The Optimal Euclidean L^p-Sobolev logarithmic inequality *

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We prove an optimal logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^d)$. Explicit minimizers are given. This result is connected with best constants of a special class of Gagliardo-Nirenberg type inequalities.

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1. MAIN RESULTS

The Euclidean logarithmic Sobolev inequality states that for any function $u \in W^{1,2}(\mathbb{R}^d)$ with $\int |u|^2\, dx = 1$,

$$\int |u|^2 \log |u|\, dx \leq \frac{d}{4} \log \left[\frac{2}{\pi d e} \int |\nabla u|^2\, dx \right] , \tag{1}$$

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Here and in what follows, the symbol $\int$ with no limits specified indicates integration in entire $\mathbb{R}^d$. Stated in this form, inequality (1) appears in the work by Weissler [21]. It is optimal and equivalent to the Gross logarithmic inequality [12] with respect to Gaussian weight,

$$\int |g|^2 \log |g| \, d\mu \leq \int |\nabla g|^2 \, d\mu \tag{2}$$

where $d\mu(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}} dx$ and $\int |g|^2 \, d\mu = 1$. Extremals for (1) are precisely the Gaussians $u(x) = (\pi\sigma)^{-\frac{d}{2}} e^{-\frac{1}{4}\sigma|x-\bar{x}|^2}$ with $\sigma > 0$, $\bar{x} \in \mathbb{R}^d$, see [8]. Different proofs of these estimates have appeared in the literature, see for instance [1, 19, 4]. Geometric and probabilistic implications as well as extensions of these inequalities have been the subject of many works, we refer the reader to [13, 5] for results and further references. It is natural to ask for the validity of a corresponding $W^{1,p}$-analogue of estimate (1). Adams [1] found a class of general L^p-weighted logarithmic inequalities which generalized (2). For $p = 1$, Beckner in [5] finds the optimal inequality

$$\int |u| \log |u| \; dx \leq d \log \left[\mathcal{L}_1 \int |\nabla u| \; dx \right] ,$$

for any $u \in W^{1,1}(\mathbb{R}^d)$ such that $\int |u| \; dx = 1$, with $\mathcal{L}_1 = \frac{1}{\sqrt{\pi} d} \left[\Gamma\left(\frac{d}{2}+1\right) \right]^{\frac{1}{d}}$, the extremals being characteristic functions of balls. While finding non-optimal L^p-versions of the logarithmic Sobolev inequality is not difficult, as we illustrate below, the methods developed in the works above metioned do not seem to apply for general $1 < p < d$. This open question is answered in the following result.

Theorem 1.1. *Let us assume $1 < p < d$. Then for any $u \in W^{1,p}(\mathbb{R}^d)$ with $\int |u|^p \; dx = 1$ we have,*

$$\int |u|^p \log |u| \; dx \leq \frac{d}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \; dx \right] \tag{3}$$

where
$$\mathcal{L}_p = \frac{p}{d} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \right]^{\frac{p}{d}} . \tag{4}$$

Inequality (3) is optimal and equality holds if and only if for some $\sigma > 0$ and $\bar{x} \in \mathbb{R}^d$

$$u(x) = \pi^{-\frac{d}{2}} \sigma^{-d\,\frac{p-1}{p}} \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \, e^{-\frac{1}{\sigma}|x-\bar{x}|^{\frac{p}{p-1}}} \quad \forall\, x \in \mathbb{R}^d . \tag{5}$$

Before proceeding, we will see that obtaining a non-optimal constant in this inequality is a relatively simple matter. A first observation is that we have the validity of the following logarithmic interpolation inequality:

Assume $1 \le p < s < +\infty$. *Then for any* $u \in L^p(\mathbb{R}^d) \cap L^s(\mathbb{R}^d)$, *we have*

$$\int u^p \log\left(\frac{|u|}{\|u\|_p}\right)\, dx \le \frac{s}{s-p}\|u\|_p \log\left(\frac{\|u\|_s}{\|u\|_p}\right)\,. \tag{6}$$

Indeed, let us consider Hölder's inequality $\|u\|_q \le \|u\|_p^\alpha \|u\|_s^{1-\alpha}$ with $\alpha = \frac{p}{q}\,\frac{s-q}{s-p}$, $p \le q \le s$, and let us take logarithm of both sides. Then we obtain

$$\log\left(\frac{\|u\|_q}{\|u\|_p}\right) + (\alpha-1)\log\left(\frac{\|u\|_p}{\|u\|_s}\right) \le 0\,.$$

Since this inequality trivializes to an equality when $q = p$, we may differentiate it with respect to q at $q = p$ and (6) immediately follows. Here and in what follows we denote for any $q > 0$, $\|v\|_q = \left(\int |v|^q\, dx\right)^{1/q}$.

Now, let us apply (6) with $1 \le p < d$, $s = \frac{d\,p}{d-p}$. Using Sobolev inequality we obtain, as noticed by Beckner [5], the inequality

$$\forall\, u \in W^{1,p}(\mathbb{R}^d) \qquad \int |u|^p \log|u|\, dx \le \frac{d}{p^2}\log\left[C_p \int |\nabla u|^p\, dx\right]\,,$$

where C_p is Talenti's constant [17]:

$$C_p = \frac{1}{d}\left(\frac{p-1}{d-p}\right)^{p-1} \pi^{-\frac{p}{2}} \left(\frac{\Gamma(d)\,\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{p})\,\Gamma(d\,\frac{p-1}{p}+1)}\right)^{\frac{p}{d}}\,.$$

The best constant in (3) given by (4) is strictly smaller than C_p for $p > 1$. but equality holds in the limit $p \to 1$ and C_p and $\mathcal{L}_p$ are asymptotically equivalent as $d \to +\infty$. Stirling's formula: $\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\, x^x e^{-x}$ as $x \to +\infty$, indeed gives

$$\frac{C_p}{\mathcal{L}_p} = \frac{1}{p} e^{p-1}(d-p)^{-(p-1)}\left(\frac{\Gamma(d)}{\Gamma(\frac{d}{p})}\right)^{\frac{p}{d}} \to 1 \quad \text{as } d \to +\infty\,.$$

Our approach in the proof of Theorem 1.1 consists of finding Inequality (3) as a limiting case of a family of Gagliardo-Nirenberg inequalities which are also optimal and of independent interest. In order to state that result, we need to introduce some notation. We designate by $\mathcal{D}^{p,q}$ the completion of the space of smooth compactly supported functions on $\mathbb{R}^d$ for the norm $\|\cdot\|_{p,q}$ defined by $\|u\|_{p,q} = \|\nabla u\|_p + \|u\|_q$. Given $1 < p < q$, let us consider the number

$$r = p\,\frac{q-1}{p-1}\ . \tag{7}$$

Theorem 1.2. *Assume that $1 < p < d$, $p < q \leq \frac{p(d-1)}{d-p}$. Then for all $u \in \mathcal{D}^{p,q}$,*

$$\|u\|_r \leq \mathcal{S}\, \|\nabla u\|_p^{\theta}\, \|u\|_q^{1-\theta}. \tag{8}$$

Here r is given by (7),

$$\theta = \frac{(q-p)\,d}{(q-1)\,(d\,p - (d-p)\,q)}\ , \tag{9}$$

and with $\delta = d\,p - q\,(d-p) > 0$, the optimal constant $\mathcal{S}$ takes the explicit form:

$$\mathcal{S} = \left(\frac{q-p}{p\sqrt{\pi}}\right)^{\theta} \left(\frac{p\,q}{d\,(q-p)}\right)^{\frac{\theta}{p}} \left(\frac{\delta}{p\,q}\right)^{\frac{1}{r}} \left(\frac{\Gamma(q\,\frac{p-1}{q-p})\,\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{p-1}{p}\,\frac{\delta}{q-p})\,\Gamma(d\,\frac{p-1}{p}+1)}\right)^{\frac{\theta}{d}}.$$

Equality holds in (8) if and only if for some $\alpha \in \mathbb{R}$, $\beta > 0$, $\bar{x} \in \mathbb{R}^d$,

$$u(x) = \alpha\left(1 + \beta\,|x-\bar{x}|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}} \quad \forall\, x \in \mathbb{R}^d\ . \tag{10}$$

Let us observe that when $q = p\,\frac{d-1}{d-p}$, we have $\theta = 1$, and $r = \frac{d\,p}{d-p}$, the critical Sobolev exponent. Inequality (8) then becomes the optimal Sobolev inequality with $\mathcal{S} = C_p$, as found by Aubin and Talenti in [2, 17]. On the other hand, as already quoted, estimate (3) corresponds to the limit $q \downarrow p$ in (8). These Gagliardo-Nirenberg inequalities thus interpolate in optimal way between the Sobolev and the logarithmic Sobolev inequalities.

Approximation of best constants have been studied in [14]. The idea of taking a derivative with respect to some parameter in a family of inequalities has been used in different settings in [3, 5]. Optimal Gagliardo-Nirenberg inequalities for $p = 2$ were used in the study of intermediate asymptotics of fast diffusion and porous medium equations [11] (the limit $q \to 2$ corresponds to inequality (1) used for the heat equation).

The proof of Theorem 1.2 in §3 is carried out by direct minimization in a similar spirit as that in [2, 17], except that we shall rely on a nontrivial uniqueness result of radial solutions of equations involving the p-Laplacian recently found by Serrin and Tang in [16]. Identification of all extremals use symmetry results of Gidas-Ni-Nirenberg type for the p-Laplacian [9, 7]. Note that as in the case $p = 2$ [11], when $q < p$, Theorem 1.2 has a corresponding version which we discuss at the end of this paper.

2. PROOF OF THEOREM 1.1

We will carry out the proof of Theorem 1.1 based on Theorem 1.2, except that we postpone the characterization of the minimizers for the end of next section.

Let $u \in W^{1,p}(\mathbb{R}^d) \setminus \{0\}$. Then $u \in \mathcal{D}^{p,q}$ for any $q \in [p, \frac{p(d-1)}{d-p}]$. Taking logarithm to both both sides of inequality (8) at this u, we get

$$\frac{1}{\theta} \log \left(\frac{\|u\|_r}{\|u\|_q} \right) - \frac{1}{\theta} \log \mathcal{S} \leq \log \left(\frac{\|\nabla u\|_p}{\|u\|_q} \right) . \tag{11}$$

Note that

$$\theta = \left(\frac{d}{(p-1)p^2} + o(1) \right) (q-p) \quad \text{as } q \downarrow p .$$

Since, we recall, $r = p\,\frac{q-1}{p-1}$, a direct computation of the first term in the left hand side of (11) yields

$$\frac{p}{d} \int \frac{u^p}{\|u\|_p^p} \log \left(\frac{u^p}{\|u\|_p^p} \right) dx - \lim_{q \downarrow p} \frac{1}{\theta} \log \mathcal{S} \leq \log \left(\frac{\|\nabla u\|_p}{\|u\|_p} \right) . \tag{12}$$

Now we compute $\lim_{q\downarrow p} \frac{1}{\theta} \log \mathcal{S}$. To do so, we choose for $\mathcal{S}$ the extremal function:

$$w_q(x) = \left(1 + \frac{q-p}{p-1} |x|^{\frac{p}{p-1}} \right)^{-\frac{p-1}{q-p}} ,$$

which converges to $w(x) = e^{-|x|^{\frac{p}{p-1}}}$ as $q \downarrow p$. Thus

$$\lim_{q \downarrow p} \frac{1}{\theta} \log \mathcal{S} = -\log \left(\frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{(p-1)p^2}{d} \lim_{q \downarrow p} \frac{\log \left(\frac{\|w_q\|_r}{\|w_q\|_q} \right)}{q-p} = I + II .$$

Now,

$$II = \frac{1}{d} \int \frac{w^p}{\|w\|_p^p} \log \left(\frac{w^p}{\|w\|_p^p} \right) dx \,+\, \frac{(p-1)p^2}{d} \, (III - IV) ,$$

where

$$III = \lim_{q\to p} \frac{1}{q-p}\left[\log\left(\frac{\|w_q\|_r}{\|w_q\|_q}\right)\right], \quad IV = \lim_{q\to p} \frac{1}{q-p}\left[\log\left(\frac{\|w\|_r}{\|w\|_q}\right)\right].$$

Now,

$$III = -\frac{1}{(p-1)p^2}\log(\|w\|_p^p) + \frac{1}{p}\lim_{q\to p}\frac{1}{q-p}\left[\log(\|w_q\|_r^r) - \log(\|w_q\|_q^q)\right].$$

Thus

$$III = -\frac{1}{(p-1)p^2}\log(\|w\|_p^p) + \frac{1}{p(p-1)}\frac{1}{\|w\|_p^p}\int pw^{p-1}\,dx$$

Exactly the same computation yields $III = IV$. Hence

$$\lim_{q\downarrow p}\frac{1}{\theta}\log\mathcal{S} = -\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right) + \frac{1}{d}\int\frac{w^p}{\|w\|_p^p}\log\left(\frac{w^p}{\|w\|_p^p}\right)dx \equiv \frac{1}{p}\log\mathcal{L}_p\,.$$

Using the facts

$$\int e^{-|x|^a}\,dx = \frac{2\pi^{\frac{d}{2}}}{a}\frac{\Gamma(\frac{d}{a})}{\Gamma(\frac{d}{2})}, \quad \int e^{-|x|^a}|x|^a\,dx = \frac{d}{a}\int e^{-|x|^a}\,dx\,,$$

we find that $\mathcal{L}_p$ satisfies (4). Then inequality (3) readily follows from (12).

The optimality of $\mathcal{L}_p$ and the fact that functions of the form (5) are extremals follow at once the from teh optimality of the extremals for $\mathcal{S}$. We postpone to the end of next section the proof of the fact that functions (5) constitute all extremals for $\mathcal{L}_p$. □

3. PROOF OF THEOREM 1.2

We shall assume in the proof that $q < \frac{p(d-1)}{d-p}$ since the case of equality corresponds precisely to the usual optimal Sobolev inequality.

Let us consider the functional defined in $\mathcal{D}^{p,q}$ as

$$J(u) = \frac{1}{p}\int|\nabla u|^p dx \;+\; \frac{1}{q}\int|u|^q dx$$

Given a number $K > 0$ which we will fix later, let us consider the set

$$\mathcal{M}_K = \{u\in\mathcal{D}^{p,q}\ /\int|u|^r = K\}\,.$$

Let us set

$$c^* = \inf_{u \in \mathcal{M}_K} J(u) \,. \tag{13}$$

Using Sobolev and Hölder's inequalities, we easily see that $c^* > 0$. Moreover, this number is attained:

LEMMA 3.1. *There exists a radially symmetric, non-negative function $\bar{u} \in \mathcal{M}_K$, $\bar{u} = \bar{u}(|x|)$ such that $J(\bar{u}) = c^*$.*

Let us assume for the moment the validity of this fact and let us use it to establish estimate (8). We want to identify the minimizer $\bar{u}$ predicted by Lemma 3.1 for a special choice of K. By the Lagrange multiplier rule, $\bar{u}$ is a positive ground state radial solution of an equation of the form

$$-\Delta_p u + u^{q-1} - \mu u^{r-1} = 0 \quad \text{in } \mathbb{R}^d. \tag{14}$$

for certain $\mu > 0$. Here Δ_p stands for the standard p-Laplacian operator, $\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$. Now, the transformation

$$w(x) = \mu^{\frac{1}{r-q}} u(\mu^{\frac{q-p}{p(r-q)}} x)$$

takes equation (14) into

$$-\Delta_p w + w^{q-1} - w^{r-1} = 0 \quad \text{in } \mathbb{R}^d, \tag{15}$$

Equation (15) has a explicit solution given by

$$w_*(x) = \alpha(1 + \beta|x|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}$$

with $\alpha = \left(\dfrac{p\,(q-1)}{p\,(d-1) - q\,(d-p)}\right)^{\frac{p-1}{q-p}}$, $\beta = (q-1)\left(\dfrac{q-p}{p\,(d-1) - q\,(d-p)}\right)^{\frac{p}{p-1}}$.

At this point we invoke a result by Serrin and Tang in [16], which ensures that the radial positive ground state solution of (15) is unique. Therefore we must have $\bar{u}(x) = \mu^{-\frac{1}{r-q}} w_*(\mu^{-\frac{q-p}{p(r-q)}} x)$. Now,

$$\int \bar{u}(x)^r dx = \mu^{\frac{d(q-p)}{p(r-q)} - \frac{r}{r-q}} \int w_*(x)^r dx = K.$$

At this point we make the convenient choice of K in the definition of $\mathcal{M}_K$, $K \equiv \int w_*(x)^r dx$. Then we find that, necessarily $\mu = 1$, and $\bar{u} = w_*$. Thus we have the inequality $J(w_*) \leq J(u)$ for all $u \in \mathcal{M}_K$. Now, given such a

u, we consider for $\lambda > 0$ the function $u_\lambda = \lambda^{\frac{d}{r}} u(\lambda x)$. Then $J(w_*) \leq J(u_\lambda)$ or

$$J(u^*) \leq \lambda^{\frac{(d-p)}{r}(\frac{pd}{d-p}-r)} \int \frac{|\nabla u|^p}{p} dx \; + \; \lambda^{-(1-\frac{q}{r})d} \int \frac{|u|^q}{q} dx \tag{16}$$

for all $\lambda > 0$. Minimizing the right hand side of (16) in λ we obtain the existence of an optimal positive constant $\mathcal{T}$ depending only on p, q and d such that

$$\mathcal{T} \leq \|\nabla u\|_p^\theta \|u\|_q^{1-\theta} \tag{17}$$

for all $u \in \mathcal{M}_K$, where θ is given by (9) and (17) is reached with equality at $u = w_*$. ¿From here, the optimal inequality (8) readily follows, as well as the fact that the functions (10) are extremals for it. The computation of the optimal constant $\mathcal{S}$ can be carried out directly using properties of the Gamma function.

Let us now prove Lemma 3.1. Although this is a relatively standard fact, we provide a self-contained argument along the lines of [6], see also [15]. Using Schwarz' symmetrization, it suffices to seek the minimizers within the subset of $\mathcal{M}_K$ of non-negative radial functions $u(|x|)$ which are decreasing and go to zero as $|x| \to \infty$. Let us consider a minimizing sequence u_n for J on $\mathcal{M}_K$, constituted by radially symmetric decreasing functions. Then u_n may be assumed to converge weakly in $\mathcal{D}^{p,q}$ and in L^r to some $\bar{u}$, and strongly in L^r over compact sets. By semicontinuity, $\bar{u}$ is a minimizer of (13) if we show that $u_n \to \bar{u}$ strongly in $L^r(\mathbb{R}^d)$. This is an immediate consequence of the following result, which is a variation of the well-known Strauss compactness lemma.

Lemma 3.2. *Let p, q and r be given numbers as in the statement of Theorem 1.1. Then there exist positive constants C and σ such that for all $u \in D^{p,q}$,*

$$\forall \, \rho > 0 \; , \quad \int_{|x|>\rho} |u|^r \, dx \leq C \, \|u\|_{p,q} \, \rho^{-\sigma} \; .$$

Proof. Let us write

$$u^p(\rho) = -p \int_\rho^\infty u(s)^{p-1} u'(s) \frac{1}{s^{d-1}} s^{d-1} \, ds \; .$$

Let t be defined by the relation $\frac{p-1}{q} + \frac{1}{p} + \frac{1}{t} = 1$, namely $\frac{1}{t} = (p-1)(\frac{1}{p} - \frac{1}{q})$. Using that $p < d$ and Hölder's inequality we find

$$u(\rho) \leq C \, \|\nabla u\|_p^{\frac{1}{p}} \, \|u\|_q^{\frac{p-1}{p}} \, \rho^{-b} \; ,$$

where $b = ((d-1)t - d)/pt$. Now, by interpolation,

$$\left(\int_{|x|>\rho} |u|^r \, dx \right)^{\frac{1}{r}} \leq \|u\|_q^{(1-\alpha)} \left(\int_{|x|>\rho} |u|^{\frac{dp}{d-p}} \, dx \right)^{\alpha \, \frac{(d-p)}{dp}} \leq C \, \rho^{\alpha \frac{d-p}{dp}(d-1-\frac{bdp}{d-p})}$$

for a certain $\alpha > 0$. Now, we directly check that $b\frac{d\,p}{d-p} > d$ if and only if $(p-1)\,t > d$, *i.e.* $q < dp/(d-p)$. This relation is automatically satisfied by assumption, and the lemma thus follows. □

It only remains to prove that all extremals need to be radially symmetric around some point and are therefore of the given form (10). It is a standard matter that weak solutions of equation (14) are at least of class C^1 [18]. Minimizers need to be strictly one-signed, for if u is a minimizer so is $|u|$ and the result follows from a strong maximum principle for the p-Laplacian in [20]. For $1 < p < 2$, radial symmetry of positive solutions is a special case of a result contained in [10] which is an extension of [9]. There, radial symmetry is proven for an equation of the form $\Delta_p u + f(u) = 0$ for f nonincreasing near $u = 0$ and locally Lipschitz in $(0, \infty)$. For $p > 2$, symmetry follows from Theorem 7.3 in [7]. In that result the assumption $u \in W^{1,p}(\mathbb{R}^d)$ was used, however examining the proof one sees that only "enough decay" is needed. Let us make this more precise. First of all, we observe that the solution u is uniformly small outside a large ball. Indeed, a standard Moser iteration yields an interior estimate in concentric balls of fixed radii, estimating L^∞-norm of the solution in terms of $W^{1,p}$ norm in the larger ball. The latter quantity gets small far from the origin since $u \in \mathcal{D}^{p,q}$. Let us say $u(x) < \varepsilon_0$, sufficiently small, for all $|x| > R_0$:

$$-\Delta_p u + \frac{1}{2} u^{q-1} \leq 0 \quad \text{for } R_0 \leq |x| \; .$$

It is readily checked that the function $w(x) = K\,|x|^{-\frac{p}{q-p}}$ satisfies $-\Delta_p w + \frac{1}{2} w^{q-1} \geq 0$ for $x \neq 0$ and any K sufficiently large. Now, if we choose K so that $w(R_0) > \varepsilon_0$, we obtain by integral comparison $u \leq w$ for $|x| > R_0$. Now, using $\eta^2 u$ as a test function in the equation satisfied by u, where η is a cut-off function which equals zero for $|x| < R/2$, is equal to one for $|x| > R$, and $|\nabla \eta| \leq C/R$, we obtain that

$$\int_{|x|>R} |\nabla u|^p dx \leq C \int_{|x|>R/2} u^q dx \leq C R^{d - \frac{q}{p} q - p}.$$

We can then find a sequence $R = R_n \to \infty$ along which

$$\int_{|x|=R_n} |\nabla u|^p d\sigma \leq C R_n^{d-1-\frac{qp}{q-p}}.$$

$$(\int_{|x|=R_n} |\nabla u|^p d\sigma)^{\frac{p-1}{p}} (\int_{|x|=R_n} |w|^p d\sigma)^{\frac{1}{p}} \leq C R_n^{d-1-\frac{q(p-1)}{q-p}-\frac{p}{q-p}} .$$

The exponent in the last term is negative thanks to $q < \frac{dp}{d-p}$, hence that quantity goes to zero. This fact suffices for the the argument in [7] p. 201 to go through and the proof of the theorem is concluded. □

End of the proof of Theorem1.1. Finally let us show that all minimizers corresponding to the logarithmic Sobolev inequality are also given by the functions (5). In this case, the extremals correspond after scaling to the positive ground state solutions of a problem of the form

$$\Delta_p u + u^{p-1} \log u = 0 .$$

Again radial symmetry follows from [9] for $1 < p < 2$ and [7] for $p > 2$. The uniqueness result of [16] applies to show that the radial solution is unique. □

We end this paper by stating a family of optimal Galiardo-Nirenberg inequalities when $q < p < d$.

THEOREM 3.1. *Assume that $1 < p < d$, $1 < q < p$. There exists a constant $\mathcal{S}$ such that for all $u \in \mathcal{D}^{p,q}$,*

$$\|u\|_q \leq \mathcal{S} \, \|\nabla u\|_p^\theta \, \|u\|_r^{1-\theta} ,$$

where r is given by (7), $\theta = \frac{(p-q)\,d}{q\,(d\,(p-q)+p\,(q-1))}$ and with $\delta = d\,p - q\,(d-p) > 0$, the optimal constant $\mathcal{S}$ takes the explicit form:

$$\mathcal{S} = \left(\frac{p-q}{p\sqrt{\pi}}\right)^\theta \left(\frac{p\,q}{d\,(p-q)}\right)^{\frac{\theta}{p}} \left(\frac{p\,q}{\delta}\right)^{\frac{1-\theta}{r}} \left(\frac{\Gamma(\frac{p-1}{p}\,\frac{\delta}{p-q}+1)\,\Gamma(\frac{d}{2}+1)}{\Gamma(q\,\frac{p-1}{p-q}+1)\,\Gamma(d\,\frac{p-1}{p}+1)}\right)^{\frac{\theta}{d}} .$$

If $q > 2 - \frac{1}{p}$, equality holds if and only if for some $\alpha \in \mathbb{R}$, $\beta > 0$, $\bar{x} \in \mathbb{R}^d$,

$$u(x) = \alpha \left(1 - \beta\,|x-\bar{x}|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{q-p}} \quad \forall\, x \in \mathbb{R}^d .$$

Proof. Notice that the extremals are compactly supported functions. A minimisation procedure similar to that of Theorem 1.2 can be carried out for the functional $\frac{1}{p}\int_{\mathbb{R}^d} |\nabla u|^p \; dx + \frac{1}{r}\int_{\mathbb{R}^d} |u|^r \; dx$ under an appropriate constraint on $\|u\|_q$. One can prove the existence of a radial minimizer using approximations on balls and an appropriate scaling argument as in

[11]. If $q > 2 - \frac{1}{p}$ (which means $r > 1$), this minimizer is a radial ground state solution of $-\Delta_p u + u^{r-1} - u^{q-1} = 0$. Using the symmetry and uniqueness results above quoted, one can then show that there is no other radial minimizer. □

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