A CUMULANT BASED ALGORITHM FOR THE IDENTIFICATION OF INPUT OUTPUT QUADRATIC SYSTEMS

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ABSTRACT

A parameter estimation algorithm is developed for the identification of an input output quadratic model. The excitation is a zero mean white Gaussian input and the output is corrupted by additive measurement noise. Input output crosscumulants up to fifth order are employed and the identification problem of the unknown model parameters is reduced to the solution of succesive linear systems of equations that are solved iteratively. Simulation results are provided for different SNR's illustrating the performance of the algorithm and confirming the theoretical set up.

1 INTRODUCTION

Identification of nonlinear discrete time input output models plays an increasingly central role in today's applications in many different areas of science and engineering [1],[2]. Realistic situations where nonlinearities are present and must be taken into account, include communication channels, control of idustrial processes, time series analysis and others. The Volterra functional representation is a useful approach to model the above type of systems. The nonparametric identification case has been treated extensively in [10],[11],[12], and closed form crosscumulant solutions are provided. In this work we are concerned with the parametric identification of a special class of polynomial models [3], the input output quadratic models, generalizing previous work on identification of bilinear systems [4]. In contrast to the existing methods which depend on adaptive least squares techniques [5],[6],[7] we have developed a cumuland based algorithm that is capable of directly computing the uknown model coefficients. The approach reduces the identification problem to the solution of linear systems of equations that are computed iteratively at each step of the estimation process.

2 PROBLEM STATEMENT

The plant we seek to identify is of the following single input single output quadratic type :

$$z(n) = \sum_{i=1}^{k_1} a_i z(n-i) + \sum_{i=0}^{k_2} b_i u(n-i) +$$

$$\sum_{i=1}^{k_3}\sum_{j=i}^{k_3} A_{ij} z(n-i)z(n-j) + \sum_{i=0}^{k_4}\sum_{j=i}^{k_4} B_{ij} u(n-i)u(n-j) +$$

$$+ \sum_{i=1}^{k_5}\sum_{j=1}^{k_5} c_{ij} z(n-i)u(n-j)$$

and

$$y(n) = z(n) + \eta(n) \tag{1}$$

The input $u(n)$ is a stationary zero mean white Gaussian process and the measurement noise $\eta(n)$ is zero mean white and independent of the input. It is assumed that $B_{00} \neq 0, B_{0k_4} \neq 0$. The p-th order cumulant sequence of a stationary random signal $x(k)$ [8],[9] is denoted by:

$$c_{px}(k_1, k_2, \ldots, k_{p-1}) =$$

$$\text{cum}[x(n), x(n-k_1), x(n-k_2)\ldots, x(n-k_{p-1})]$$

Since the input is a stationary zero mean white Gaussian random process:

$$c_{2u}(n) = \gamma_2 \delta(n) = \begin{cases} \gamma_2 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and $c_{ku}(n_1, n_2, \ldots, n_{k-1}) = 0$ for all $k > 2$. Moreover

$$\text{cum}[y(n), y(k), u(l)] = \text{cum}[z(n), z(k), u(l)]$$

It is assumed that both input and output sequences are stationary. To compute the coefficients of the quadratic model sufficient crosscumulant information between the output and copies of the input is generated up to fifth order and in particular $c_{yu}(l)$, $c_{yuu}(l_1, l_2)$, $c_{yuuu}(l_1, l_2, l_3)$, $c_{yuuuu}(l_1, l_2, l_3, l_4)$ which are evaluated on specific cumulant slices. The Leonov-Shiryaev theorem is repeatedly applied to arrive at the appropriate relations.

It is remarked that the input output quadratic type model admits a Volterra series expansion of the following form:

$$y(n) = \sum_{k_1=0}^{\infty} h_1(k_1)u(n-k_1)+$$

$$+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} h_2(k_1,k_2)u(n-k_1)u(n-k_2)+$$

$$+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} h_3(k_1,k_2,k_3)u(n-k_1)u(n-k_2)u(n-k_3)$$

$$+\cdots+\eta(n)$$

2.1 A useful lemma

The above Volterra representation is a key ingredient to prove the following lemma which establishes some causality properties that are essential for the derivation of the algorithm.

Lemma

$$\begin{array}{ll} c_{yu}(l) = 0, & l < 0 \\ c_{yuu}(l_1,l_2) = 0, & l_1 < 0,\ or\ l_2 < 0 \\ c_{yuuu}(l_1,l_2,l_3) = 0 & l_1 \le 0,\ or\ l_2 \le 0,\ or\ l_3 \le 0 \\ c_{yuuuu}(l_1,l_2,l_3,l_4) = 0 & l_1 \le 0,\ or\ \ l_2 \le 0, \\ & or\ \ l_3 \le 0,\ or\ l_4 \le 0 \end{array}$$

3 IDENTIFICATION STEPS

Step 1. Computation of $b_0,\ B_{0l}, 0 \le l \le k_4$

This is the initial step which is realized only in the first iteration.

$$\mathbf{b_0} = \frac{c_{yu}(0)}{\gamma_2}, \mathbf{B_{00}} = \frac{c_{yuu}(0,0)}{2\gamma_2^2}$$

$$\mathbf{B_{0l}} = \frac{c_{yuu}(0,l)}{\gamma_2^2} \qquad 1 \le l \le k_4 \qquad (2)$$

The above equations correspond to the linear system (R_1, Q_1) where R_1 is a diagonal matrix coefficient.

From the initial model assumptions $c_{yuu}(0,0) \ne 0$ and $c_{yuu}(0,k_4) \ne 0$ since $B_{00} \ne 0$ and $B_{0k_4} \ne 0$. These conditions constitute sufficient persistence excitation conditions for the identification algorithm.

Step 2. Computation of $A_{l_1l_2},\ 1 \le l_1 \le k_3,\ l_1 \le l_2 \le k_3$

Step 2 is common to all iterations. In fact iterations are indexed by l_1. Thus the first iteration engages step 2 to determine the first row A_{1l_2}, the second iteration in step 2 computes the second row A_{2l_2} and so forth. The linear system $R_2(l_1)x(l_1) = Q_2(l_1)$ is generated where $R_2(l_1)$ is columnwise

$$R_2(l_1) = \left(\ \mathbf{r_1}\quad \mathbf{r_2}\ldots\quad \mathbf{r_3}\ \right)$$

$\mathbf{r_1}$ is the column

$$\begin{pmatrix} 6c_{yuu}^2(0,0) \\ 2c_{yuu}(0,0)c_{yuu}(1,1) + 4c_{yuu}^2(0,1) \\ \vdots \\ 2c_{yuu}(0,0)c_{yuu}(k_3-l_1,k_3-l_1) + 4c_{yuu}^2(0,k_3-l_1) \end{pmatrix}$$

$\mathbf{r_2}$, $\mathbf{r_3}$ are respectively, the columns

$$\begin{pmatrix} 0 \\ c_{yuu}^2(0,0) \\ \vdots \\ c_{yuu}(0,0)c_{yuu}(k_3-l_1,k_3-l_1) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_{yuu}^2(0,0) \end{pmatrix}$$

and

$$x(l_1) = \left(A_{l_1l_1}\ A_{l_1l_1+1}\ \ldots\ A_{l_1k_3}\right)^T$$

$$Q_2(l_1) = \left(q_{l_1}\ q_{l_1+1}\ \ldots\ q_{k_3}\right)^T \qquad (3)$$

For fixed l_1, the right hand side $Q_2(l_1) = c_{yuuuu}(l_1,l_1,l_2,l_2) - L_4(l_1,l_1,l_2,l_2)$ is a known column vector. $L_4(l_1,l_1,l_2,l_2)$ is a function of crosscumulants and previously computed parameters.

Step 3. Computation of $c_{l_1l_2},\ 1 \le l_1 \le k_5,\ l_1 \le l_2 \le k_5$

As before step 3 enters all iterations indexed by l_1. The upper triangular part of C is determined via a set of linear systems having the same triangular matrix. The triangular system $R_3(l_1)x(l_1) = Q_3(l_1)$ is generated and $R_3(l_1)$ has the form

$$\begin{pmatrix} 3\gamma_2 c_{yuu}(0,0) & 0 & \ldots & 0 \\ 2\gamma_2 c_{yuu}(0,1) & \gamma_2 c_{yuu}(0,0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2\gamma_2 c_{yuu}(0,k_5-l_1) & 0 & \ldots & \gamma_2 c_{yuu}(0,0) \end{pmatrix}$$

with

$$x(l_1) = \left(c_{l_1l_1}\ c_{l_1l_1+1}\ \ldots\ c_{l_1k_5}\right)^T$$

$$Q_3(l_1) = \left(q_{l_1}\ q_{l_1+1}\ \ldots\ q_{k_5}\right)^T$$

where, for fixed l_1, the right hand side

$$Q_3(l_1) = c_{yuuu}(l_1,l_1,l_2) - L_3(l_1,l_1,l_2) \qquad (4)$$

is a known column vector. $L_3(l_1,l_1,l_2)$ is a function of crosscumulants and previously computed parameters.

Step 4. Computation of $c_{l_2l_1},\ 1 \le l_1 \le k_5,\ l_1 < l_2 \le k_5$

Iterations are indexed by l_1 and provide the columns of C succesively. To determine the lower triangular part of C, the linear system $R_4(l_1)x(l_1) = Q_4(l_1)$ is generated where

$$R_4(l_1) = \left(\ \mathbf{r_1}\quad \mathbf{r_2}\ldots\quad \mathbf{r_3}\ \right)$$

The column vector $\mathbf{r_1}$ is

$$\begin{pmatrix} \gamma_2 c_{yuu}(0,0) \\ \gamma_2 c_{yuu}(1,1) \\ \vdots \\ \gamma_2 c_{yuu}(k_5-l_1-1, k_5-l_1-1) \end{pmatrix}$$

$\mathbf{r_2}$, $\mathbf{r_3}$ are respectively the columns

$$\begin{pmatrix} 0 \\ \gamma_2 c_{yuu}(0,0) \\ \vdots \\ \gamma_2 c_{yuu}(k_5-l_1-2, k_5-l_1-2) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \gamma_2 c_{yuu}(0,0) \end{pmatrix}$$

with

$$x(l_1) = (c_{l_1+1l_1} \; c_{l_1+2l_1} \; \dots \; c_{k_5 l_1})^T$$

$$Q_4(l_1) = (q_{l_1+1} \; q_{l_1+2} \; \dots \; q_{k_5})^T \quad (5)$$

For fixed l_1, $Q_4(l_1) = c_{yuuu}(l_1, l_2, l_2) - L_3(l_1, l_2, l_2)$ is a known column vector.

Step 5. Computation of a_{l_1} and $B_{l_1 l_2}$, $1 \le l_1 \le k_4$, $l_1 \le l_2 \le k_4$

Specific crosscumulant expressions are evaluated on the above specified slices as well as at the points $(l_1, k_4 + l_1)$. The block of a is determined in conjunction with the upper half part of B through a series of systems $R_5(l_1)x(l_1) = Q_5(l_1)$ where:

$$R_5(l_1) = \begin{pmatrix} c_{yuu}(0,k_4) & 0 & 0 & \dots & 0 \\ c_{yuu}(0,0) & 2\gamma_2^2 & 0 & \dots & 0 \\ c_{yuu}(0,1) & 0 & \gamma_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{yuu}(0,k_4-l_1) & 0 & 0 & \dots & \gamma_2^2 \end{pmatrix}$$

$$x(l_1) = (a_{l_1} \; B_{l_1 l_1} \; B_{l_1 l_1+1} \dots \; B_{l_1 k_4})^T$$

$$Q_5(l_1) = (p \; q_{l_1} \; q_{l_1+1} \dots \; q_{k_4})^T \quad (6)$$

where for fixed l_1, $Q_5(l_1) = c_{yuu}(l_1, l_2) - L_2(l_1, l_2)$ is a known column vector with

$$p = c_{yuu}(l_1, k_4 + l_1) - L_2(l_1, k_4 + l_1)$$

and

$$q_m = c_{yuu}(l_1, m) - L_2(l_1, m), \;\; l_1 \le m \le k_4$$

Step 6. Computation of b_l, $1 \le l \le k_2$

The b_l coefficients are directly computed from

$$b_l = \frac{1}{\gamma_2}[c_{yu}(l) - L_1(l)]$$

As in previous steps $L_1(l)$ is a function of crosscumulants and previously determined coefficients. Clearly $R_6 = \gamma_2 I$. The description of the algorithm is complete.

4 SIMULATION RESULTS

Consider the following input output quadratic model:

$$z(n) = a_1 z(n-1) + b_0 u(n) + \sum_{i=1}^{2} A_{1j} z(n-1) z(n-j)$$

$$+ \sum_{i=0}^{1}\sum_{j=i}^{1} B_{ij} u(n-i) u(n-j) + \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij} z(n-i) u(n-j)$$

and $y(n) = z(n) + \eta(n)$ with $B_{00} \neq 0$ and $B_{01} \neq 0$.

Simulation results for eleven coefficients are summarized in the following tables for two SNR's at 10dB and 30dB.

Table 1: N=40,000 samples, M=500 runs, Gaussian white input

Parameters	**True Value**	**Mean**	**Variance**
		SNR=30dB	
a1	0.1	0.0989	9.60×10^{-3}
b0	1	0.9999	1.08×10^{-4}
B00	1	1.0001	2.45×10^{-4}
B01	-0.5	-0.4985	4.47×10^{-4}
B11	-0.2	-0.1994	8.20×10^{-3}
A11	-0.01	-0.0099	8.75×10^{-5}
A12	0.05	0.0504	3.88×10^{-4}
c11	0.15	0.1494	4.82×10^{-4}
c12	-0.2	-0.2002	7.45×10^{-4}
c21	-0.1	-0.1008	6.02×10^{-4}
c22	-0.13	-0.1308	2.92×10^{-4}

Table 2: N=40,000 samples, M=500 runs, Gaussian white input

Parameters	**True Value**	**Mean**	**Variance**
		SNR=10dB	
a1	0.1	0.1033	1.07×10^{-2}
b0	1	1.0004	1.21×10^{-4}
B00	1	0.9998	2.17×10^{-4}
B01	-0.5	-0.4994	4.54×10^{-4}
B11	-0.2	-0.2021	9.0×10^{-3}
A11	-0.01	-0.0104	8.01×10^{-5}
A12	0.05	0.0497	3.56×10^{-4}
c11	0.15	0.1504	5.00×10^{-4}
c12	-0.2	-0.1984	8.18×10^{-4}
c21	-0.1	-0.1005	5.56×10^{-4}
c22	-0.13	-0.1301	2.98×10^{-4}

The insensitivity of the algorithm to independent measurement noise is evident.

Table 3: N=5,000 samples, M=500 runs, Gaussian white input

Parameters	True Value	Mean	Variance
		SNR=30dB	
a1	0.1	0.0892	7.91×10^{-2}
b0	1	0.9985	9.14×10^{-4}
B00	1	0.9999	2.10×10^{-3}
B01	-0.5	-0.5034	3.47×10^{-3}
B11	-0.2	-0.1895	5.82×10^{-2}
A11	-0.01	-0.0111	7.34×10^{-4}
A12	0.05	0.0534	2.80×10^{-3}
c11	0.15	0.1525	3.70×10^{-3}
c12	-0.2	-0.2042	6.00×10^{-3}
c21	-0.1	-0.1062	4.40×10^{-3}
c22	-0.13	-0.1350	4.01×10^{-3}

Table 4: N=5,000 samples, M=500 runs, Gaussian white input

Parameters	True Value	Mean	Variance
		SNR=10dB	
a1	0.1	0.1043	8.5×10^{-2}
b0	1	1.0002	9.49×10^{-4}
B00	1	1.0007	2.10×10^{-3}
B01	-0.5	-0.5012	3.50×10^{-3}
B11	-0.2	-0.2015	7.18×10^{-2}
A11	-0.01	-0.0117	6.35×10^{-4}
A12	0.05	0.0484	3.20×10^{-3}
c11	0.15	0.1526	3.90×10^{-3}
c12	-0.2	-0.1974	6.20×10^{-3}
c21	-0.1	-0.1005	5.10×10^{-3}
c22	-0.13	-0.1303	2.80×10^{-3}

5 Conclusion

Identification of quadratic models has been addressed in this paper. An algorithm based on input output crosscumulant information up to order 5 has been developed using zero mean white Gaussian input. The proposed scheme is an exact estimation method and yields the desired parameters via a series of nonsingular triangular linear systems. We were able to directly identify the unknown coefficients by isolating each one separately, using appropriate input shifts in the crosscumulant expressions and thus forming linear systems of equations of lower triangular structure. Simulations were performed for different SNR's at 30dB and 10dB and for different sample lengths. The overall performance of the algorithm was quite satisfactory in the presence of independent measurement noise, confirming the theoretical development.

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