Nonlinear diffusions, hypercontractivity and the optimal L^p-Euclidean logarithmic Sobolev inequality

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Abstract

The equation $u_t = \Delta_p(u^{1/(p-1)})$ for $p > 1$ is a nonlinear generalization of the heat equation which is also homogeneous, of degree 1. For large time asymptotics, its links with the optimal L^p-Euclidean logarithmic Sobolev inequality have recently been investigated. Here we focus on the existence and the uniqueness of the solutions to the Cauchy problem and on the regularization properties (hypercontractivity and ultracontractivity) of the equation using the L^p-Euclidean logarithmic Sobolev inequality. A large deviation result based on a Hamilton-Jacobi equation and also related to the L^p-Euclidean logarithmic Sobolev inequality is then stated.

Key words: Optimal L^p-Euclidean logarithmic Sobolev inequality, Sobolev inequality, nonlinear parabolic equations, degenerate parabolic problems, entropy, existence, Cauchy problem, uniqueness, regularization, hypercontractivity, ultracontractivity, large deviations, Hamilton-Jacobi equations

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Introduction

A semi-group $(P_t)_{t\geq 0}$ is said to be *hypercontractive* with contraction function $t \mapsto q(t)$ if and only if q is increasing and if for any admissible f,

$$\|P_t f\|_{q(t)} \leq C(t)\,\|f\|_{q(0)} \quad \forall\, t \geq 0$$

for some continuous function $t \mapsto C(t)$. It is *ultracontractive* if for some $q \geq 1$

$$\|P_t f\|_\infty \leq C(t)\,\|f\|_q \quad \forall\, t > 0\ .$$

It is the purpose of Gross' and Varopoulos' Theorems [23,32] to prove such properties for diffusion processes. This question introduces in a very natural way the logarithmic Sobolev inequality

$$\int f^2\,\log(f^2)\,d\mu \leq C^* \int |\nabla f|^2\,d\mu \quad \forall\, f \in H^1(\mathbb{R}^n) \text{ s.t. } \int f^2\,d\mu = 1\ ,$$

for some positive constant C^*, where μ is a measure on $\mathbb{R}^n$ which is invariant under the action of $(P_t)_{t\geq 0}$. In the case of the semi-group associated with the heat equation, $d\mu$ is the Lebesgue measure and the above inequality is the Euclidean logarithmic Sobolev inequality, with $C^* = 2$. This inequality can be reformulated in a form which is optimal under scalings [33] as

$$\int f^2 \log(f^2)\,dx \leq \frac{n}{2}\log\left[\frac{2}{\pi n e}\int |\nabla f|^2\,dx\right] \quad \forall f \in W^{1,2}(\mathbb{R}^n) \text{ s.t. } \|f\|_2 = 1\ .$$

Here we consider the semi-group generated by the nonlinear diffusion equation

$$u_t = \Delta_p(u^{1/(p-1)})$$

with $\Delta_p w := \mathrm{div}\,(|\nabla w|^{p-2}\nabla w)$ for some $p > 1$ and prove that the associated semi-group is hyper- and ultra-contractive. The inequality which generalizes the Euclidean logarithmic Sobolev inequality is the *optimal L^p-Euclidean logarithmic Sobolev inequality*

$$\int f^p \log(f^p)\,dx \leq \frac{n}{p}\log\left[\mathcal{L}_p \int |\nabla f|^p\,dx\right] \quad \forall f \in W^{1,p}(\mathbb{R}^n) \text{ s.t. } \|f\|_p = 1\ ,$$

which has been introduced recently [18] and then extended in [22] (also see [14]). This inequality holds for some positive and optimal constant $\mathcal{L}_p$ (see Theorem 4 below for more details). The *entropy*, which corresponds to the left hand side of the inequality, plays a crucial role for the existence and the uniqueness of a global solution to the Cauchy problem.

This paper is organized as follows. In Section 1, we state our main results and introduce the optimal L^p-Euclidean logarithmic Sobolev inequality. The existence and the uniqueness of a global solution is established in Section 2. Section 3 is devoted to hypercontractivity and Section 4 to connections with large deviations and the Hamilton-Jacobi equation

$$v_t + \frac{1}{p}\,|\nabla v|^p = 0\ ,$$

for which the optimal L^p-Euclidean logarithmic Sobolev inequality also plays an important role. Note that this equation and its regularity properties have been the subject of an earlier study of the third author [22].

1 Main results

Consider a global solution to the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \end{cases} \tag{1}$$

for some nonnegative initial data f. Note that $\Delta_p u^m = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right)$ is homogeneous of degree one if and only if $m = 1/(p-1)$ (we shall take advantage of this fact in the proof of Theorem 1). If one considers the equation $u_t = \Delta_p u^m$, the case $m \neq 1/(p-1)$ has interesting scaling properties related to Gagliardo-Nirenberg inequalities. The optimal L^p-Euclidean logarithmic Sobolev inequality appears then as a limit case [17–19] of these inequalities when $m \to 1/(p-1)$.

By $\|u\|_p$, $p \neq 0$, we denote the quantity $(\int |u|^p\,dx)^{1/p}$ and unless it is explicitely specified, integrals are taken over $\mathbb{R}^n$. We also write $p^* = p/(p-1)$ for the Hölder conjugate exponent of p, if $p \in (1, +\infty)$.

Our first result is a global existence and uniqueness result. See the beginning of Section 2 for some comments on the literature and on our strategy of proof.

Theorem 1 *Let $p > 1$ and assume that f is a nonnegative function in $L^1(\mathbb{R}^n)$ such that $|x|^{p^*} f$ and $f \log f$ belong to $L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}^+_t, L^1(\mathbb{R}^n_x))$ of (1) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}^+_t, W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n_x))$.*

Here by *weak solution* of (1) we simply mean a solution in the sense of the distributions. The *a priori* estimate on the *entropy term* $\int u \log u \, dx$ plays a crucial role in the proof. Concerning regularity, our main result is the following *hypercontractivity* property.

Theorem 2 *Let α, $\beta \in [1, +\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in Theorem 1, if moreover $f \in L^\alpha(\mathrm{I\!R}^n)$, any solution of (1) with initial data f satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}} \quad \forall \, t > 0$$

with

$$A(n,p,\alpha,\beta) = \Big(\mathcal{C}_1 \, (\beta - \alpha)\Big)^{\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}} \, \mathcal{C}_2^{\frac{n}{p}} \ ,$$

$$\mathcal{C}_1 = n \, \mathcal{L}_p \, e^{p-1} \, \frac{(p-1)^{p-1}}{p^{p+1}} \ , \quad \mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}} \, \frac{\beta^{\frac{1-p}{\beta} - \frac{1}{\alpha} + 1}}{\alpha^{\frac{1-p}{\alpha} - \frac{1}{\beta} + 1}} \ .$$

See Theorem 4 below for a definition of $\mathcal{L}_p$. Note that for $p = 2$, with $\mathcal{L}_2 = \frac{2}{\pi \, n \, e}$, one recovers the classical estimates of the heat equation (see for instance [3,23,28,32]). A similar result holds for α, $\beta \in (0, 1]$ with $\beta \leq \alpha$ and at a formal level for $\beta \leq \alpha < 0$: see Theorems 10, 11 in Section 3. As a special case of Theorem 2, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 3 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathrm{I\!R}^n)$ satisfying the same assumptions as in Theorem 1 with $\alpha = 1$. Then for any $t > 0$*

$$\|u(\cdot,t)\|_\infty \leq \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} \ .$$

The main tool in our approach is the following optimal L^p-Euclidean logarithmic Sobolev inequality.

Theorem 4 [18,22] *Let $p \in (1, +\infty)$. Then for any $w \in W^{1,p}(\mathrm{I\!R}^n)$ with $\int |w|^p \, dx = 1$ we have,*

$$\int |w|^p \log |w|^p \, dx \leq \frac{n}{p} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx\right] \ , \tag{2}$$

with

$$\mathcal{L}_p = \frac{p}{n}\,\left(\frac{p-1}{e}\right)^{p-1}\,\pi^{-\frac{p}{2}}\,\left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)}\right]^{\frac{p}{n}}\,.$$

Inequality (2) is optimal and it is an equality if

$$w(x) = \left(\pi^{\frac{n}{2}}\left(\frac{\sigma}{p}\right)^{\frac{n}{p^*}}\frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)}\right)^{-1/p}\;e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}\quad\forall\,x\in\mathbb{R}^n$$

for any $p>1$*,* $\sigma>0$ *and* $\bar{x}\in\mathbb{R}^n$*. For* $p\in(1,n)$ *the equality holds only if* w *takes the above form.*

For our purpose, it is more convenient to use this inequality in a non homogeneous form, which is based on the fact that

$$\inf_{\mu>0}\left[\frac{n}{p}\,\log\left(\frac{n}{p\mu}\right)+\mu\,\frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right] = n\,\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right)+\frac{n}{p}\,.$$

Corollary 5 [17] *For any* $w\in W^{1,p}(\mathbb{R}^n)$*,* $w\neq 0$*, for any* $\mu>0$*,*

$$p\int|w|^p\,\log\left(\frac{|w|}{\|w\|_p}\right)\,dx+\frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int|w|^p\,dx\le\mu\int|\nabla w|^p\,dx\,.$$

Inequality (2) has been established in [18] for $1<p<n$ in view of the description of the intermediate asymptotics of (1) in $\mathbb{R}^n$ (see [17], and [30] for the asymptotic behaviour in the bounded case). It has been linked to optimal regularization properties of the Hamilton-Jacobi equation

$$v_t+\frac{1}{p}\,|\nabla v|^{\frac{1}{p}}=0 \tag{3}$$

and extended to any $p\in(1,+\infty)$ in [22]. Also see [21] for a previous work on hypercontractivity and properties of the Hamilton-Jacobi equation in case $p=2$, and [29,7,3,14,13,15] for connections with optimal mass transport, which have been recently investigated.

For earlier results concerning the standard logarithmic Sobolev inequality ($p=2$), one should refer to [23] (in the form of Corollary 5), to [33] for the form which is invariant under scalings (Theorem 4, $p=2$) and to [10] for the expression of all optimal functions. In case $p=1$, Inequality (2) was stated in [27] and the expression of the optimal functions has been established in [4].

2 Proof of Theorem 1

Existence and uniqueness of solutions to quasilinear parabolic equations have been extensively studied. However, as far as we know, the available results deal only with bounded domains. A standard reference when there is no external potential is the paper by Alt and Luckhaus [2]. See [31,30,11] for more recent results and further references. Very recently, Agueh in [1] adapted the strategy of steepest descent of the entropy with respect to a convex cost functional of Jordan, Kinderlehrer and Otto [24] to quasilinear parabolic equations. Their approach relies on mass transportation techniques and is certainly the right one from an abstract point of view. It covers Equation (1) in the case of a bounded domain. Here we choose to give a more direct proof for the existence and the uniqueness, which strongly relies on *a priori* estimates for the *entropy* $\int u \log u \, dx$ (this denomination makes sense both from probabilistic and physical points of view). As a last preliminary remark, let us note that because of the homogeneity of the equation, we can use the notion of weak solution defined in Section 1 although the initial data is essentially in $L^1(\mathbb{R}^n)$, so that we dont need to introduce any renormalization procedure.

Since (1) is 1-homogenous, in the sense that $\mu\, u$ is a solution corresponding to the initial data $\mu\, f$ for any $\mu > 0$ whenever u is a solution corresponding to an initial data f, there is no restriction to assume that $\int f \, dx = 1$. It is also straightforward to check that u is a solution of (1) if and only if v is a solution of

$$\begin{cases} v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases} \tag{4}$$

provided u and v are related by the transformation

$$u(x,t) = \frac{1}{R(t)^n}\, v(\xi, \tau)\,,\ \xi = \frac{x}{R(t)}\,,\ \tau(t) = \log R(t)\,,\ R(t) = (1 + p\, t)^{1/p}$$

(see [17,19] for more details and consequences for large time asymptotics). Let

$$v_\infty(\xi) = \pi^{-\frac{n}{2}} \Big(\frac{p}{\sigma}\Big)^{n/p^*} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{p^*}+1)}\ \exp(-\frac{p}{\sigma}\, |x|^{p^*})$$

with $\sigma = (p^*)^2$. For any nonnegative constant μ, $\mu\, v_\infty$ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) = 0$$

such that $\int v_\infty \, dx = \mu$. We may rewrite (4) as

$$\begin{cases} v_\tau = \nabla_\xi \left[v \left(\left|\frac{\nabla_\xi v}{v}\right|^{p-2} \frac{\nabla_\xi v}{v} - \left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^{p-2} \frac{\nabla_\xi v_\infty}{v_\infty} \right) \right] & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases}$$

The next step consists in regularizing the problem. First we replace the initial data f by

$$f^{\varepsilon_0} = N_{\varepsilon_0}^{-1} \chi_{\varepsilon_0} * \min(f_0 + \varepsilon_0 v_\infty, {\varepsilon_0}^{-1} v_\infty) \; , \quad \varepsilon_0 \in (0,1)$$

where $\chi_{\varepsilon_0} = {\varepsilon_0}^{-n} \chi(\cdot/\varepsilon_0)$ is a regularizing function, χ is a C^∞ with compact support function, with values in $[0,1]$, such that $\chi(x) \equiv 1$ if $|x| \leq 1$ and $\chi(x) \equiv 0$ if $|x| \geq 2$. The normalization constant N_{ε_0} is chosen such that $\int f^{\varepsilon_0} \, dx = 1$. We can also replace the equation for v by a regularized one:

$$\begin{cases} v_\tau = \nabla_\xi \left[v \left(\left[(1-\varepsilon) \left| \frac{\nabla_\xi (v + \eta v_\infty)}{v + \eta v_\infty} \right|^2 + \frac{\varepsilon}{(1+\eta)2} \left| \frac{\nabla_\xi v_\infty}{v_\infty} \right|^2 \right]^{\frac{p}{2}-1} \frac{\nabla_\xi v}{v} - \left| \frac{\nabla_\xi v_\infty}{(1+\eta) v_\infty} \right|^{p-2} \frac{\nabla_\xi v_\infty}{v_\infty} \right) \right] \\ v(\cdot, \tau = 0) = f^{\varepsilon_0} \end{cases}$$

for some positive regularizing parameters ϵ and η. Note that v_∞ is still a stationary solution. To emphasize the dependence in the various regularization parameters, we shall denote this solution by $v^{\varepsilon_0}_{\varepsilon,\eta}$. The standard theory [26] applies since this is a quasilinear parabolic equation of the form

$$v_\tau = \nabla_\xi \cdot [a(\xi, v, \nabla_\xi v)]$$

for which the right hand side is locally (in ξ) uniformly elliptic. To be precise, one should first solve the problem on a bounded domain (it is now strictly elliptic), say a large centered ball B_R of radius R, with Dirichlet boundary conditions $v = v_\infty$ on ∂B_R (the initial data also has to be modified accordingly), and then let $R \to +\infty$.

The solution is smooth and the Maximum Principle applies. The functions $\varepsilon_0 N_{\varepsilon_0}^{-1} v_\infty$ and $(\varepsilon_0 N_{\varepsilon_0})^{-1} v_\infty$ are respectively lower and upper stationary solutions:

$$\frac{\varepsilon_0}{N_{\varepsilon_0}} v_\infty(\xi) \leq v^{\varepsilon_0}_{\varepsilon,\eta}(\tau, \xi) \leq \frac{1}{\varepsilon_0 \, N_{\varepsilon_0}} v_\infty(\xi) \quad \forall \, (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+ \tag{5}$$

uniformly with respect to ε, $\eta > 0$ so that we may let $\eta \to 0$ and keep the above estimate. Note that a similar uniform in ε and η (but local in ξ) estimate holds for $(v^{\varepsilon_0}_{\varepsilon,\eta})^{-1} \left| \nabla_\xi v^{\varepsilon_0}_{\varepsilon,\eta} \right|$. Details are left to the reader.

Now we may build an entropy estimate as follows:

$$\frac{d}{d\tau}\int v_{\varepsilon,0}^{\varepsilon_0}\log\Big(\frac{v_{\varepsilon,0}^{\varepsilon_0}}{v_\infty}\Big)\,d\xi = -\int\left[\frac{\nabla_\xi v_{\varepsilon,0}^{\varepsilon_0}}{v_{\varepsilon,0}^{\varepsilon_0}}-\frac{\nabla_\xi v_\infty}{v_\infty}\right]$$
$$\cdot\left[v_{\varepsilon,0}^{\varepsilon_0}\left(\left[(1-\varepsilon)\left|\frac{\nabla_\xi v_{\varepsilon,0}^{\varepsilon_0}}{v_{\varepsilon,0}^{\varepsilon_0}}\right|^2+\varepsilon\left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^2\right]^{\frac{p}{2}-1}\frac{\nabla_\xi v_{\varepsilon,0}^{\varepsilon_0}}{v_{\varepsilon,0}^{\varepsilon_0}}-\left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^{p-2}\frac{\nabla_\xi v_\infty}{v_\infty}\right)\right]d\xi$$

(which by the way proves that $v_{\varepsilon,0}^{\varepsilon_0}$ converges to v_∞ as $\tau\to+\infty$). Because of (5), such an estimate passes to the limit in integral form as $\varepsilon\to 0$:

$$\int v^{\varepsilon_0}\log\Big(\frac{v^{\varepsilon_0}}{v_\infty}\Big)\,d\xi \le \int f^{\varepsilon_0}\log\Big(\frac{f^{\varepsilon_0}}{v_\infty}\Big)\,d\xi$$
$$-\int_0^\tau\!\!\int v^{\varepsilon_0}\left(\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}-\frac{\nabla v_\infty}{v_\infty}\right)\cdot\left(\left|\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}\right|^{p-2}\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}-\left|\frac{\nabla v_\infty}{v_\infty}\right|^{p-2}\frac{\nabla v_\infty}{v_\infty}\right)d\xi\,d\tau\,, \tag{6}$$

where $v^{\varepsilon_0}:=v_{0,0}^{\varepsilon_0}$ is now a solution of

$$\begin{cases} {v^{\varepsilon_0}}_\tau=\nabla_\xi\left[v^{\varepsilon_0}\left(\left|\frac{\nabla_\xi v^{\varepsilon_0}}{v^{\varepsilon_0}}\right|^{p-2}\frac{\nabla_\xi v^{\varepsilon_0}}{v^{\varepsilon_0}}-\left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^{p-2}\frac{\nabla_\xi v_\infty}{v_\infty}\right)\right]\\ v^{\varepsilon_0}(\cdot,\tau=0)=f^{\varepsilon_0}\end{cases}$$

satisfying (5) and such that $(v^{\varepsilon_0})^{-1}\,|\nabla_\xi v^{\varepsilon_0}|$ is locally bounded in ξ (however this estimate is certainly not true uniformly with respect to ε_0).

We may now go back to the original variables, t and x. Let u^{ε_0} be the solution of Equation (1) with initial data f^{ε_0} and consider $u_\infty=\frac{1}{R(t)^n}\,v(\frac{x}{R(t)},\log R(t))$. Since

$$\int u\log\Big(\frac{u}{u_\infty}\Big)dx=\int u\log u\,dx+(p-1)(R(t))^{-p^*}\int|x|^{p^*}u\,dx+\sigma(t)\int u\,dx$$

for some C^1 function σ, it is sufficient to study the first term of the right hand side to pass to the limit as $\varepsilon_0\to 0$ in the entropy inequality, *i.e.*,

$$\frac{d}{dt}\int u^{\varepsilon_0}\log u^{\varepsilon_0}\,dx=-\frac{1}{p-1}\int\left|p^*\,\nabla(u^{\varepsilon_0})^{1/p}\right|^p\,dx\,.$$

A crucial remark is the following lemma, which has been stated in [5] (also see [6]) for $p=2$ and in [20] in the other cases. For completeness, we give a proof of it.

Lemma 6 [20] *On the space $\{u\in L^1(\mathbb{R}^n)\ :\ u^{1/p}\in W^{1,p}(\mathbb{R}^n)\}$, the functional $F[u]:=\int|\nabla u^\alpha|^p\,dx$ is convex for any $p>1$, $\alpha\in[\frac{1}{p},1]$.*

Proof. For any two given nonnegative C^1 with compact support functions u_1, u_2, let

$$u^t = t\,u_2 + (1-t)\,u_1 = u_1 + t\,v \text{ with } v = u_2 - u_1\ , \quad f(t) = F[u^t]\ .$$

It is readily checked that $f(t)$ is finite for any $t \in [0,1]$ and twice differentiable. For simplicity, we shall write u instead of u^t in the computations. Define

$$\begin{aligned}
X &= \alpha\, u^{\alpha-1}\,\nabla u \\
Y &= \alpha\, u^{\alpha-2}\,[(\alpha-1)\,v\,\nabla u + \,u\,\nabla v] \\
Z &= \alpha\,(\alpha-1)\,u^{\alpha-3}\,[(\alpha-2)\,v^2\,\nabla u + \,2\,u\,v\,\nabla v]
\end{aligned}$$

Then

$$\begin{aligned}
f''(t) &= p\int |X|^{p-4}\left[(p-2)\,(x\cdot Y)^2 + |X|^2(|Y|^2 + X\cdot Z)\right]dx \\
&= p\,\alpha^4\!\int |X|^{p-4}u^{4\alpha-6}\frac{A^2}{v^2}\left[{\scriptstyle (\alpha-1)((\alpha-1)p-1)}A^2 + {\scriptstyle 2p(\alpha-1)}AB + {\scriptstyle (p-1)}B^2\right]dx
\end{aligned}$$

where $A = v\nabla u$ and $B = u\nabla v$. The quantity ${\scriptstyle (\alpha-1)((\alpha-1)p-1)}A^2 + {\scriptstyle 2p(\alpha-1)}AB + {\scriptstyle (p-1)}B^2$ is nonnegative for any A, $B \in \mathbb{R}^n$ if and only if $0 \geq [p(\alpha-1)]^2 - (p-1)\,(\alpha-1)((\alpha-1)p-1) = (\alpha\,p-1)(\alpha-1)$. $\square$

In the case of Equation (1) the entropy production term is therefore convex. Thus the entropy inequality (6) passes to the limit as $\varepsilon_0 \to 0$. By the Dunford-Pettis criterion, u^{ε_0} converges to some function u weakly in $L^1(\mathbb{R}^n \times \mathbb{R}^+_{\mathrm{loc}})$. Moreover, because of the divergence form of the right hand side of the equation, we have

$$\frac{d}{dt}\int u^{\varepsilon_0}\,dx = 0$$

so that $\int u\,dx$ is also conserved. Since

$$(p-1)\nabla u^{1/(p-1)} = p\,u^{1/(p(p-1))}\nabla u^{1/p}\ ,$$

we obtain

$$\|\nabla u^{1/(p-1)}\|_{p-1} \leq p^*\|u\|_1^{1/(p(p-1))}\,\|\nabla u^{1/p}\|_p$$

by Hölder's inequality (this even makes sense for $p \in (1,2)$ since the Hölder exponents are p and p^*). There is no difficulty to check that $u(\cdot,0) = f$ and that u^{ε_0} strongly converges to u in $L^1_{\rm loc}(\mathbb{R}^n \times \mathbb{R}^+)$. It remains to make sure that u is a solution of (1). Since $\nabla (u^{\varepsilon_0})^{1/p}$ weakly converges to $\nabla u^{1/p}$ in $L^\infty(\mathbb{R}^+_{\rm loc}, L^p_{\rm loc}(\mathbb{R}^n)))$, if $p \geq 2$, $\nabla (u^{\varepsilon_0})^{1/(p-1)}$ weakly converges to $\nabla u^{1/(p-1)}$ in $L^\infty(\mathbb{R}^+_{\rm loc}, L^{p-1}_{\rm loc}(\mathbb{R}^n)))$. This is enough to give a sense to $\Delta_p u$ and prove that u satisfies (1) in the distribution sense. The adaptations to be made if $p \in (1,2)$ are left to the reader. This concludes the proof of existence.

Remark 7 *The entropy decays exponentially since*

$$\frac{d}{dt} \int u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx = -\frac{1}{p-1} \int |p^*\,\nabla u^{1/p}|^p\,dx$$

and Corollary 5 applied with $w = u^{1/p}$, $\mu = \frac{n\,\mathcal{L}_p}{p\,e}$ gives

$$\frac{d}{dt} \int u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx \leq -\frac{(p*)^{p+1} e}{n\,\mathcal{L}_p} \int u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx\;.$$

For a more precise description of the asymptotic behaviour, see [17,19].

It is remarkable that the entropy, or to be precise, the *relative entropy*, turns out to be the right tool for uniqueness as well. Consider two solutions u_1 and u_2 of (1). A simple computation shows that

$$\begin{aligned}
&\frac{d}{dt} \int u_1 \log\Big(\frac{u_1}{u_2}\Big)\,dx \\
&= \int \Big(1 + \log\Big(\frac{u_1}{u_2}\Big)\Big)\;(u_1)_t\,dx - \int \Big(\frac{u_1}{u_2}\Big)\;(u_2)_t\,dx \\
&= -{\scriptstyle (p-1)}^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx\;.
\end{aligned}$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1}\;\|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \leq \int u_1(\cdot,t) \log\Big(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\Big)\,dx \leq \int f \log\Big(\frac{f}{f}\Big)\,dx = 0$$

by the Csiszár-Kullback inequality [16,25].

Remark 8 *The computation we have used above for proving the uniqueness is exactly the same as for the existence proof, with $u_1 = u$ and $u_2 = u_\infty$. This is why the detailed justification of the computation has been omitted. All terms make sense at least in the integrated in t sense. In the stationary case, similar computations have been used extensively, see [8] for an example in case $p = 2$.*

3 Proof of Theorem 2

As a preliminary result, let us note that the quantity $\int u^q \log u\, dx$ makes sense.

Lemma 9 *Let q, Q be such that $1 \le q < Q$ and assume that $u \in L^1 \cap L^Q(\mathbb{R}^n)$ is a nonnegative function such that $|x|^{p^*} u \in L^1(\mathbb{R}^n)$. Then $u^q \log u$ belongs to $L^1(\mathbb{R}^n)$.*

Proof. On the one hand, let $U = \exp(-|x|^{p^* \frac{Q-q}{Q-1}})$. Then

$$\int u^q \log u \; dx = \int u^q \log \Big(\frac{u}{U}\Big) \; dx + \int |x|^{p^* \frac{Q-q}{Q-1}} \, u^q \; dx \; .$$

The first term of the right hand side is bounded from below by Jensen's inequality:

$$\int u^q \log \Big(\frac{u}{U}\Big) \; dx = \frac{1}{q} \int u^q \log \left(\frac{u^q}{U^q}\right) \; dx \ge \frac{1}{q} \int u^q \, dx \, \log \left(\frac{\int u^q \, dx}{\int U^q \, dx}\right)$$

and the second term, which is nonnegative, makes sense because of Hölder's inequality:

$$\int |x|^{p^* \frac{Q-q}{Q-1}} \, u^q \; dx \le \left(\int |x|^{p^*} u \, dx\right)^{\frac{Q-q}{Q-1}} \left(\int u^Q \, dx\right)^{\frac{q-1}{Q-1}} \; .$$

On the other hand (see [9,18])

$$\int u^q \log u \; dx \le \frac{1}{Q-q} \int u^q \, dx \, \log \left(\frac{\int u^Q \, dx}{\int u^q \, dx}\right) \; ,$$

as can be checked using Hölder's interpolation of $\|u\|_r$ between $\|u\|_q$ and $\|u\|_Q$ for some $r \in [q, Q)$ and deriving with respect to r at $r = q$. $\square$

Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq} \int u^q \, dx = \int u^q \log u \; dx \; . \tag{7}$$

Consider now a solution u of (1). For a given $q \in [1, +\infty)$,

$$\frac{d}{dt} \int u^q \, dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p \, dx \; . \tag{8}$$

Assume that q depends on t and let $F(t) = \|u(\cdot,t)\|_{q(t)}$. Let $\;' = \frac{d}{dt}$. A combination of (7) and (8) gives

$$\frac{F'}{F} = \frac{q'}{q^2}\left[\int \frac{u^q}{F^q}\log\left(\frac{u^q}{F^q}\right)dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}}\,\frac{1}{F^q}\int u^{q-p}|\nabla u|^p\,dx\right] .$$

Since $\int u^{q-p}|\nabla u|^p\,dx = \left(\frac{p}{q}\right)^p \int |\nabla u^{q/p}|^p\,dx$, Corollary 5 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } \; A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)ds$$
$$\text{and } \; \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}} .$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 ,$$

which means that

$$q(t) = \frac{1}{a\,t+b}$$

for some a, $b \in \mathbb{R}$. Thus A is given by

$$A(t) = -\frac{n}{p}\int_0^t a\,\log\left(\frac{a\,\mathcal{K}_p}{(a\,s+b)^{p-1}(a\,s+b-1)}\right)ds$$

and an identification of $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$. Note that $a = -q'q^{-2} < 0$. Let $\varphi(u) = (p-1)\,u\,\log u - (1-u)\log(1-u) - p\,u$. Then

$$\begin{aligned} A(t) &= -\tfrac{n}{p}\,a\textstyle\int_0^t \left[\log(-a\,\mathcal{K}_p) - \varphi'(as+b)\right]ds \\ &= \tfrac{n}{p}\,\tfrac{\beta-\alpha}{\alpha\,\beta}\log\left(\tfrac{\beta-\alpha}{\alpha\,\beta}\,\tfrac{\mathcal{K}_p}{t}\right) + \tfrac{n}{p}\left[\varphi\left(\tfrac{1}{\beta}\right) - \varphi\left(\tfrac{1}{\alpha}\right)\right] . \end{aligned}$$

This ends the proof of Theorem 2. □

With a minor adaptation of the above proof, one can state a result similar to the one of Theorem 2 in the case α, $\beta \in (0,1]$ with $\beta \leq \alpha$ and at a formal level in the case $\beta \leq \alpha < 0$ (in both cases, $a > 0$). Since the sign of q' is changed, the inequality is reversed, compared to Theorem 2: such results are not hypercontractivity properties any more. In the second case, the existence of a solution is not covered by Theorem 1 and is, as far as we know, an open question. With $\varphi(u) = (p-1)\,u\,\log u + (u-1)\log(u-1) - p\,u$, one gets the following result.

Theorem 10 *Let α, $\beta \in (0,1]$ with $\beta \leq \alpha$. Under the same assumptions as in Theorem 1, any solution u of (1) with initial data f such that f^α belongs to $L^1(\mathbb{R}^n)$ satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \geq \|f\|_\alpha\, A(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0$$

with

$$A(n,p,\alpha,\beta) = \Big(\mathcal{C}_1\,(\alpha-\beta)\Big)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}\ ,$$

$$\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}\ , \quad \mathcal{C}_2 = \frac{(1-\beta)^{\frac{1-\beta}{\beta}}}{(1-\alpha)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}\ .$$

Here $\mathcal{C}_2$ has the same expression as in Theorem 2 and one can write

$$A(n,p,\alpha,\beta) = \Big(\mathcal{C}_1\,|\beta-\alpha|\Big)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}\ , \quad \mathcal{C}_2 = \frac{|\beta-1|^{\frac{1-\beta}{\beta}}}{|\alpha-1|^{\frac{1-\alpha}{\alpha}}}\,\frac{|\beta|^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{|\alpha|^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}} \tag{9}$$

in order to have a general expression which is valid for both results.

At a formal level (existence of a global solution is not known), it is even possible to state a result for negative exponents α and β. Note indeed that in such a case, the boundedness of $\int u_0^\alpha\,dx$ is incompatible with the requirement: $u_0 \in L^1(\mathbb{R}^n)$. The following result is obtained by adapting the proof of Theorem 2 to the case $\varphi(u) = (p-1)\,u\,\log(-u) - (1-u)\log(1-u) - p\,u$.

Theorem 11 *Let α, $\beta < 0$ with $\beta \leq \alpha$. Any C^2 global solution u of (1) with initial data f such that f^α belongs to $L^1(\mathbb{R}^n)$ satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \geq \|f\|_\alpha\, A(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0$$

where $A(n,p,\alpha,\beta)$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by (9).

4 Large deviations and Hamilton-Jacobi equations

Consider a solution of

$$\begin{cases} v_t + \frac{1}{p}\,|\nabla v|^p = \frac{1}{p-1}\,p^{\frac{2-p}{p-1}}\,\varepsilon^{p^*}\Delta_p v \qquad (x,t)\in \mathbb{R}^n\times\mathbb{R}^+ \\ v(\cdot,t=0)=g \end{cases} \tag{10}$$

The following lemma shows what is the relation of (10) and (1).

Lemma 12 *Let $\varepsilon>0$. Then v is a C^2 solution of (10) if and only if*

$$u = e^{-\frac{1}{\lambda\varepsilon^{p^*}}\,v} \quad \textit{with } \lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$$

is a C^2 positive solution of

$$u_t = \varepsilon^p\,\Delta_p(u^{1/(p-1)}) \tag{11}$$

with initial data $f = e^{-\frac{1}{\lambda\varepsilon^{p^*}}\,g}$.

In the limit case $\varepsilon=0$,

$$Q_t^p g(x) := v(x,t) = \inf_{y\in\mathbb{R}^n}\left\{ g(y) + \frac{t}{p^*}\left|\frac{x-y}{t}\right|^{p^*}\right\}$$

is a solution known as the Hopf-Lax solution of the Hamilton-Jacobi equation (3):

$$v_t + \frac{1}{p}\,|\nabla v|^p = 0\ .$$

Let $P_t^p f(x) := u(x,t)$ whenever u is a solution of (1) with initial data f. Because of the convergence of the solutions of (10) to the solutions of (3), by Lemma 12 we get the following result.

Theorem 13 *With the above notations and assumptions, for any C^2 function g,*

$$Q_t^p g(x) = \lim_{\varepsilon\to 0}\left[-\lambda\,\varepsilon^{p^*}\log\left(P^p_{\varepsilon^p t}\left(e^{-\frac{g}{\lambda\,\varepsilon^{p^*}}}\right)\right)\right] \quad \forall\, t>0\ .$$

In other words, this essentially means that the family $(P^p_{\varepsilon^p t})_{\varepsilon>0}$ satisfies a *large deviation principle* of order ε^{p^*} and rate function $\frac{1}{p^*\, t^{p^*-1}}|\, x - \cdot\, |^{p^*}$.

This provides a new proof of the main result of [22].

Corollary 14 *Let* $\lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$. *For any* α, β *with* $0 \le \alpha \le \beta$, *we may write*

$$\|e^{Q^p_t g}\|_\beta \le \|e^g\|_\alpha \, B(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0\ ,$$

with

$$B(n,p,\alpha,\beta) = \Big((\beta-\alpha)\,\lambda^{p-1}\mathcal{C}_1\Big)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \left(\frac{\alpha^{\frac{p-1}{\alpha}+\frac{1}{\beta}}}{\beta^{\frac{p-1}{\beta}+\frac{1}{\alpha}}}\right)^{\frac{n}{p}} .$$

Proof. We may first rewrite Theorem 10 as

$$\|P^p_\tau f\|_\gamma \ge \|f\|_\delta \left(\frac{\mathcal{C}_1}{\tau}\right)^{\frac{n}{p}\,\frac{\gamma-\delta}{\gamma\delta}} \left\{(\delta-\gamma)^{\frac{\gamma-\delta}{\gamma\delta}}\,\frac{(1-\gamma)^{\frac{1-\gamma}{\gamma}}}{(1-\delta)^{\frac{1-\delta}{\delta}}}\,\frac{(\gamma)^{\frac{1-p}{\gamma}-\frac{1}{\delta}+1}}{(\delta)^{\frac{1-p}{\delta}-\frac{1}{\gamma}+1}}\right\}^{\frac{n}{p}} ,$$

where we replaced α, β and t by δ, γ and τ respectively. Take now $f = e^{-\frac{h}{\lambda\,\varepsilon^{p^*}}}$, $\tau = \varepsilon^p t$, $\delta = \lambda\,\varepsilon^{p^*}\alpha$ and $\gamma = \lambda\,\varepsilon^{p^*}\beta$ and raise the above expression to the power $\lambda\,\varepsilon^{p^*}$. Taking the limit $\varepsilon \to 0$ we obtain,

$$\|e^{-h}\|_\beta \le \|e^{-Q^p_t h}\|_\alpha \, B(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0\ .$$

The result then holds by taking $h = -Q^p_t(g)$ and by using the following inequality: $-Q^p_t(-Q^p_t(g)) \le g$. $\square$

Remark 15 *If instead of Theorem 10, we use Theorem 11, we obtain a direct but formal proof of the Corollary 14. The proof is similar to the one of Corollary 14. According to Theorem 10,*

$$\|P^p_\tau f\|_\delta \ge \|f\|_\gamma \left(\frac{\mathcal{C}_1}{\tau}\right)^{\frac{n}{p}\,\frac{\delta-\gamma}{\gamma\delta}} \left\{(\gamma-\delta)^{\frac{\delta-\gamma}{\gamma\delta}}\,\frac{(1-\delta)^{\frac{1-\delta}{\delta}}}{(1-\gamma)^{\frac{1-\gamma}{\gamma}}}\,\frac{(-\delta)^{\frac{1-p}{\delta}-\frac{1}{\gamma}+1}}{(-\gamma)^{\frac{1-p}{\gamma}-\frac{1}{\delta}+1}}\right\}^{\frac{n}{p}} ,$$

where we replaced α, β *and* t *by* γ, δ *and* τ *respectively. Take now* $f = e^{-\frac{g}{\lambda\,\varepsilon^{p^*}}}$, $\tau = \varepsilon^p t$, $\gamma = -\lambda\,\varepsilon^{p^*}\alpha$ *and* $\delta = -\lambda\,\varepsilon^{p^*}\beta$ *and raise the above expression to the power* $-\lambda\,\varepsilon^{p^*}$. *The result then holds by taking the limit* $\varepsilon \to 0$.

Conclusion

As a conclusion, let us summarize the main results. The three following identities have been established:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p\, dx = 1$,

$$\int |w|^p \log |w|\, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p\, dx \right] .$$

(ii) With the notation P_t^p for the semigroup associated to (1), *i.e.* $u_t = \Delta_p(u^{1/(p-1)})$,

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha\, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} .$$

(iii) With the notation Q_t^p for the semigroup associated to (3), *i.e.* $v_t + \frac{1}{p}\,|\nabla v|^p = 0$,

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha\, B(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} .$$

The first identity is the optimal L^p-Euclidean logarithmic Sobolev inequality (2), see [18,22]. The equivalence (i) $\Longleftrightarrow$ (iii) has been established in [22]. In this paper, what we have seen is that (i) $\Longrightarrow$ (ii) and that (ii) $\Longrightarrow$ (iii). Going back to the proof of Theorem 2, it is not difficult to check that (ii) $\Longrightarrow$ (i), so that the constants in (ii) are optimal.

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