; Blue lines: unstable solutions through torus bifurcations; Green lines: unstable solutions through saddle-node bifurcations; Red lines: unstable solutions through branch point bifurcations. In (b), Gray area: stable region of the period-1 solutions; black line: torus bifurcation; green line: saddle-node bifurcation; red line: branch point bifurcation.

curves obtained for different values of α and with the following set of parameter values: $\zeta_m = 0.01$, $\zeta_e = 0.1$, $\xi = 5.0$, $\beta = 1.0$, $\mu = \omega_e/\omega_m$, $\omega_m = 1$, $\omega_e = 5$. The black sections of the curves refer to the stable regions, while the gray, red and

green ones correspond to unstable ranges delimited by tori, branch points and saddle-node bifurcations, respectively. The relevant stability chart in the Ω - α plane is shown in Fig. 1(b). It can be observed that the area delimited by the torus bifurcations occurring at the right of the resonance represents the widest unstable region for the system behaving in strongly nonlinear regime (i.e. for high values of α , see also [11]), and it turns out to be also the most robust one, persisting also for low values of the α parameter, down to $\alpha \cong 0.08$ where it disappears. Moreover, the occurrence of torus bifurcations points out the presence of confined quasiperiodic or chaotic solutions, whose numerical characterization is presented in a companion paper. This suggests the opportunity to develop an analytical investigation, in order to get a more in-depth description of this area in terms of its existence and stability conditions, with also the possibility to identify the relevant quasiperiodic or chaotic motions.

3. Multiple scales formulation

The multiple scales method [16] is employed by introducing three independent time scales, i.e. the fast one, T_0 , and the slower ones, T_1, T_2 , as follows:

$$T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t \quad (2)$$

where ϵ is a small dimensionless ordering parameter and, consistently, expressing the time derivatives as

$$\begin{aligned} d/dt &= D_0 + \epsilon D_1 + \epsilon^2 D_2 \\ d^2/dt^2 &= D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 \\ &\quad + 2\epsilon^2 D_0 D_2 \end{aligned} \quad (3)$$

Both mechanical x and electrical (charge) q displacements are scaled as small perturbations of the reference position:

$$\begin{aligned} x(t) &= \epsilon x_1(T_0, T_1, T_2) + \epsilon^2 x_2(T_0, T_1, T_2) \\ &\quad + \epsilon^3 x_3(T_0, T_1, T_2) \\ q(t) &= \epsilon q_1(T_0, T_1, T_2) + \epsilon^2 q_2(T_0, T_1, T_2) \\ &\quad + \epsilon^3 q_3(T_0, T_1, T_2) \end{aligned} \quad (4)$$

In order to properly describe the dynamics of the weakly coupled system in presence of possible strong forcing amplitude, the parameters are scaled to obtain the forced damped electrical equation at the generating order: $\beta \rightarrow \epsilon \hat{\beta}$, $\zeta_m \rightarrow \epsilon^2 \hat{\zeta}_m$. Moreover, attention is focused on studying the primary resonance $\Omega = \omega_e$, with electrical-to-mechanical frequency ratio $\mu = 0.2$, implying $\omega_e = 5\omega_m$. The following set of perturbation equations is obtained:

$$\begin{aligned} \text{order } \epsilon^1 : \\ D_0^2 x_1 + \omega_m^2 x_1 &= 0 \\ D_0^2 q_1 + 2\zeta_e \omega_e D_0 q_1 + \omega_e^2 q_1 &= \omega_e^2 \beta \sin \Omega t \end{aligned} \quad (5)$$

order ϵ^2 :

$$\begin{aligned} D_0^2 x_2 + \omega_m^2 x_2 &= -2D_0 D_1 x_1 + \frac{\xi \alpha \omega_m^4 D_0 q_1^2}{\omega_e^4} \\ D_0^2 q_2 + 2\zeta_e \omega_e D_0 q_2 + \omega_e^2 q_2 &= -2D_0 D_1 q_1 - 2\zeta_e \omega_e D_1 q_1 \\ &\quad - \alpha(D_0^2 q_1 x_1 + D_0 x_1 D_0 q_1) \end{aligned} \quad (6)$$

order ϵ^3 :

$$\begin{aligned} D_0^2 x_3 + \omega_m^2 x_3 &= -2D_0 D_1 x_2 - D_1^2 x_1 - 2\zeta_m \omega_m D_0 x_1 - 2D_0 D_2 x_1 \\ &\quad + \frac{2\xi \alpha \omega_m^4 D_0 q_1 D_0 q_2}{\omega_e^4} + \frac{2\xi \alpha \omega_m^4 D_0 q_1 D_1 q_1}{\omega_e^4} \\ D_0^2 q_3 + 2\zeta_e \omega_e D_0 q_3 + \omega_e^2 q_3 &= -D_1^2 q_1 - 2D_0 D_1 q_2 - 2D_0 D_2 q_1 \\ &\quad - \alpha(D_0 x_2 D_0 q_1 + D_0 x_1 D_0 q_2 + D_0 q_1 D_1 x_1 + D_0 x_1 D_1 q_1 + D_0^2 q_2 x_1 \\ &\quad + 2D_0 D_1 q_1 x_1 + D_0^2 q_1 x_2) - 2\zeta_e \omega_e (D_1 q_2 + D_2 q_1) \end{aligned} \quad (7)$$

At the first order, the damped forced electrical equation furnishes the particular solution

$$q_1 = (\Gamma_1 + i\Gamma_2)e^{i\Omega T_0} + \text{c.c.} \quad (8)$$

where

$$\begin{aligned} \Gamma_1 &= -\frac{\beta \zeta_e \omega_e^3 \Omega}{\Omega^4 + 2(2\zeta_e^2 - 1)\Omega^2 \omega_e^2 + \omega_e^4} \\ \Gamma_2 &= \frac{\beta \omega_e^2 (\Omega^2 - \omega_e^2)}{2(\Omega^4 + 2(2\zeta_e^2 - 1)\Omega^2 \omega_e^2 + \omega_e^4)} \end{aligned} \quad (9)$$

define the known amplitude, not dependent on the time scales. Differently, the solution of the mechanical equation reads

$$x_1 = A(T_1, T_2)e^{i\omega_m T_0} + \text{c.c.} \quad (10)$$

with $A(T_1, T_2)$ undetermined function of the slow time scales and c.c. complex conjugate terms (the overbar will denote the complex conjugate and i is the imaginary unit). By substituting x_1 and q_1 in the first equation of (6), and imposing the solvability conditions implies

$$D_1 A = 0 \quad (11)$$

so that $A = A(T_2)$. The particular solutions at this order are

$$\begin{aligned} x_2 &= (c_{11} + ic_{12})e^{2i\Omega T_0} + c_{13} + \text{c.c.} \\ q_2 &= A((c_{14} + ic_{15})e^{i(\omega_m + \Omega)T_0} + (c_{16} - ic_{17})e^{i(\omega_m - \Omega)T_0}) + \text{c.c.} \end{aligned} \quad (12)$$

In view of Eqs. (8),(10),(11),(12), the solvability condition of the mechanical problem at the third order (7) yields

$$\begin{aligned} D_2 A = & \frac{1}{\omega_e^4} \omega_m A \left(\xi \alpha \omega_m^2 \Omega \left(-i \Omega (\Gamma_1 (c_{14} + c_{16} + i(c_{15} - c_{17}))) \right. \right. \\ & + \Gamma_2 (c_{15} + c_{17} + i(c_{16} - c_{14}))) + \omega_m (-\Gamma_2 (c_{14} + c_{16} + i(c_{15} - c_{17}))) \\ & \left. \left. + \Gamma_1 (c_{15} + c_{17} + i(c_{16} - c_{14}))) \right) - \zeta_m \omega_e^4 \right) \end{aligned} \quad (13)$$

while the particular solutions read

$$\begin{aligned} x_3 = & A(c_{31} + i c_{32}) e^{i(\omega_m + 2\Omega)T_0} + A(c_{33} + i c_{34}) e^{i(\omega_m - 2\Omega)T_0} + \text{c.c.} \\ q_3 = & (c_{21} - i c_{22}) e^{i\Omega T_0} + (c_{23} + i c_{24}) e^{3i\Omega T_0} + A^2 \left(e^{i(2\omega_m + \Omega)T_0} (c_{27} - i c_{28}) \right. \\ & \left. + e^{i(2\omega_m - \Omega)T_0} (c_{25} - i c_{26}) \right) + A \bar{A} (c_{29} - i c_{30}) e^{i\Omega T_0} + \text{c.c.} \end{aligned} \quad (14)$$

The expressions of all c_{ij} coefficients are reported in Appendix A. According to the usual reconstitution procedure, the amplitude derivatives with respect to time t are obtained from (3)

$$\dot{A} = \epsilon D_1 A + \epsilon^2 D_2 A \quad (15)$$

The ϵ parameter is completely reabsorbed through a backward rescaling, and, by recalling (11), the complex amplitude modulation equation (AME, or bifurcation equation) for the mechanical variable coincides with (13), i.e. $\dot{A} = D_2 A$. To conveniently express equation (13) in Cartesian form, the following coordinates transformation is applied

$$A(t) = \frac{1}{2}(r(t) + i s(t)), \quad \bar{A}(t) = \frac{1}{2}(r(t) - i s(t)) \quad (16)$$

and, by substituting (16) in (13) the following system is obtained:

$$\begin{aligned} \dot{r} = & \xi \alpha^2 \beta^2 f_1(\Omega) s + (\xi \alpha^2 \beta^2 f_2(\Omega) - \zeta_m \omega_m) r \\ \dot{s} = & -\xi \alpha^2 \beta^2 f_1(\Omega) r + (\xi \alpha^2 \beta^2 f_2(\Omega) - \zeta_m \omega_m) s \end{aligned} \quad (17)$$

in which the functions $f_1(\Omega)$, $f_2(\Omega)$ introduced are reported in Appendix A. The equilibria of the asymptotic system (17) coincide with the trivial rest position $(r, s) = (0, 0)$, i.e. the mechanical amplitude is identically null, meaning that to obtain the frequency-response curve the amplitudes of the reconstructed electrical and mechanical solutions have to be computed. By recalling the series expansion (4) of x and q , contributions up to the third order are taken into account by making use of (8), (10), (12), (14), and applying a backward rescaling. Guided by expressions of generating solutions (8), contributions of frequency ω_m and Ω are selected for the mechanical and electrical solutions, respectively,

Figure 2: Comparison between numerical (black) and asymptotic (green) responses with respect to the mechanical amplitude X (a) and the electrical amplitude Q (b), with detection of instability regions, for $\mu = 0.2$ and $\beta = 1.0$. Gray diamond: torus bifurcation; green square: Hopf bifurcation.

to obtain

$$\begin{aligned}
 x(t) &= c_{13} + r \cos(\omega_m t) - s \sin(\omega_m t) \\
 q(t) &= \frac{1}{2} \left(4(c_{21} + \Gamma_1) + c_{29}(r^2 + s^2) \right) \cos(\Omega t) \\
 &\quad + \frac{1}{2} \left(4(c_{22} - \Gamma_2) + c_{30}(r^2 + s^2) \right) \sin(\Omega t)
 \end{aligned} \tag{18}$$

The relevant amplitudes can be expressed as

$$\begin{aligned}
 X &= c_{13} + \sqrt{r^2 + s^2} \\
 Q &= \frac{1}{2} \sqrt{(4(\Gamma_1 + c_{21}) + c_{29}(r^2 + s^2))^2 + (4(c_{22} - \Gamma_2) + c_{30}(r^2 + s^2))^2}
 \end{aligned} \tag{19}$$

It is worth noting that due to the trivial equilibrium of the asymptotic system, terms related to r and s can be neglected.

The comparison between the analytical frequency-response curve given by Eqs. (19) and the one obtained by integrating the ODE system (1) is shown in Fig. 2. It shows a very good agreement of the results, from both a qualitative and quantitative viewpoint, highlighting the ability of the asymptotic approach to accurately identify the unstable region. To better understand its nature, the stability analysis of the asymptotic system is addressed in the next Section.

4. Stability analysis and its dependence from system parameters

To investigate the stability of the asymptotic responses, the eigenvalues of the Jacobian matrix of system (17) are studied:

$$\lambda_{1,2} = \xi \alpha^2 \beta^2 f_2(\Omega) - \zeta_m \omega_m \pm i \xi \alpha^2 \beta^2 f_1(\Omega) \tag{20}$$

Figure 3: Analytical stability thresholds in the Ω - ω_e plane, for different values of the nonlinear coupling α .

where $f_i(\Omega)$ ($i = 1, 2$) are described in Appendix A. The system loses stability through a Hopf bifurcation when the real part of the complex eigenvalues becomes positive, with the critical case defined by the following equation:

$$\xi\alpha^2\beta^2f_2(\Omega) - \zeta_m\omega_m = 0 \quad (21)$$

For positive left-hand side of equation (21), the asymptotic system displays the birth of periodic solutions which, in the ODEs system, correspond to quasiperiodic and eventually chaotic responses, depending on the possible presence of other secondary bifurcations in the post-critical behavior. This post-critical behavior was detected also in [9], where it is referred to as self-excited oscillations.

In order to get information about the position of the unstable region in the frequency-response curve, it is of interest to investigate the critical threshold (21) as a function of the electrical and external forcing frequencies. The outcomes are reported in Fig. 3 for different values of the nonlinear coupling α ; they point out that this kind of instability can occur only in the half-plane on the right of the dashed curve $\Omega = \omega_e$, corresponding to the right hand slope of the resonance curve (i.e. $\Omega > \omega_e$). This circumstance is also confirmed by the analyses performed in the strongly nonlinear regime (see, e.g., Fig. 1(a)).

Equation (21) can be also used to obtain indications about the values to be considered for the system parameters in order to guarantee, or alternatively avoid, the presence of the instability region under analysis. To focus on the role played by the electrical and mechanical damping, the behavior chart reported in Fig. 4 can be computed by zeroing the derivative of Eq. (21) with respect to Ω . Thus, the chart provides the minimum values of the nonlinear coupling α needed to ensure the arise of the unstable area. The results highlight the major role played by the electrical damping in shifting the occurrence of instability towards higher values of the nonlinear coupling. By doubling its reference value

(i.e. $\zeta_e = 0.1$) a shift of α by about 124% (from 0.085 to 0.19) is observed; the analogous increase of the mechanical damping (i.e. from $\zeta_m = 0.01$ to $\zeta_m = 0.02$) moves the onset of instability to a 41% higher coupling value α .

140 In order to validate the analytical findings, the stability threshold obtained by solving eq. (21) is compared with the torus bifurcation threshold derived from the ODEs system (1), as reported in Fig. 5. The good agreement between analytical (dashed green) and numerical (solid gray) results is confirmed, especially for low values of the nonlinear coupling. As the α parameter increases, 145 slight differences can be detected, mostly located at the left side branch of the curve. They are related to the presence of the nonlinear inertial coupling, which strongly modifies the full system resonance curve peak location, which is shifted

Figure 4: In the ζ_e - ζ_m plane, analytical detection of the minimum α values needed to ensure the presence of the instability region.

Figure 5: Asymptotic analytical (dashed green) and numerical (solid gray) stability threshold in the Ω - $\alpha(\beta)$ plane, at $\frac{\omega_m}{\omega_e} = \mu = 0.2$.

150 towards lower frequency values as the coupling increases, as already illustrated in Fig. 1(a). As a consequence, the left boundary of the instability region moves accordingly. Conversely, in the formulation of the asymptotic model the non-linearity is treated as perturbation of the underlying linear system, as such it is unable to chase the position of the resonance curve, which maintains its peak around the electrical frequency value (which coincides to 5 in the reported figures), corresponding to the left limit for the unstable area.

155 5. Post-critical behavior: quasiperiodic oscillations

In order to study the post-critical behavior of the asymptotic system arisen from the Hopf bifurcations previously detected, the multiple scales analysis is developed up to the fifth order. Accordingly, two further slow scales are considered, i.e., $T_3 = \epsilon^3 t$, $T_4 = \epsilon^4 t$, and the mechanical and electrical expansions (4) are enriched by two additional orders:

$$\begin{aligned} x(t) &= \epsilon x_1(T_0, \dots, T_4) + \epsilon^2 x_2(T_0, \dots, T_4) + \epsilon^3 x_3(T_0, \dots, T_4) \\ &\quad + \epsilon^4 x_4(T_0, \dots, T_4) + \epsilon^5 x_5(T_0, \dots, T_4) \\ q(t) &= \epsilon q_1(T_0, \dots, T_4) + \epsilon^2 q_2(T_0, \dots, T_4) + \epsilon^3 q_3(T_0, \dots, T_4) \\ &\quad + \epsilon^4 q_4(T_0, \dots, T_4) + \epsilon^5 q_5(T_0, \dots, T_4) \end{aligned} \quad (22)$$

The perturbation equations at the fourth and fifth orders are then obtained as:

order ϵ^4 :

$$\begin{aligned} D_0^2 x_4 + x_4 \omega_m^2 &= f_4(x_1, x_2, x_3, q_1, q_2, q_3) \\ D_0^2 q_4 + 2\zeta_e \omega_e D_0 q_4 + \omega_e^2 q_4 &= g_4(x_1, x_2, x_3, q_1, q_2, q_3) \end{aligned} \quad (23)$$

order ϵ^5 :

$$\begin{aligned} D_0^2 x_5 + \omega_m^2 x_5 &= f_5(x_1, x_2, x_3, x_4, q_1, q_2, q_3, q_4) \\ D_0^2 q_5 + 2\zeta_e \omega_e D_0 q_5 + \omega_e^2 q_5 &= g_5(x_1, x_2, x_3, x_4, q_1, q_2, q_3, q_4) \end{aligned} \quad (24)$$

where f_i , g_i ($i = 4, 5$) are given in Appendix B. Substitution into system (23) of the obtained expressions of the mechanical and electrical solutions at the 3 previous orders (8), (10), (12), (14), allows to select the secular terms at the fourth order

$$\begin{aligned} \frac{2}{\omega_e^4} &\left(\omega_e^4 (D_1 D_2 A + i \omega_m D_3 A) + D_1 A \left(\xi \alpha (-c_{15} (\Gamma_1 - i \Gamma_2) \right. \right. \\ &\quad \left. \left. + (c_{16} - i c_{17}) (\Gamma_2 - i \Gamma_1) + c_{14} (\Gamma_2 + i \Gamma_1)) \Omega \omega_m^4 + \zeta_m \omega_m \omega_e^4 \right) \right) = 0 \end{aligned} \quad (25)$$

which furnish, by means of (11),

$$D_3 A = 0 \quad (26)$$

The particular solutions at this order read:

$$\begin{aligned}
x_4 = & (c_{41} - ic_{42} + c_{43}A\bar{A} - ic_{44}A\bar{A})e^{2i\Omega T_0} + (c_{45} - ic_{46})e^{4i\Omega T_0} \\
& + (c_{47} - ic_{48})A^2e^{2iT_0(\Omega+\omega_m)} + (c_{49} - ic_{50})A^2e^{2i\omega_m T_0} \\
& + (c_{51} + ic_{52})A^2e^{-2iT_0(\Omega-\omega_m)} + c_{53}A\bar{A} \\
& + 4\xi\alpha\omega_m^2\Omega^2(\Gamma_1c_{21} - \Gamma_2c_{22})/\omega_e^4 + \text{c.c.} \\
q_4 = & (c_{60} + ic_{61})Ae^{-iT_0(3\Omega-\omega_m)} + (c_{62} + ic_{63})Ae^{3iT_0(3\Omega+\omega_m)} \\
& + (c_{64} + ic_{65})A^3e^{iT_0(\Omega+3\omega_m)} + (c_{66} + ic_{67})A^3e^{-iT_0(\Omega-3\omega_m)} \\
& (c_{68} + ic_{69} + (c_{70} + ic_{71})A\bar{A})Ae^{iT_0(\Omega+\omega_m)} \\
& + (c_{72} + ic_{73} + (c_{74} + ic_{75})A\bar{A})Ae^{-iT_0(\Omega-\omega_m)} + \text{c.c.}
\end{aligned} \tag{27}$$

Moving to the fifth order, the solvability condition for the mechanical amplitude, obtained by means of Eqs. (8), (10), (11),(12),(13), (14),(26),(27), is expressed as follows:

$$D_4A = (c_{80} + ic_{81})A + (c_{82} + ic_{83})A^2\bar{A} \tag{28}$$

Derivatives of the mechanical amplitude with respect to time scales T_3 (26) and T_4 (28) are added to Eq. (15), and after reabsorption of ϵ parameter the complex mechanical amplitude becomes

$$\begin{aligned}
\dot{A} = & \frac{1}{\omega_e^4}A\left(\xi\alpha\omega_m^3\Omega\left(-i\Omega\left(\Gamma_1(c_{14} + c_{16} + i(c_{15} - c_{17}))\right) + \Gamma_2(c_{15} + c_{17} + i(c_{16} - c_{14}))\right)\right) \\
& + \omega_m\left(-\Gamma_2(c_{14} + c_{16} + i(c_{15} - c_{17})) + \Gamma_1(c_{15} + c_{17} + i(c_{16} - c_{14}))\right) \\
& + (c_{80} + ic_{81})\omega_e^4 - \zeta_m\omega_m\omega_e^4\Big) + (c_{82} + ic_{83})A^2\bar{A}
\end{aligned} \tag{29}$$

which, in Cartesian coordinates reads

$$\begin{aligned}
\dot{r} = & d_1r^3 + d_2s^3 + d_1rs^2 + d_2r^2s + d_3r + d_4s \\
\dot{s} = & -d_2r^3 + d_1s^3 - d_2rs^2 + d_1r^2s - d_4r + d_3s
\end{aligned} \tag{30}$$

With respect to the AMEs obtained at the third order (17), system (30) is enriched by the cubic terms furnished by the contribution of the fifth order secular terms (28), which also modify the expressions of the linear coefficients d_3 , d_4 . As far as the reconstructed solutions up to the third order of Eq. (19), the fourth order solutions (27) show to alter the constant term of the mechanical amplitude, while no enrichment is provided to the terms of frequency Ω in the electrical solution. Consequently, the updated amplitude of the mechanical variable becomes

$$X = c_{13} + \frac{1}{4}c_{53}(r^2 + s^2) + \frac{4\xi\alpha\Omega^2\omega_m^2}{\omega_e^4}(c_{21}\Gamma_1 - c_{22}\Gamma_2) + \sqrt{r^2 + s^2} \tag{31}$$

Figure 6: Comparison between numerical (black), third order asymptotic (green) and fifth order asymptotic (red) responses with respect to the mechanical amplitude X (a) and the electrical amplitude Q (b), with detection of instability regions, for $\mu = 0.2$ and $\beta = 1.0$. Gray diamond: torus bifurcation; green/red square: Hopf bifurcation.

which, together with the asymptotic system (30), provides the results represented with red curves in Fig. 6. The comparison with the numerical outcomes (black) and with that of the third-order AMEs (green), displays a further improvement in the accuracy of the analytical solutions mostly pertaining, as expected, to the mechanical response, which now reproduces almost perfectly the numerical curve. The enhancement in the detection of the bifurcations marking the unstable region can be also observed. The latter aspect can be better observed in Fig. 7 where the bifurcation threshold, with the related enlargement, is portrayed in the Ω - α plane. The unstable region lower bound is more precisely detected by the fifth-order AMEs (dotted red) with respect to the third-order ones (dashed green), especially with regard to the right branch which is coinciding with the numerical results (solid gray). Moreover, differently from the third order case (Fig. 5), the left branch is better approximated, with a good precision for values of the coupling parameter α lower than 0.15, and with an opposite trend with respect to the third-order threshold as α increases.

System (30) allows to describe also the behavior inside the unstable region, where numerical investigations point out the presence of a confined region of quasiperiodic solutions, as shown by the Poincaré maps at different values of the forcing frequency represented in gray in Fig. 8. Here, the maximum and minimum values of the switched branch arising from the asymptotically detected Hopf bifurcations are reported too (red curves), highlighting a very good accordance between numerical and analytical results, from both a qualitative and quantitative viewpoint. In fact, AMEs (30) enable us to accurately catch the amplitudes of the quasiperiodic responses, and to capture their non-symmetric evolution inside the detected region. The qualitative description of a quasiperiodic solution (Fig. 9) and of a 6-period solution (Fig. 10) occurring inside the investigated region is shown. The asymptotic system, beyond confirming the accuracy of the amplitude values obtained, also with respect to the time evolution of the solutions (red curves in figure), allow to detect the qualitative difference

Figure 7: Third order asymptotic (dashed green), fifth order asymptotic (dotted red) and numerical (solid gray) stability thresholds in the Ω - $\alpha(\beta)$ plane (a), and enlargement (b), at $\frac{\omega_m}{\omega_e} = \mu = 0.2$.

Figure 8: Comparison between fifth order asymptotic amplitude (red) and numerical Poincaré maps of the mechanical quasiperiodic solutions, at $\alpha = 0.1$.

185 between mechanical and electrical responses. As a matter of fact, the electrical
phase portraits, and the relevant FFT, are markedly quasiperiodic (Fig. 9) and
highly periodic (Fig. 10), while the mechanical response is always dominated
by a single frequency corresponding to its natural frequency. This also explains
the lack of oscillating responses in the post-critical behavior provided by the
190 asymptotic system (see red curves of Figs. 8, 9) which, as already observed,
provides information about the sole mechanical amplitude. In this regard, it
is worth noting that, coherently with the obtained results, while the analyti-
cal characterization of the critical points (20) would suggest the presence of a

Figure 9: Characterization of the quasiperiodic response at $\Omega = 5.8$ and $\alpha = 0.1$. Temporal evolution (a), phase portraits with Poincaré maps in the mechanical (b) and electrical (c) planes, and FFT of the mechanical (d) and electrical (e) response. In (a), the comparison between the numerical (gray) responses and the fifth order asymptotic (red) amplitudes is reported.

pitchfork bifurcation, Hopf bifurcations are here detected. The reason of this
 195 discrepancy is related to the Cartesian coordinates choice. These coordinates
 are convenient to study systems with trivial solution, however, differently from
 the polar ones, they do not filter out the oscillating component of the response.
 Nevertheless, the stability analysis herein developed is consistent with the ac-
 200 tual system behavior which, due to the contribution of the electrical solution,
 confirms the occurrence of a torus bifurcation, corresponding to a Hopf in the
 relevant asymptotic model.

6. Conclusions

The weakly nonlinear dynamics of a linear oscillator nonlinearly coupled with
 a linear electric circuit is studied in conditions of primary electrical resonance.
 205 Willing to investigate the stability of the frequency-amplitude responses, the
 multiple scale analysis up to the third order is applied to the system of 2
 d.o.f. ODEs to obtain Cartesian Amplitude Modulation Equations (AMEs) en-
 abling to reconstruct the particular mechanical and electrical solutions at each
 scaling order.

210 The validation, carried out by comparison with the numerical integration,
 confirms the accuracy of the asymptotic approach in qualitatively and quanti-
 tatively grasping the mechanical and electrical responses of the coupled system,
 and their stability regions. Moreover, the analytical expression of the critical

Figure 10: Characterization of the 6-period response at $\Omega = 6.036$ and $\alpha = 0.1$. Temporal evolution (a), phase portraits with Poincaré maps in the mechanical (b) and electrical (c) planes, and FFT of the mechanical (d) and electrical (e) response. In (a), the comparison between the numerical (gray) responses and the fifth order asymptotic (red) amplitudes is reported.

stability threshold allows to verify the influence of the main parameters on the system dynamical response; stability charts are also derived to properly tune the system parameters in order to predict and control the onset of instability.

With the aim to achieve a comprehensive description of the system dynamics, the post-critical behavior is also investigated by enriching the asymptotic approach with two further orders. The ensuing fifth order AMEs allow to catch rather accurately the numerical amplitudes of the quasiperiodic solutions originating from the torus bifurcations of the ODEs system, as well as to improve further the third order results related to the description of the frequency-response curves.

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Appendix A.

Expressions of c_{ij} and $f_i(\Omega)$ coefficients introduced in Sect. 3:

$$\begin{aligned}
c_{11} &= \frac{\xi\alpha(\Gamma_1 - \Gamma_2)(\Gamma_1 + \Gamma_2)\Omega^2\omega_m^4}{(4\Omega^2 - \omega_m^2)\omega_e^4} \\
c_{12} &= \frac{2\xi\alpha\Gamma_1\Gamma_2\Omega^2\omega_m^4}{(4\Omega^2 - \omega_m^2)\omega_e^4} \\
c_{13} &= \frac{2\xi\alpha(\Gamma_1^2 + \Gamma_2^2)\Omega^2\omega_m^2}{\omega_e^4} \\
c_{14} &= \frac{\alpha\Omega(\Omega + \omega_m)(-\Gamma_1(\Omega + \omega_m)^2 + 2\Gamma_2\zeta_e\omega_e(\Omega + \omega_m) + \omega_e^2\Gamma_1)}{(\Omega + \omega_m)^4 + 2(2\zeta_e^2 - 1)(\Omega + \omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{15} &= \frac{\alpha\Omega(\Omega + \omega_m)(-\Gamma_2(\Omega + \omega_m)^2 - 2\Gamma_1\zeta_e\omega_e(\Omega + \omega_m) + \omega_e^2\Gamma_2)}{(\Omega + \omega_m)^4 + 2(2\zeta_e^2 - 1)(\Omega + \omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{16} &= \frac{\alpha\Omega(\Omega - \omega_m)(-\Gamma_1(\Omega - \omega_m)^2 + 2\Gamma_2\zeta_e\omega_e(\Omega - \omega_m) + \omega_e^2\Gamma_1)}{(\Omega - \omega_m)^4 + 2(2\zeta_e^2 - 1)(\Omega - \omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{17} &= \frac{\alpha\Omega(\Omega - \omega_m)(-\Gamma_2(\Omega - \omega_m)^2 - 2\Gamma_1\zeta_e\omega_e(\Omega - \omega_m) + \omega_e^2\Gamma_2)}{(\Omega - \omega_m)^4 + 2(2\zeta_e^2 - 1)(\Omega - \omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{21} &= (\alpha\Omega^2((\Gamma_1(c_{11} - c_{13}) + c_{12}\Gamma_2)\Omega^2 + 2((c_{11} + c_{13})\Gamma_2 - c_{12}\Gamma_1)\zeta_e\Omega\omega_e \\
&\quad - ((c_{11} - c_{13})\Gamma_1 + c_{12}\Gamma_2)\omega_e^2))/(\Omega^4 + 2(2\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4) \\
c_{22} &= -(\alpha\Omega^2((-\Gamma_2(c_{11} + c_{13}) + c_{12}\Gamma_1)\Omega^2 + 2((c_{11} - c_{13})\Gamma_1 + c_{12}\Gamma_2)\zeta_e\Omega\omega_e \\
&\quad - ((c_{11} - c_{13})\Gamma_1 + c_{12}\Gamma_2)\omega_e^2))/(\Omega^4 + 2(2\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4) \\
c_{22} &= -(\alpha\Omega^2((-\Gamma_2(c_{11} + c_{13}) + c_{12}\Gamma_1)\Omega^2 + 2((c_{11} - c_{13})\Gamma_1 + c_{12}\Gamma_2)\zeta_e\Omega\omega_e \\
&\quad + ((c_{11} + c_{13})\Gamma_2 - c_{12}\Gamma_1)\omega_e^2))/(\Omega^4 + 2(2\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4) \\
c_{23} &= \frac{3\alpha\Omega^2(9(-c_{11}\Gamma_1 + c_{12}\Gamma_2)\Omega^2 + 6(c_{12}\Gamma_1 + c_{11}\Gamma_2)\zeta_e\omega\omega_e + (c_{11}\Gamma_1 - c_{12}\Gamma_2)\omega_e^2)}{81\Omega^4 + 18(2\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4} \\
c_{24} &= -\frac{3\alpha\Omega^2(9(c_{12}\Gamma_1 + c_{11}\Gamma_2)\Omega^2 + 6(c_{11}\Gamma_1 - c_{12}\Gamma_2)\zeta_e\omega\omega_e - (c_{12}\Gamma_1 + c_{11}\Gamma_2)\omega_e^2)}{81\Omega^4 + 18(2\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4} \\
c_{25} &= -\frac{\alpha(\Omega - 2\omega_m)(\Omega - \omega_m)(c_{16}(\Omega - 2\omega_m)^2 - 2c_{17}\zeta_e(\Omega - 2\omega_m)\omega_e - c_{16}\omega_e^2)}{(\Omega - 2\omega_m)^4 + 2(\zeta_e^2 - 1)(\Omega - 2\omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{26} &= -\frac{\alpha(\Omega - 2\omega_m)(\Omega - \omega_m)(c_{17}(\Omega - 2\omega_m)^2 - 2c_{16}\zeta_e(\Omega - 2\omega_m)\omega_e - c_{17}\omega_e^2)}{(\Omega - 2\omega_m)^4 + 2(\zeta_e^2 - 1)(\Omega - 2\omega_m)^2\omega_e^2 + \omega_e^4}
\end{aligned}$$

$$\begin{aligned}
c_{27} &= \frac{\alpha(\Omega + 2\omega_m)(\Omega + \omega_m)(-c_{14}(\Omega + 2\omega_m)^2 + 2c_{15}\zeta_e(\Omega + 2\omega_m)\omega_e + c_{14}\omega_e^2)}{(\Omega + 2\omega_m)^4 + 2(\zeta_e^2 - 1)(\Omega + 2\omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{28} &= \frac{\alpha(\Omega + 2\omega_m)(\Omega + \omega_m)(c_{15}(\Omega + 2\omega_m)^2 + 2c_{14}\zeta_e(\Omega + 2\omega_m)\omega_e - c_{15}\omega_e^2)}{(\Omega + 2\omega_m)^4 + 2(\zeta_e^2 - 1)(\Omega + 2\omega_m)^2\omega_e^2 + \omega_e^4} \\
c_{29} &= \frac{\alpha\Omega}{\Omega^4 + 2(\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4}(-\Omega^2((c_{14} + c_{16})\Omega + (c_{14} - c_{16})\omega_m) \\
&\quad + 2\zeta_e\Omega((c_{15} + c_{17})\Omega + (c_{15} - c_{17})\omega_m)\omega_e + ((c_{14} + c_{16})\Omega + (c_{14} - c_{16})\omega_m)\omega_e^2) \\
c_{30} &= \frac{\alpha\Omega}{\Omega^4 + 2(\zeta_e^2 - 1)\Omega^2\omega_e^2 + \omega_e^4}(\Omega^2((c_{15} + c_{17})\Omega + (c_{15} - c_{17})\omega_m) \\
&\quad + 2\zeta_e\Omega((c_{14} + c_{16})\Omega + (c_{14} - c_{16})\omega_m)\omega_e - ((c_{15} + c_{17})\Omega + (c_{15} - c_{17})\omega_m)\omega_e^2) \\
c_{31} &= \frac{\xi\alpha(c_{14}\Gamma_1 - c_{15}\Gamma_2)\omega_m^4}{2\omega_e^4} \\
c_{32} &= \frac{\xi\alpha(c_{15}\Gamma_1 + c_{14}\Gamma_2)\omega_m^4}{2\omega_e^4} \\
c_{33} &= \frac{\xi\alpha(c_{16}\Gamma_1 - c_{17}\Gamma_2)\omega_m^4}{2\omega_e^4} \\
c_{34} &= \frac{\xi\alpha(c_{17}\Gamma_1 + c_{16}\Gamma_2)\omega_m^4}{2\omega_e^4} \\
f_1(\Omega) &= (\omega_m^3(-\Omega^2(\Omega^2 - \omega_m^2)^4 - (-3 + 4\zeta_e^2)(\Omega^2 + \omega_m^2)(\Omega^3 - \Omega\omega_m^2)^2\omega_e^2 \\
&\quad + \Omega^2((-3 + 4\zeta_e^2)\Omega^4 - 2(1 + 4\zeta_e^2)\Omega^2\omega_m^2 + (-3 + 4\zeta_e^2)\omega_m^4)\omega_e^4 + \Omega^2(\Omega^2 + \omega_m^2)\omega_e^6))/ \\
&\quad (2(\Omega^4 + 2(-1 + 2\zeta_e^2)\Omega^2\omega_e^2 + \omega_e^4)((\Omega - \omega_m)^4 + 2(-1 + 2\zeta_e^2)(\Omega - \omega_m)^2\omega_e^2 + \omega_e^4) \\
&\quad ((\Omega - \omega_m)^4 + 2(-1 + 2\zeta_e^2)(\omega + \omega_m)^2\omega_e^2 + \omega_e^4)) \\
f_2(\Omega) &= (\zeta_e\Omega^2\omega_m^4\omega_e((\Omega^2 - \omega_m^2)^3 - 2(-1 + 2\zeta_e^2)(\Omega^2 - \omega_m^2)^2\omega_e^2 - (3\Omega^2 + \omega_m^2)\omega_e^4))/ \\
&\quad ((\Omega^4 + 2(-1 + 2\zeta_e^2)\Omega^2\omega_e^2 + \omega_e^4)((\Omega - \omega_m)^4 + 2(-1 + 2\zeta_e^2)(\Omega - \omega_m)^2\omega_e^2 + \omega_e^4) \\
&\quad ((\Omega + \omega_m)^4 + 2(-1 + 2\zeta_e^2)(\Omega + \omega_m)^2\omega_e^2 + \omega_e^4))
\end{aligned}$$

Appendix B.

Expressions of f_i and g_i ($i = 4, 5$) coefficients introduced in Eqs. (23),(24):

$$\begin{aligned}
f_4(x_1, x_2, x_3, q_1, q_2, q_3) &= -D_1^2 x_2 - 2\left(D_0 D_1 x_3 + 2D_0 D_2 x_2 + D_1 D_2 x_1 + D_0 D_3 x_1\right) \\
&\quad - 2\zeta_m \omega_m \left(D_0 x_2 + D_1 x_1\right) + (\xi\alpha\omega_m^4)/\omega_e^4 \left(D_0 q_2^2 + D_1 q_1^2 \right. \\
&\quad \left. + 2D_0 q_1 D_0 q_3 + 2D_0 q_2 D_1 q_1 + 2D_0 q_1 D_1 q_2 + 2D_0 q_1 D_2 q_1\right)
\end{aligned}$$

$$\begin{aligned}
g_4(x_1, x_2, x_3, q_1, q_2, q_3) = & -D_1^2 q_2 - 2D_0 D_1 q_3 - 2D_0 D_2 q_2 - 2D_1 D_2 q_1 - 2D_0 D_3 q_1 \\
& - \alpha \left(D_0^2 q_3 x_1 + D_1^2 q_1 x_1 + D_0^2 q_2 x_2 + D_0^2 q_1 x_3 + D_0 q_3 D_0 x_1 + D_0 q_2 D_0 x_2 \right. \\
& + D_0 q_1 D_0 x_3 + D_0 x_2 D_1 q_1 + D_0 x_1 D_1 q_2 + D_0 q_2 D_1 x_1 + D_1 q_1 D_1 x_1 + D_0 q_1 D_1 x_2 \\
& + D_0 x_1 D_2 q_1 + D_0 q_1 D_2 x_1 + 2D_0 D_1 q_2 x_1 + 2D_0 D_2 q_1 x_1 + 2D_0 D_1 q_1 x_2 \Big) \\
& - 2\zeta_e \omega_e \left(D_1 q_3 + D_2 q_2 + D_3 q_1 \right)
\end{aligned}$$

$$\begin{aligned}
f_5(x_1, x_2, x_3, x_4, q_1, q_2, q_3, q_4) = & -D_1^2 x_3 - D_2^2 x_1 - 2 \left(D_0 D_1 x_4 + D_0 D_2 x_3 \right. \\
& + D_1 D_2 x_2 + D_0 D_3 x_2 + D_1 D_3 x_1 + D_0 D_4 x_1 \Big) - 2\zeta_m \omega_m \left(D_0 x_3 + D_1 x_2 + D_2 x_1 \right) \\
& + (2\xi \alpha \omega_m^4) / \omega_e^4 \left(D_0 q_2 D_0 q_3 + D_0 q_1 D_0 q_4 + D_0 q_3 D_1 q_1 + D_0 q_2 D_1 q_2 + D_1 q_1 D_1 q_2 \right. \\
& + D_0 q_1 D_1 q_3 + D_0 q_2 D_2 q_1 + D_1 q_1 D_2 q_1 + D_0 q_1 D_2 q_2 + D_0 q_1 D_3 q_1 \Big)
\end{aligned}$$

$$\begin{aligned}
g_5(x_1, x_2, x_3, x_4, q_1, q_2, q_3, q_4) = & -D_1^2 q_3 - D_2^2 q_1 - 2 \left(D_0 D_1 q_4 + D_0 D_2 q_3 \right. \\
& + D_1 D_2 q_2 + D_0 D_3 q_2 + D_1 D_3 q_1 + D_0 D_4 q_1 \Big) - \alpha \left(D_1^2 q_2 x_1 + D_1^2 q_1 x_2 \right. \\
& + D_0^2 q_2 x_3 + D_0^2 q_1 x_4 + D_0^2 q_3 x_2 + D_0^2 q_4 x_1 + D_0 q_4 D_0 x_1 + D_0 q_3 D_0 x_2 \\
& + D_0 q_2 D_0 x_3 + D_0 q_1 D_0 x_4 + D_0 x_3 D_1 q_1 + D_0 x_2 D_1 q_2 + D_0 x_1 D_1 q_3 \\
& + D_0 q_3 D_1 x_1 + D_1 q_2 D_1 x_1 + D_0 q_2 D_1 x_2 + D_1 q_1 D_1 x_2 + D_0 q_1 D_1 x_3 \\
& + D_0 x_2 D_2 q_1 + D_1 x_1 D_2 q_1 + D_0 x_1 D_2 q_2 + D_0 q_2 D_2 x_1 + D_1 q_1 D_2 x_1 + D_0 q_1 D_2 x_2 \\
& + D_0 x_1 D_3 q_1 + D_0 q_1 D_3 x_1 + 2D_0 D_1 q_3 x_1 + 2D_0 D_2 q_2 x_1 + 2D_1 D_2 q_1 x_1 \\
& + 2D_0 D_3 q_1 x_1 + 2D_0 D_1 q_2 x_2 + 2D_0 D_2 q_1 x_2 + 2D_0 D_1 q_1 x_3 \Big) \\
& - 2\zeta_e \omega_e \left(D_1 q_4 + D_2 q_3 + D_3 q_2 + D_4 q_1 \right)
\end{aligned}$$

References

- 230 [1] R. H. Bishop, Mechatronic systems, sensors, and actuators: fundamentals and modeling, CRC press, 2007.
- [2] M. F. Rahman, N. C. Cheung, K. W. Lim, Position estimation in solenoid actuators, IEEE Trans Ind Appl 32 (3) (1996) 552–559.
- 235 [3] V. Giurgiutiu, S. E. Lyshevski, Micromechatronics: Modeling, analysis, and design with MATLAB, CRC Press, 2016.
- [4] R. K. Govindarajan, G. Venkataramanan, Servo control of solenoid actuators using augmented feedback linearization, in: Control and Modeling for Power Electronics (COMPEL), 2017 IEEE 18th Workshop on, IEEE, 2017, pp. 1–6.

- 240 [5] A. Dobrucki, B. Merit, V. Lemarquand, G. Lemarquand, Modeling of the nonlinear distortion in electrodynamic loudspeakers caused by the voice-coil inductance, in: 10ème Congrès Français d’Acoustique, 2010.
- [6] N. Kesavamurthy, G. Rao, A study of bethenod’s phenomenon, *IEEE Trans Circuit Theory* 19 (2) (1972) 215–218.
- 245 [7] D. D. Kana, Parametric coupling in a nonlinear electromechanical system, *J Eng Ind* 89 (4) (1967) 839–847.
- [8] P. S. Landa, Y. B. Duboshinskiĭ, Self-oscillatory systems with high-frequency energy sources, *Sov Phys Uspekhi* 32 (8) (1989) 723.
- [9] P. S. Landa, *Nonlinear oscillations and waves in dynamical systems*, Vol. 360, Springer Science & Business Media, 2013.
- 250 [10] P. Koval’chuk, L. Kruk, Forced nonlinear vibrations of cylindrical shells filled with a liquid, *Int Appl Mech* 38 (11) (2002) 1388–1393.
- [11] I. T. Georgiou, F. Romeo, Multi-physics dynamics of a mechanical oscillator coupled to an electro-magnetic circuit, *Int J Nonlin Mech* 70 (2015) 153–164.
- 255 [12] F. Romeo, I. T. Georgiou, Multiphysics chaotic interaction in a coupled electro-magneto-mechanical system, in: *ASME Proceedings IMECE*, 2014, p. V04BT04A018.
- [13] A. Luongo, D. Zulli, G. Piccardo, Analytical and numerical approaches to nonlinear galloping of internally resonant suspended cables, *J Sound Vib* 315 (3) (2008) 375–393.
- 260 [14] V. Settimi, O. Gottlieb, G. Rega, Asymptotic analysis of a noncontact AFM microcantilever sensor with external feedback control, *Nonlinear Dynam* 79 (4) (2015) 2675–2698.
- [15] J. Warminski, D. Zulli, G. Rega, J. Latałski, Revisited modelling and multimodal nonlinear oscillations of a sagged cable under support motion, *Mechanica* 51 (11) (2016) 2541–2575.
- 265 [16] A. H. Nayfeh, *Perturbation methods*, John Wiley & Sons, 2008.