DETECTION OF THE SOURCE NUMBER BY THE GERSCHGORIN DISKS

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ABSTRACT

The inclusion regions of the eigenvalues enable us to work out efficient criteria for the estimation of the number of sinusoids. To exploit those regions, it is necessary first to transform the covariance matrix. That is why we put forward a transformation based on an approximation of the eigenvalues and eigenvectors, so as to obtain the radii and the centers of the Gerschgorin disks, the disks being the studied inclusion regions. We show that the introduction of information concerning the radii and the centers in the detection criteria improves their performances tremendously. A new criterion using the Euclidean distance (and called GDE_{dist}) is also suggested.

1 INTRODUCTION

The estimation of the source number is one of the most crucial problems in many major applications of signal processing like RADAR, SONAR, MNR and, generally speaking, in the fields of array processing and spectral analysis. Indeed, we must extract the signal, supposed to be here under the form of sinusoidal sources, from a noisy signal generally condensed in a covariance matrix. For the high resolution methods (MUSIC, ESPRIT), this is achieved by an eigendecomposition of the covariance matrix into two distinct signal and noise subspaces. The worst situation is the underestimation of the signal subspace where an amount of the signal information is definitely lost. To cope with this problem, several criteria have been proposed in literature. Recently, we have suggested developing a new family based on an inclusion region of eigenvalues called the Gerschgorin disks. Our objective is to limit or suppress all the restrictive hypotheses for a practical application. To that purpose, we try to act upstream the theory of matrices and not only downstream after the factorisation or the decomposition of matrices. To exploit the Gerschgorin disks, it is necessary first to transform the covariance matrix. That is why we put forward a transformation based on an approximation of the eigenvalues and eigenvectors, so as to obtain the radii and the centers of the Gerschgorin disks. We show that the introduction concerning the radii and the centers in the detection criteria improves their performances tremendously. A new criterion using the Euclidean distance and called GDE_{dist} is also suggested.

2 THE INCLUSION REGIONS

By definition, an inclusion region of a matrix A is a region of the complex plan that contains at least one of its eigenvalues. The inclusion regions are at the basis of a lot of methods in numerical analysis enabling us to locate and evaluate the perturbations of the eigenvalues. The most common regions are the Gerschgorin and Ostrowski regions. They take into account the evolutions of the eigenvalues that are submitted to a perturbation E such as $A \rightarrow A + E$. The square matrix A of dimension (N,N) can be written $A = D + B$ where $D = diag(a_{ii}, \dots, a_{NN})$ is the main diagonal of A while B contains the off-diagonal elements of A. With $A_\epsilon = D + \epsilon B$, for all $\epsilon \in \mathbb{C}$, then if ϵ is small enough, the eigenvalues of A are located in the small neighbourhood of the diagonal elements $a_{11}, \dots, a_{NN}$. A way to describe this neighbourhood is the theorem of Gerschgorin [1] where the eigenvalues λ of $A = [a_{ij}]$ of order N belong to the union of the N Gerschgorin disks described by :

$$\bigcup_{i=1}^{N} \left\{ \lambda \in \mathbb{C} : \mid \lambda - a_{ii} \mid \leq R_i = \sum_{j=1, j\neq i}^{N} \mid a_{ij} \mid \right\} \qquad (1)$$

Each disk D_i is defined in the complex plan by a radius R_i and a center $O_i = a_{ii}$. The radii are calculated from the sum of each row element of A except the diagonal element. Moreover, if the disks are pairwise disjoint, then each one captures exactly one eigenvalue. However, the direct application of the Gerschgorin theorem on a covariance matrix C gives no indication about the source number because the radii are longer and the disks are intertwined. For example, we consider a signal $x(t)$ composed of one sinusoid (source number M=2 from the Euler relation), with a normalized frequency 0.25, embedded in a white Gaussian noise $n(t)$ of unitary variance such as :

$$x(t) = A_1 \sin(2\pi f_1 t) + n(t) \qquad (2)$$

with t = 1, ... ,8, $A_1 = 4.47$ (10 dB), $f_1 = 0,25$. The covariance matrix C is chosen of dimension (4,4) and under the modified covariance form. So, it is hermitian and its eigenvalues are real. The elements of C are :

$$C = \begin{pmatrix} 130.11 & 4.15 & -119.7 & -26.29 \\ 4.15 & 121.41 & 9.98 & -119.7 \\ -119.7 & 9.98 & 121.41 & 4.15 \\ -26.29 & -119.7 & 4.15 & 130.11 \end{pmatrix} \quad (3)$$

The direct application of the Gerschgorin theorem does not enable us to separate the signal and noise subspaces. That is why we present two methods based on the unitary transformations of C to exploit the inclusion regions. Indeed, a unitary transformation does not modify the eigenvalues.

3 UNITARY TRANSFORMATIONS OF THE COVARIANCE MATRIX

The aim of those transformations is to improve the localization of the eigenvalues in reducing the surface of the inclusion regions, that is to say to search for a diagonal form for the covariance matrix C. The first possibility has already been presented in previous papers [1][2][3]. We briefly recall its principle. Let a unitary matrix U and a partitioned matrix C be defined by :

$$U = \begin{pmatrix} U_1 & \mathbf{0} \\ \mathbf{0}^H & 1 \end{pmatrix}, \; C = \begin{pmatrix} C_1 & \mathbf{c} \\ \mathbf{c}^H & c_{NN} \end{pmatrix} \quad (4)$$

where the vector $\mathbf{c}$ is the last column of C except the element c_{NN}. If we apply U to the partitioned matrix C, we obtain the transformed matrix called C_T, such as $C_T = U^H C\, U$ and we obtain :

$$C_T = \begin{pmatrix} S_1 & U_1^H \mathbf{c} \\ \mathbf{c}^H U_1 & c_{NN} \end{pmatrix} = \begin{pmatrix} \lambda_1^{'} & & & R_1 \\ & \lambda_2^{'} & & R_2 \\ & & \ddots & \vdots \\ R_1^H & R_2^H & \cdots & c_{NN} \end{pmatrix} \quad (5)$$

The matrix S_1 is diagonal and contains the eigenvalues $\lambda_i^{'}$ of the submatrix C_1 obtained after the eigendecomposition $C_1 = U_1 S_1 U_1^H$, where U_1 is the matrix of the eigenvectors u_i of C_1 such as $U_1 = [u_1 \cdots u_{N-1}]$. From the interlacing property of the eigenvalues, the values $\lambda_i^{'}$ are arranged in the same order as the eigenvalues λ_i of C. So, from the matrix C_T, we can deduce the following radii and centers :

$$R_i = \mid \mathbf{u}_i^H \; \mathbf{c} \mid , \; O_i = \lambda_i^{'} \quad (6)$$

for i=1, ... , N-1. With the values of C given in (3),the Gerschgorin disks of C_T are represented on the figure 1. One value is near the origin (0,0), the two other values can be associated to the signal subspace. But, if the disks are numerous, this figure is not readable. Moreover, a disk center remote from zero can have a small radius and a disk center near zero a long radius but it is less likely. Thus we prefer a representation of the radii in accordance with the disk centers where the values near the origin are supposed to be associated to the noise subspace (see figure 2). Both the radii and the centers are important and must be taken into account in the detection criteria.

The second way to obtain a diagonal form of the covariance matrix consists in doing the approximation of the eigendecomposition with the Fourier vectors (or steering vectors). Let a matrix $A = (a_{ij})$ be of dimension (N,N) calculated after the following unitary transformation of C :

$$A = Q^H C\, Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{q}_i^H C\, \mathbf{q}_j \quad (7)$$

where $Q = [\mathbf{q}_1 \cdots \mathbf{q}_N]$ and the Fourier vectors $\mathbf{q}_i = [1 \; e^{j2\pi f_i} \dots e^{j2\pi f_i (N-1)}]^T / \sqrt{N}$ for i=1, ... , N, such as $\|\mathbf{q}_i\|_2 = 1$. Each vector $\mathbf{q}_i$ is orthogonal to the others. The N normalized frequencies f_i are linearly distinct from 1/N on the interval [0 , 1-1/N]. With the property $Q^H Q = I$, the eigenvalues of A are the same as those of C. If we take the equality $A = D + B$, the matrix D can be written :

$$D = \begin{pmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{NN} \end{pmatrix} \quad (8)$$

where $a_{ii} = \sum_{i=1}^{N} \mathbf{q}_i^H C \mathbf{q}_i = \hat{\lambda}_i$ from the definition $C\mathbf{q} = \hat{\lambda}\mathbf{q}$. The values $\hat{\lambda}_i$ are the estimated eigenvalues of C associated to the estimated eigenvectors $\mathbf{q}_i$. The matrix B is as follows :

$$B = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1j} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{i1} & & & 0 \end{pmatrix} \quad (9)$$

If Q is an eigenvector basis of C, then the diagonal D contains the eigenvalues of C and the elements of B are zero. In this case, the radii R_i are also zero. If Q is an approximation of this basis, the main diagonal of D estimates the eigenvalues of C and the elements of B represent limits for the deviation of each diagonal element from its exact value that is an exact eigenvalue. For the instance (3), we apply the theorem of Gerschgorin to the matrix A to obtain two distinct sets of disks (see figure 3).

4 DETECTION CRITERIA

To estimate the source number, we suggest developing criteria using the radius and center information of the disks. A heuristic criterion using the Euclidean distance is put forward. This distance is normalized to take into account the contributions of the radii and of the centers, let the following distance be called $dist(i)$:

$$dist(i) = \sqrt{(O_i/O_{max})^2 + (R_i/R_{max})^2} \qquad (10)$$

where O_{max} and R_{max} are respectively the maximal values of centers and radii of the Gerschgorin disks, where i = 1, ..., P. In the case of the matrix C, P = N-1 and, in the case of the matrix A, P = N. The distances $dist(i)$ are sorted in a decreasing order. The proposed heuristic criterion needs a reasonable arbitrary threshold by subtracting the mean of distances from each of them, let the criterion GDE_{dist} be described by :

$$GDE_{dist}(k) = dist(k) - \frac{F(L)}{P}\sum_{i=1}^{P} dist(i) \qquad (11)$$

with k = 1, ..., P and $F(L)$ is a constant value or an adjustable function according to the sample number L of the signal. In fact, it is possible to make $F(L)$ vary from 0 to 2 values by short successive steps (for example $\triangle = 0.01$) and to estimate M for each step with the GDE_{dist} criterion. Then, we retain the largest stages where the estimated value M remains similar. We apply this method to the well-known Marple signal using the second transformation proposed in section 3 [4]. This signal of 64 samples is composed of 4 complex sinusoids in a colored noise. Two sinusoids are very close to normalized frequencies 0.2 and 0.21 and they are more powerful by 20 dB than the two others placed at the normalized frequencies 0.1 and -0.15. The representation of radii and disk centers (see figure 4) from the Gerschgorin theorem applied to A matrix shows the 4 sources. By changing $F(L)$ as described previously, the largest stage provides the right number of sources (see figure 5). To simplify the calculation in the multiple simulations, we take $F(L) = 1$. When the first negative value of the criterion has been reached, the estimated source number M becomes M = k-1. Other criteria can be developed.

5 SIMULATION RESULTS

The simulations are carried out in the same conditions as those described in the reference [3], with a signal $x(t)$ of 16 samples composed of 2 sinusoids (M=4) of normalized frequencies $f_1 = 0.25$, $f_2 = 0.3$ in a white Gaussian noise $n(t)$ of unitary variance, such as :

$$x(t) = A_1 \sin(2\pi f_1 t) + A_2 \sin(2\pi f_2 t) + n(t) \qquad (12)$$

where t=1, 2, ..., 16. For each result, 200 independent simulations are carried out. The estimated matrix C of dimension (8,8) is under the modified covariance. In the first simulation, $A_1 = A_2$ and those amplitudes vary from -5 dB t +15 dB. In the second simulation, A_1 is fixed to 4.47 (10 dB) and A_2 varies from 10 dB to 20 dB. The nonwhite Gaussian noise is obtained through an AR(1) filter of coefficient 0.9. From the figure 6, the GDE_{dist} criterion associated to the matrix A gives better results (4 to 6 dB) than that associated to C. In the case where $A_1 \neq A_2$ (see figure 7), GDE_{dist} with A gives the best detection rate that is 100 %. The results are superior to those obtained in [3].

6 CONCLUSION

The Gerschgorin disks provide further information for the source number estimation criteria. In numerous cases, such as signals with few samples or sources of different powers, this information can be crucial to estimate the source number because the eigenvalues are no longer significant. Other studies are in progress.

References

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[3] O. Caspary and P. Nus, "New criteria based on Gerschgorin radii for source number estimation," Proc. of EUSIPCO, Rhodes, Greek, 8-11 September, pp. 77-80, 1998.

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* Caption of figures 6 and 7 :

GDE_{dist} on C, white noise : $-o-$
GDE_{dist} on C, colored noise : $\cdots o \cdots$
GDE_{dist} on A, white noise : $- * -$
GDE_{dist} on A, colored noise : $\cdots * \cdots$

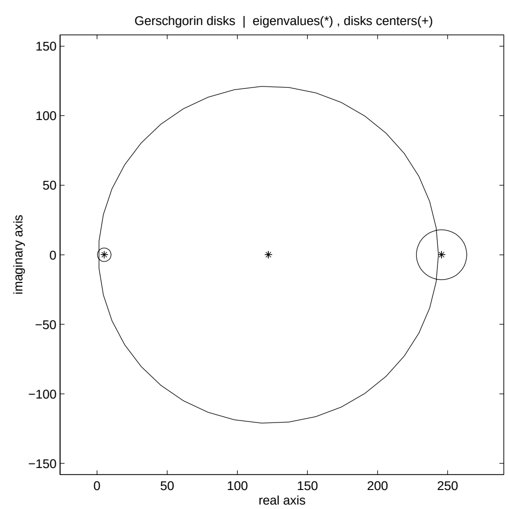

Figure 1: Gerschgorin disks of C_T

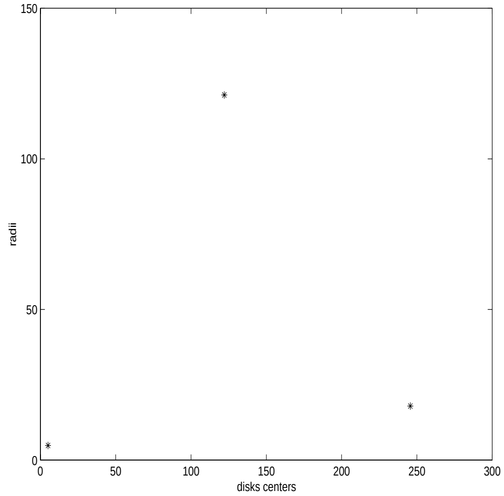

Figure 2: Gerschgorin disks of C_T, 2D representation

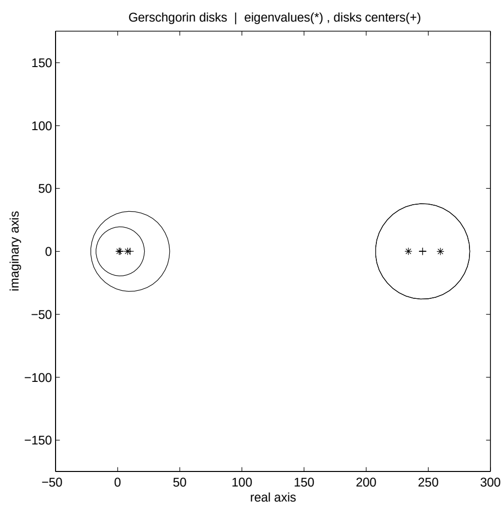

Figure 3: Gerschgorin disks of A

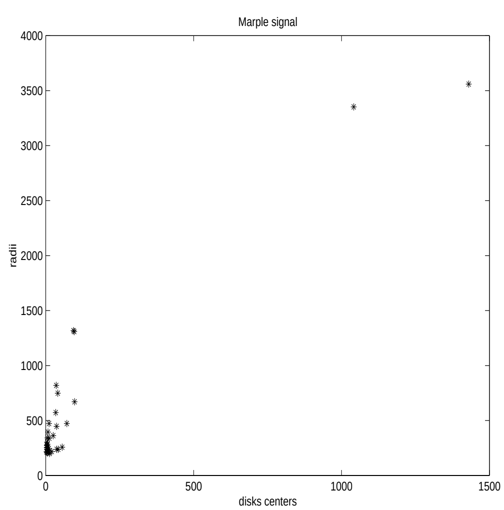

Figure 4: Gerschgorin disks of A (Marple signal)

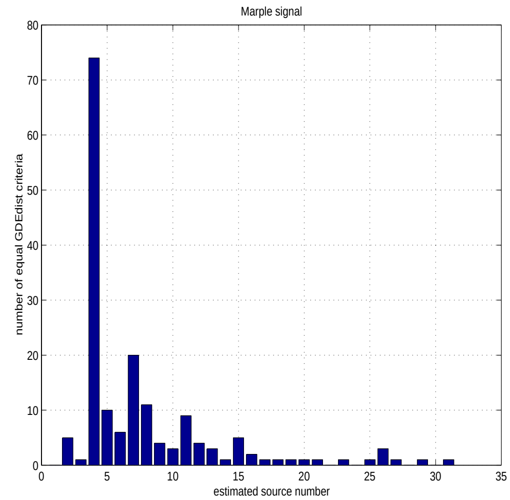

Figure 5: Method of stages (Marple signal)

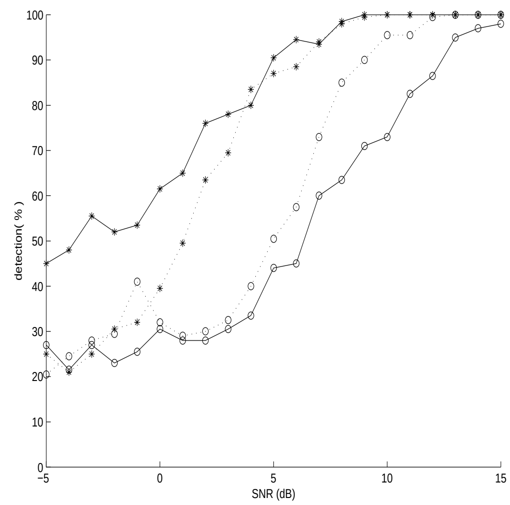

Figure 6: GDE_{dist} criterion (identical amplitudes)*

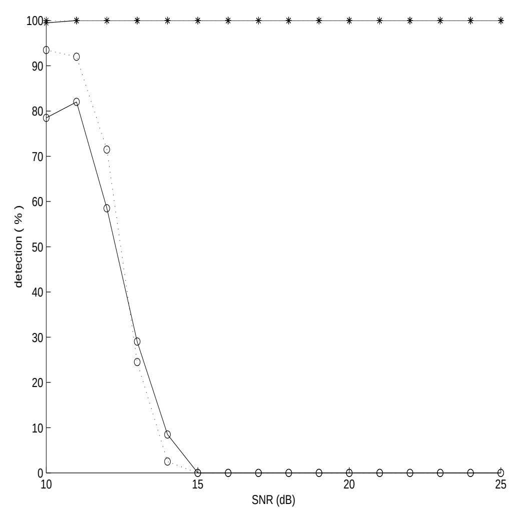

Figure 7: GDE_{dist} criterion (different amplitudes)*